The problem of establishing a realistic dimensioning criterion is discussed and solutions are proposed.

The theoretical part of the study consists in a presentation of a general mathematical model for lost-call-cleared systems. This model covers all possible traffic source configurations, with or without unbalance. For practical telephone applications, this model is particularized for the one-stage full availability groups and the state probability distribution in the stationary state is given with the most general formulation.

I. INTRODUCTION

The problem of unbalanced traffic sources can be analyzed and developed from two complementary approaches. In the first one, presented in [1], small groups of randomly chosen traffic sources were studied. The possible dangers of such a random choice were stressed through the fact that the traffic of the group may exceed the maximum value, leading to possibilities of large congestion values in the loss-system to which the group under consideration is connected.

In the second approach, which constitutes the object of the present study, small groups of selectively chosen sources are considered; this selective choice is made according to the individual traffic of each source, which is assumed to be a-priori known. These unbalanced groups are connected to lost call-cleared systems and the resulting congestions are studied on both numerical and theoretical aspects.

The numerical study is performed on some typical examples of one-stage-full-availability groups and link-systems with concentration in the first switching stage; most of the results can be considered as general ones for these configurations.

A theoretical mathematical model is developed in the appendix, covering all possible traffic source configurations. The use of the Markov process theory allows obtaining explicit formulae for the state probability distribution in the stationary state. For telephone applications, this model is particularized for the one-stage-full-availability groups for which the state-probability distribution is given in the most general way. According to each specific example of unbalanced source configuration, it is then possible to derive the exact formulae for the congestions in terms of the state probabilities and, further, to compute them.

II. ONE STAGE FULL-AVAILABILITY GROUPS WITH UNBALANCED TRAFFIC SOURCES

II.1. Statement of the problem

One stage full-availability groups are considered here; they are characterized by their number \( K \) (finite) of devices and their total number \( K \) of traffic sources \((K > R \text{ and } K \text{ finite or not})\). These groups work under the "lost-call-cleared" assumptions. The objectives are to establish the expressions of the occupancy probability distributions and to derive the blocking formulae under the following alternative conditions:

- either the \( K \) sources are strictly identical, with an offered traffic \( E_0 \) per source or, reversely, the \( K \) sources are not identical, insofar as their traffic characteristics are concerned.
- The first case leads to the very well-known "Engset" or "Erlang" occupancy probability distributions, according to whether \( K > R \) or \( K \leq R \). In the Engset case, the call and time congestions are respectively denoted \( C^* \) and \( B^* \) and, further on, many references will be made to these congestions.

For non-identical traffic sources, the most general case is obtained when considering \( n \) homogeneous groups of sources \( G_1, \ldots, G_n \), with each \( G_i \) characterized by the following parameters:
For practical applications, this general case is mainly particularized in the case of \( n = 2 \) homogeneous groups of sources, representing either 2 groups of \( K_1 \) and \( K_2 \) identical subscribers, each group with a different traffic offered \( A_1 \) and \( A_2 \), or one group of \( K_2 \) subscribers (traffic \( A_1 \)) mixed with one group of \( K_1 \) subscribers (traffic \( A_2 \)), or 2 groups of \( K_1 \) and \( K_2 \) trunks belonging to 2 different routes (traffic \( A_1 \) and \( A_2 \)).

Further on, this will be denoted by:

- \( C_i \) the weighed call-congestion, calculated with \( C_1 \) and \( C_2 \) through the formula \( C = A_1 C_1 + A_2 C_2 \)
- \( B \) the time congestion (probability that the \( B \) devices are simultaneously busy)

A lot of numerical examples have been calculated and the results which are given hereunder may qualitatively be considered as general ones.

II.2. Case of two homogeneous groups of traffic sources

In the case of 2 homogeneous groups of sources, the complete numerical problem may be described by using the following parameters:

- \( \overline{C} \) which is the average offered traffic per source \( \overline{C} = \frac{A_1}{1} + \frac{A_2}{2} \)
- \( \overline{Z} \) which is the disequilibrium of traffic, defined by the ratio \( \frac{A_2}{K_2} / \frac{A_1}{K_1} \)

(In a general way, the index 2 will always be reserved for the group having the highest traffic \( A_2/K_2 \) per source; so we get \( \overline{Z} \geq 1 \)).

