ABSTRACT

This paper examines the mathematical model of a fully available system with repeated calls. The analytical expressions for probable characteristics of single and double line systems are shown. For general cases a numerical method has been elaborated. This method is convenient for computers. The algorithm described in this paper was used to compute the tables.

A numerical example of the evaluation of a limited availability networks with repeated calls is examined.

INTRODUCTION

In the theory of telephone systems it is usually assumed that an unanswered call is either lost (loss system) or waiting (waiting system). The system with repeated calls, where an unanswered call is repeated until it is completed or abandoned (1, 2) is for the most cases more close to reality. A system with repeated calls is general enough and includes loss and waiting systems as the limiting cases. But even for a fully available system with repeated calls and simple considerations concerning offered traffic and line occupation time distribution we cannot receive simple analytical results for probability characteristics of a general case. However it is possible to obtain the numerical solution of algebraical equilibrium equation describing the considered system. The present article offers an algorithm convenient for numerical computation of probability characteristics of a fully available system with repeated calls. This algorithm is the generalization of the algorithm described in (3) and as compared to other known calculation methods for systems with repeated calls, it is characterized by simplicity, accuracy and effectiveness for computer operation. A similar algorithm for a particular case of a system with repeated calls is described in (4).

1. Mathematical model of a fully available system with repeated call's

A fully available system of \( \nu \) lines and \( n \) waiting rooms (or to be more precise, sources of repeated calls) is considered. The system state is characterized by a pair \((j, k)\), where \( j \) is the number of engaged lines and \( k \) is the number of waiting rooms. Four types of occurrences are possible for the system to pass from the \((j, k)\) state into another:

a) the arrival of a primary call transforms the system into \((j+1, k)\) state if \( j < \nu \), or into \((j, k+1)\) state if \( j = \nu \) and \( k \neq 0 \). If \( j = \nu \) and \( k = 