II.2.1. Case of two homogeneous groups of subscribers

The figures 1 and 2 illustrate the particular case of one switching matrix with 8 outlets and 16 inlets to which are connected 14 low traffic and 2 high traffic subscribers.

The Figure 1 shows the variations of the various congestions \( C_1, C_2, \overline{C} \) and \( B \) versus the disequilibrium \( \overline{Z} \) and for a fixed value of the average traffic per source \( \overline{C} \).

For comparison purposes, the values of \( B^{*} \) and \( C^{*} \) also appear on this curve; they correspond to a homogeneous group of 16 sources, each of them offering a traffic \( \overline{C} \).

Obviously, for \( \overline{Z} = 1 \), this gives \( B = B^{*} \) on the one hand, and \( C = C^{*} = C^{*} \) on the other hand.

The figure 2 is based on this property. Here, the variation of \( C^{*} \) versus \( \overline{C} \) is represented by the heavy line curve of the figure. Each point \( C^{*}(\overline{C}) \) of this curve may be taken as the origin \( \overline{Z} = 1 \) of a secondary horizontal axis for the parameter \( \overline{C} \).

For comparison purposes, the values of \( \overline{B} \) and \( \overline{C} \) also appear on this curve; they correspond to a homogeneous group of 16 sources, each of them offering a traffic \( \overline{C} \).

Obviously, for \( \overline{Z} = 1 \), this gives \( \overline{B} = \overline{B}^{*} \) on the one hand, and \( \overline{C} = \overline{C}^{*} = \overline{C}^{*} \) on the other hand.

The complete set of curves appearing on this figure 2 may be interpolated or extrapolated in order to get results for any value of the parameters \( \overline{B} \) and \( \overline{C} \). More particularly, it is now possible to answer the following reverse problem, which is of a great practical interest: for a given value of the congestion and a given value of the disequilibrium of traffic, what is the maximum value of...
the average traffic offered per source (or of the total traffic offered) which can be handled by the system?

II.2.2. Case of one group of subscribers mixed with trunks

In some folded or partly folded networks, it is common practice to mix subscribers and trunks on the same switching matrix. So, the question of determining the specific congestions for each type of source is of practical interest.

The figure 3 relates to the case of one switching matrix with 8 outlets and 16 inlets to which are connected 14 low traffic subscribers and 2 high traffic trunks belonging to the same route. It gives the variations of the various congestions \( C_i \) (for the subscribers), \( C'_i \) (for the trunks), \( C \) (weighed call congestion) and \( B \) (time congestion) versus the disequilibrium of traffic \( \varepsilon \) and for a fixed value of the average traffic per source.

One particular point may be remarked on this figure. This is the fact that the previously mentioned properties \( B(\varepsilon,\varepsilon = 1) = C(\varepsilon,\varepsilon = 1) = C'(\varepsilon,\varepsilon = 1) = C^*(\varepsilon) \) for all \( \varepsilon \) are no longer true in the present case. Here, this gives \( B(\varepsilon,\varepsilon = 1) > B^*(\varepsilon) \) and \( C(\varepsilon,\varepsilon = 1) > C'(\varepsilon,\varepsilon = 1) > C(\varepsilon,\varepsilon = 1) > C^*(\varepsilon) \) for all \( \varepsilon \). This is due to the fact that the trunks emit their calls according to a law of emission which is different from the subscribers' law.

When \( \varepsilon \) increases, the previous inequalities are reversed and the variations of the congestions become similar to those related to the case of 2 different groups of subscribers. For practical applications, (i.e. for \( \varepsilon \geq 3 \) for instance), this particularity can be completely neglected and the numerical values obtained on figure 3 are very well approximated by the corresponding ones of figure 1.

II.3. Generalisations

The theoretical model developed in the Appendix gives the general formulae for any possible configuration of unbalanced traffic sources. The only practical problem is that of computing these formulae but this may easily be done by computer. As already said, a lot of practical cases have been studied, with more than 2 homogeneous groups of sources and the results obtained do not bring out any very new characteristic property. So, only general comments will be given in the following chapter.

However, in order to support a further conclusion, a particular numerical example will be given here. It deals with a Full-availability group with 6 outlets and 8 inlets to which are connected 8 different traffic sources, each one having its own traffic offered. The results are summarized in the following table which gives the values of call congestion \( C_i \) for each source, the weighed call-congestion \( C^* \) and the congestion \( C^* \) calculated as if all the sources were identical ones. From these results, two points may be remarked:

a/ the characteristic variation of \( C_i \) with respect to \( \varepsilon \)

b/ the congestion \( C^* \) is about 8 times greater than \( C \) and 2 times greater than the greatest congestion which is \( C_1 \).

<table>
<thead>
<tr>
<th>Source</th>
<th>Traffic offered</th>
<th>Call-Congestion</th>
</tr>
</thead>
<tbody>
<tr>
<td>S_1</td>
<td>( \varepsilon = 0.0050 )</td>
<td>( C = 0.00254 )</td>
</tr>
<tr>
<td>S_2</td>
<td>( \varepsilon = 0.0099 )</td>
<td>( C = 0.00240 )</td>
</tr>
<tr>
<td>S_3</td>
<td>( \varepsilon = 0.0196 )</td>
<td>( C = 0.00214 )</td>
</tr>
<tr>
<td>S_4</td>
<td>( \varepsilon = 0.0364 )</td>
<td>( C = 0.00169 )</td>
</tr>
<tr>
<td>S_5</td>
<td>( \varepsilon = 0.0741 )</td>
<td>( C = 0.00110 )</td>
</tr>
<tr>
<td>S_6</td>
<td>( \varepsilon = 0.1379 )</td>
<td>( C = 0.00064 )</td>
</tr>
<tr>
<td>S_7</td>
<td>( \varepsilon = 0.2428 )</td>
<td>( C = 0.00036 )</td>
</tr>
<tr>
<td>S_8</td>
<td>( \varepsilon = 0.3902 )</td>
<td>( C = 0.00020 )</td>
</tr>
<tr>
<td>S_9</td>
<td>( \varepsilon = 0.00051 )</td>
<td></td>
</tr>
</tbody>
</table>

Total traffic: \( A = 0.9181 \) erl.

Aver. traf. per source: \( C^* = 0.00024 \)

II.4. Conclusions for the one-stage full-availability groups with unbalanced sources

II.4.1. Comments on the numerical results

For the problem of mixing two different groups of sources on the same full-availability group, the following points are of practical interest:

- the general shape of the curves which are given here as particular examples may be considered as one of the characteristics of the problem.
- for a traffic disequilibrium \( \varepsilon \geq 2.5 \) or 3 (and this covers almost all the practical cases), the following general relations between the various congestions are valid:
  \[ C_i \] (for the high traffic sources) \(< C \] (weighed congestion) \(< C_{\text{trunk}} \] (for the low traffic sources) \(< C^* \] (calculated as if all the sources are identical) and \( B \) (for unbalanced sources) \(< B^* \) (for identical sources).

These inequalities apply independently of the numerical characteristics of the full-availability group which is considered.

- If the case of mixing two different classes of subscribers is compared to the case of one class of subscribers with one class of trunks, it may be remarked that, all parameters being fixed, the congestions in the former case are slightly lower than in the latter case. This gives a theoretical confirmation of the following fact (which was already well-known, but for practical reasons): it is better to distribute the various trunks of one route over the maximum possible number of switching matrices than to group them on the same switch.

For the general problem of mixing unbalanced traffic sources on the same full-availability group, the following examples may be used to conclude:

- two full-availability groups having the same characteristics are compared to each of them a certain traffic source population respectively \( P_1 \) and \( P_2 \) is connected. In the population \( P_1 \), it is assumed that all the sources have the same traffic characteristics, whereas in \( P_2 \), these characteristics may vary from one source to the other. However, the total amount of traffic is assumed to be the same in both populations. There, the general results can be stated as follows:

  a/ in the population \( P_1 \), the larger the value of the offered traffic per source, the smaller the value of the congestion encountered by this source.

  b/ in the global call-and time-congestion \( C \) and \( B \) for \( P_2 \), are, in most of the cases, lower than the corresponding \( C^* \) and \( B^* \) for \( P_1 \).

  c/ all the individual \( C_i \) in \( P_2 \) are also, in most of the cases, lower than \( C^* \) in \( P_1 \).

  The points b/ and c/ clearly express the generally favourable effect of an unbalanced traffic source distribution compared with a uniform one. More precisely:

  d/ the relative differences between \( B \) and \( B^* \) on the one hand, and between \( C \) and \( C^* \) on the other hand, are mathematical functions which increase with the variance of the offered traffic distribution considered.

II.4.2. Discussion of a dimensioning criterium

For dimensioning loss systems with identical traffic sources, the criterium which is used is the following "criterium \#1": that is: "the congestion value \( C^* \) encountered for each source, should not exceed a certain pre-determined value".

In many cases, this criterium 1 is extended and applied to dimension loss systems with unbalanced traffic sources. In this case, it should be understood as: "the congestion value \( C^* \) calculated as if all the sources are identical, should not exceed a certain pre-determined value". In the cases where the traffic sources are not too unbalanced, this criterium may be taken as an acceptable one, in the sense that the specific congestions \( C_i \) and \( C \) are close to the \( C^* \) value. Moreover, it presents the advantage of avoiding the use of the unbalanced traffic theory for determining the exact congestions.

But, in the reverse cases where there is a large unbalance between the traffic sources, this criterium 1 may
hardly be taken as an acceptable one. This is clearly illustrated in the numerical example given in II.3., in which the highest specific congestion is about 8 times greater than $C$ and 2 times greater than the greatest $C_{ij}$. So, another criterium is needed, which takes into account the unbalanced traffic effect in a more realistic way. Three of them are proposed here:

**Criterium n° 2** "The highest specific congestion $C_{ij}$ should exceed a certain pre-determined value". It is seen that this highest specific congestion is the characteristic of the sources having the lowest offered traffic. So, such a criterium, on the one hand preserves the interests of the low traffic subscribers but, on the other hand, practically conditions the dimensioning of the system to the only influence of these low traffic subscribers. This is not realistic, especially when considering that the high traffic subscribers are the most sensitive to the quality of service.

**Criterium n° 3** "The weighed congestion $\overline{C}$ should not exceed a certain pre-determined value". Due to the weighing of the specific congestions proportionally to the offered traffic, more importance is given here to the high traffic subscribers; in this way, this criterium seems more realistic. However, it does not give any indication on the possibly high congestions encountered by the low traffic subscribers.

**Criterium n° 4** Two types of dimensioning parameters are considered here, according to the relative "importance" of the sources. In the case of low traffic subscribers, an absolute parameter is taken, which should not be exceeded (such as the lost traffic or the number of lost calls during a certain busy period). For the high traffic sources, a relative parameter is taken, such as the call-congestion. All the practical problems remain in the judicious choice of the critical values to be considered. For the sake of simplicity and in order to allow easy comparisons between identical and unbalanced sources, the criterium n° 3 will be considered further on for the unbalanced traffic cases.

**II.4.3. The notion of gain in traffic capacity in the case of unbalanced traffic sources**

In this chapter, the traffic handling capacity of the full-availability groups with unbalanced sources, calculated with the criterium 3, is compared with the capacity of the corresponding groups with identical sources, calculated with the criterium 1.

More precisely, let it be assumed that, for a given full-availability group, a certain maximum congestion value $p$ should not be exceeded. If this group is fed by identical sources, the criterium 1 applied to the given value $p$, allows the determination of a maximum average traffic per source $C_m$. If now, this group is fed by unbalanced sources, the criterium 3, applied to the same reference value $p$, allows the determination of another maximum average traffic per source $C_m'$. So, it has been seen that, in most of the cases, this gives $C_m > C_m'$. So, there is a gain in traffic capacity, in the case of unbalanced sources compared with the corresponding case of identical sources. This gain may be expressed by the percentage $\gamma = \frac{C_m - C_m'}{C_m}$ and may be taken as a measure of the unbalanced source effect.

On a theoretical basis, this measure depends on all the parameters of the unbalanced source configuration and on the considered congestion reference value $p$. Here, only partial results will be presented, stressing the main points of interest.

**Figure 4** relates to a series of results applying to:

a/ a full-availability group of $R = 8$ outlets and $K = 16$ inlets to which are connected $K_1 = 14$ low traffic and $K_2 = 2$ high traffic sources.

b/ the following parameter values:

$E = 10$, $K = 32$, $K_1 = 30$, $K_2 = 2$

c/ a third group with $E = 18$, $K = 74$, $K_1 = 70$ and $K_2 = 4$

This figure shows the variations of the gain $\gamma$ versus the disequilibrium $E$ between the individual traffic of the sources of each class.

Two characteristic facts should be remarked:

* For practical applications, the gain in traffic capacity may be considered as independent of the congestion reference value $p$ which has been considered for establishing the comparison. This result may be applied for all the practical values of $p$, i.e. $p \leq 1$ or $2k$.

* The variation of the gain versus the disequilibrium of traffic may vary will be approximated by a linear variation, in the practical range of variation of $E$ i.e. for $3 \leq E \leq 10$ for instance.

These two properties have been verified on all the configurations which were studied and this kind of curve may constitute a very useful tool for practical traffic engineering purposes.

**III. LINK-SYSTEMS WITH UNBALANCED TRAFFIC SOURCES**

**III.1. Introduction**

In the previous chapters, the problem of unbalanced traffic sources was studied for the case of full-availability groups with concentration. In practice, these groups may be encountered at the level of the terminal switching stages of local exchanges or of satellite exchanges and concentrators, to which are connected the subscribers of the exchange and, possibly mixed with them, some junctors or signalling devices. In most of the cases, these terminal stages are associated with other switching stages ensuring the device-accessibility and the traffic mixing. For such systems, the grade of service specification is generally stated in the form of an overall condition, like a point-to-point or a point-to-route congestion, which depends on the complete link-system studied.

The conclusion of the preceding chapter stressed the possibility, for a given full-availability group with a maximum congestion, of handling more traffic in the case of unbalanced sources than in the case of identical sources. For the link-systems, the problem which arises is the following: if, by taking into account the unbalanced source effect on the first concentrating stage, it is desired to handle more traffic than with identical sources, it may happen that the other switching stages react to this traffic increase by a consecutive increase of the blocking on the links. So, the expected favourable effect of unbalanced sources may be reduced.

**III.2. Numerical study**

Two particular examples of link-systems are considered here; the one with 2 and the other with 3 switching stages. They are schematized on figure 5. For practical applications, both of them may be used for satellite exchanges or concentrators.

Calculations and traffic simulations have been undertaken to determine the overall blocking in the case where 100% of the traffic is concentrated on the internal links, these concentrating the high traffic of unbalanced sources compared with the corresponding case of identical sources. The average traffic per source $E$ was made varying between 0.06 and 0.10 erl. for the 2-stage link-system and between 0.10 and 0.12 for the 3-stage system. These variations cover the practical range of applications for which these examples are dimensioned.

The parameter $\gamma$, which is the gain in traffic capacity introduced in II.4.3., is taken here as the numerical estimator of the unbalanced source effect. Figure 6 illustrates, for the case of the two stage link system considered, the variation of this gain versus the disequilibrium of traffic $E$ and for some fixed values of the congestion reference value $p$ (see II.4.3.). It is remarked that this gain $\gamma$ now strongly depends on this congestion reference value $p$ (see II.4.3.).

On the other hand, the property of linear variation of $\gamma$ versus $E$ for any fixed value of $p$ is still apparent on this figure.
In the range of traffic considered for the 3-stage link system, the calculated and simulated results did not give any significant difference between the congestion values obtained with identical sources or with unbalanced sources. So, for practical applications, the effect of unbalanced sources is null in this case.

All these results may be explained in the following general comments.

III.3. General comments on the results and conclusion
Link-systems are designed for some acceptable grade of service values and this implies the practical traffic conditions in which they have to work. For satellite-exchanges and concentrators, this grade of service is mainly a matter of an overall blocking toward the parent-exchange. For this overall blocking, and according to the traffic conditions considered, we have one out of the three following possibilities.

a/ the congestion encountered on the first concentrating stage is predominant
b/ the external congestion on the trunks (Erlang blocking) is predominant
c/ both congestions have more or less the same numerical importance on the overall congestion.

Since the possible unbalanced traffic effect affects the congestion on the first concentrating stage only, the following general statement may easily be deduced: the larger the relative importance of the blocking on the first concentrating stage on the overall blocking, the larger the possible unbalanced traffic effect.

This statement can be followed in the numerical results obtained in the examples. For the 3-stage system, the considered traffic conditions correspond to the case b/ and consequently, the unbalanced effect was found negligible. The 2-stage system is gradually passing from a state near-ly corresponding to the case a/ (for \( C_0 < 0.06 \)), to a state nearly corresponding to the case b/ (for \( C_0 > 0.10 \)) and the consequent decrease of the unbalanced traffic effect has been noted.

In order to conclude comments on this chapter, a very attractive economical point must be stressed which consists in the possibility of reducing the number of trunks between a satellite exchange or a concentrator and the parent-exchange, this, by taking into account the unbalanced source effect. The following numerical example illustrates this possibility, for the case of the 2-stage system considered. With 32 identical sources per A-switch, at \( C_0 = 0.08 \), 64 trunks are needed for handling the total originated traffic \( A = 40.92 \text{ erl} \). Now, if on each A-switch 30 low traffic sources (\( C_0 = 0.051 \)) and 2 high traffic sources (\( C_0 = 0.51 \)) are connected, only 60 trunks are needed for handling the same total traffic \( A = 40.92 \text{ erl} \), with a weighed overall-congestion \( C \leq 0.002 \). So, the economy is far from negligible in this case.

However, caution must be used in trying to realize such an economy because it is based on two important practical constraints:

a/ a perfect and precise knowledge of the traffic offered per source is required and b/ a constant use is made here of the dimensioning criterium 3, which practical application in traffic engineering may be discussed.

Bibliographical reference:
"Traffic unbalances in small groups of subscribers"
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Electrical Communications Vol.43 N° 2, 1968.
I. THE MATHEMATICAL MODEL AND THE NOTATIONS

Here a mathematical model is presented which constitutes a generalization of the one-stage full-availability group theory to more complex systems working under the lost call cleared assumptions.

N classes $g_i$ of sources are considered connected to the same network. Each class $g_i$ is an homogeneous class in the sense that it is composed of $E_j$ (finite or not) independent sources which are strictly identical for traffic view point. Each class is characterized by its mathematical laws for emission of the calls, for acceptance of these calls by the network and for release of these calls. Some notations have to be given in order to define these laws.

The system is said to be in the state $(j_1, ..., j_N)$ at the date $t$ if, at this date, there are exactly $j_i$ calls in course originated by the class $g_i$, ..., and $j_N$ calls from $g_N$. There, $j = \sum_{i=1}^{N} j_i$ is the total number of calls in course in the system at the considered date $t$. Let be $R_i$ the maximum number of simultaneous calls originated by $g_i$, in course in the system, and $R$ the maximum number of calls in course in the system. The so-called "L" limitations will be used:

$$\forall i \leq R, t \forall j < R$$

The mathematical symbols $=$ and $\sum$ represent the corresponding equalities and sums with application of the "L" limitations.

Law of emission of the calls

The conditional probability, knowing that the system is in the state $(j_1, ..., j_N)$ at time $t$, that a call is attempted by anyone of the free sources of the $g_i$ class, in the time interval $(t, t+dt)$ is $\lambda_{i,j_i}$, with $\lambda_{i,j_i}$ a time independent constant which is called "the $g_i$ class-rate-attempt".

Law of acceptance of the calls

The conditional probability, knowing that the system is in the state $(j_1, ..., j_N)$ at time $t$, and a call has been attempted by a $g_i$-class source in the time interval $(t, t+dt)$, that this call is accepted is $\psi_{i,j_i}$ with $\psi_{i,j_i}$ a time independent constant such that:

$$\forall i \leq R, t \forall j < R \quad \forall i \leq R, t \forall j < R$$

$\psi_{i,j_i}$ is called the acceptance coefficient; it corresponds to a selective reaction of the network to the call-attempts it receives. This reaction may depend on a lot of factors; among them: the state of occupation of the network at the date of emission of the call-attempt, the strategy of acceptance applied for each type of call-attempt...

Law of release of the calls

The conditional probability, knowing that the system is in the state $(j_1, ..., j_N)$ at time $t$, a $g_i$ class-call was released in the time interval $(t, t+dt)$ is $\mu_{i,j_i}$, with $\mu_{i,j_i}$ a time independent constant which is called "the $g_i$ class-release coefficient".

The stationary state

Under these conditions, the system follows a homogeneous, finite, permanent and discontinuous Markov's process. In its infinitesimal evolution, this process is characterized by its transition coefficients $\lambda_{i,j_i}$, $\psi_{i,j_i}$, $\mu_{i,j_i}$, and $\mu_{i,j_i}$ and these coefficients are assumed to be such that a unique stationary state exists which corresponds to a true probability distribution. Therefore, this stationary distribution is the unique solution of the following linear system:

$$P(j_1, ..., j_N) = \sum_{i=1}^{N} (\psi_{i,j_i} \cdot \lambda_{i,j_i} + \mu_{i,j_i})$$

$$\sum_{i=1}^{N} (\psi_{i,j_i} \cdot \lambda_{i,j_i} + \mu_{i,j_i}) = P(j_1, ..., j_N) + \sum_{i=1}^{N} \mu_{i,j_i} \cdot P(j_1, ..., j_i-1, j_i+1, ..., j_N)$$

with $\sum_{i=1}^{N} P(j_1, ..., j_N) = 1$

It is possible to obtain the explicit solution of this system if the following additional assumption is made: the acceptance coefficient $\psi_{i,j_i}$ may be written under the form $\psi_{i,j_i} = \sigma_{i,j_i}$ with $\sigma_{i,R_i} = 0$

$\psi_{i,j_i}$ may be interpreted as a coefficient of selection of the calls, mainly depending on the type of the call-attempt and the number of calls of this type in course of the network.

In this case, this gives the following occupancy probability distribution

$$P(j_1, ..., j_N) = P(0, ..., 0) \prod_{i=1}^{N} \frac{\psi_{i,j_i}}{\lambda_{i,j_i}} \prod_{i=1}^{N} \frac{\psi_{i,j_i}}{\lambda_{i,j_i}}$$

with $P(0, ..., 0)$ given by $\sum_{j_1, ..., j_N = 0} P(j_1, ..., j_N) = 1$

II. ADAPTATION TO THE TELEPHONE MODELS

In the telephone models, each coefficient of the above formulae may be simplified and expressed in function of the traffic terms.

If $A_i$ denotes the total offered traffic of the class $g_i$, $C_i$ the call-congestion for the sources of $g_i$, and $A_i$ the reverse of the average $g_i$-call-holding time, this gives the following relations:

a/ $A_i = \frac{K_i - j_i}{A_i}$ for an Erlang emission process.

b/ $A_i = \frac{A_i}{(T_i - C_i) A_i}$ for a pure Erlang emission process.

c/ $A_i = a_i$ for $j_i < K_i$ and $A_i = 0$, for a truncated Erlang emission process.

(Note: For a group of $K_i$ trunks offering their calls to $E$ devices, it is said to be a pure Erlang process when $K_i > E$ and a truncated Erlang process when $K_i = E$. The formula $A_i = \frac{K_i - j_i}{A_i}$ means that the $K_i$ trunks are emitting their calls at a constant rate $A_i$, independently of the total number $j_i$ of busy trunks. In the truncated case, the $K_i$ trunks are emitting their calls at a constant rate $a_i$, only during the time that they are not fully occupied, i.e., for $j_i < K_i$. Only approximated relations between $A_i$ and the really offered traffic $A_i$ can be written; one of them is $a_i = A_i$. For computation purposes, $a_i$ is taken as the input parameter for calculating the state probabilities and $A_i$ is a-posteriori calculated with respect to these probabilities.)

On the other hand, the one-stage full availability groups are characterized by an acceptance coefficient such that $\psi_{i,j_i} = 1$ for $0 \leq j_i < R_i$ and $0 \leq j_i < R_i$ and for all $i$, $\psi_{i,j_i} = 0$ and $\psi_{i,R_i} = 0$ for all $i$

Finally the negative exponential distribution law of the holding times in each class $g_i$ (average holding time $= \frac{1}{\mu_{i,j_i}}$) corresponds to $\psi_{i,j_i} = \frac{1}{\mu_{i,j_i}}$

Taking into account these simplifications, this gives the following general formula for the occupancy probability distribution

$$P(j_1, ..., j_N) = P(0, ..., 0) \prod_{i=1}^{N} \frac{\lambda_{i,j_i}}{\psi_{i,j_i}} \prod_{i=1}^{N} \frac{\lambda_{i,j_i}}{\psi_{i,j_i}}$$

with $P(0, ..., 0)$ given by $\sum_{j_1, ..., j_N = 0} P(j_1, ..., j_N) = 1$
These formulae are of great interest because they cover the complete theory of the one-stage full-availability groups with or without identical sources.

III. SPECIFIC FORMULAE

III.1. Case of identical traffic sources

III.1.1. Engset case: \( E > R \) and \( K \) finite

The input parameter for calculation is \( b \) defined by:

\[
b = \frac{A/K}{1 - A(1-C)/K} = \frac{A}{1 - \frac{A(1-C)}{K}}
\]

where \( C \) is the call-congestion.

It gives \( A_j = (K-j)\mu b \) for \( 0 \leq j \leq R \) and the previously mentioned general formula leads to:

\[
P(j) = \frac{\binom{K}{j} b^j}{\sum_{j=0}^{R} \binom{K}{j} b^j}
\]

\( A = b \sum_{j=0}^{R} (K-j)P(j) \)

\( C = \sum_{j=0}^{R} (K-j)P(j) \)

\( B = \text{Time-congestion} = \frac{P(R)}{R} \)

III.1.2. Erlang case \( \lambda > R \)

The input parameter for calculation is the total traffic offered by the route. This gives the following formulae:

\[
P(j) = \frac{A^j/j!}{\sum_{j=0}^{R} A^j/j!} \quad \text{for} \quad 0 \leq j \leq R
\]

\( C = P(R) \)

III.2. Case of unbalanced traffic sources

The input parameters for calculations are: for each class of subscribers, \( b_i \) defined by

\[
b_i = \frac{A_i/K_i}{1 - A_i(1-C_i)/K_i}
\]

and for each class of trunks, \( a_i \) introduced in II and defined by

\[
a_i = \frac{A_i \lambda_i}{\lambda_i}
\]

(with \( a_i = A_i \) for a pure Erlang emission process and \( a_i \neq A_i \) for a truncated process)

III.2.1. Two different classes of subscribers

\[
P(j_1,j_2) = \frac{(K_1^1)(K_2^1) b_1^{j_1} b_2^{j_2}}{\sum_{j_1=0}^{R_1} (K_1^1) b_1^{j_1} b_2^{j_2}}
\]

with \( L \)

\[
0 \leq j_i \leq \min(R_i,k_i) \quad \text{for} \quad i = 1,2
\]

\( A_1 = b_1 \sum_{j_1=0}^{R_1} (K_1+j_1) \sum_{j_2=0}^{R_2} P(j_1,j_2) \)

\( C_1 = 1 + \frac{1}{b_1} - \frac{K_1}{A_1} \)

Similar formulae for \( A_2 \) and \( C_2 \)

\( B \) (time-congestion) = \sum_{j_1=0}^{R_1} P(j_1, j_2)

III.2.2. One class of \( X_1 \) subscribers mixed with \( X_2 \) trunks

\[
P(j_1,j_2) = \frac{(K_1^1)(K_2^1) b_1^{j_1} b_2^{j_2}}{\sum_{j_1=0}^{R_1} (K_1^1) b_1^{j_1} b_2^{j_2}}
\]

\( A_1 \) and \( C_1 \) are still given by the respective formulae (III.2.1.2.) and (III.2.1.3.)

\[
A_2 = \left\{ \begin{array}{ll}
A_2 & \text{if} \quad X_2 > R \quad \text{(pure Erlang process)} \\
A_2 \left[ 1 - \sum_{j_1=0}^{R_1} P(j_1, K_2) \right] & \text{if} \quad X_2 \leq R
\end{array} \right.
\]

\( C_2 = \left\{ \begin{array}{ll}
C_2 & \text{if} \quad X_2 > R \\
1 - \sum_{j_1=0}^{R_1} P(j_1, K_2) & \text{if} \quad X_2 \leq R
\end{array} \right.
\]

III.2.3. \( N \) distinct traffic sources

For each source \( S_i \), the input parameter considered is \( b_i \), as previously defined.

\[
P(j_1, \ldots, j_N) = \sum_{i=1}^{N} \prod_{k=1}^{R} b_i^{j_k}
\]

with \( L \)

\[
0 \leq j_k \leq \min(R_k,k_k) \quad \text{for} \quad i = 1,2, \ldots, N
\]

\( A_1 = b_i \sum_{j_i=0}^{R_1} P(j_1, \ldots, j_i, \ldots, j_N) \)

and \( C_i = 1 + \frac{1}{b_i} - \frac{K_i}{A_i} \)

III.3. Conclusions

This list of particular cases is not exhaustive. As already said, the general formulae given in II for the state probability distribution cover all possible source configurations, with or without unbalanced traffic, thus providing a very useful theoretical tool. A judicious choice of the input parameters for calculation allows a very easy computation of these probabilities and of the resulting congestions and traffic offered, for each specific case.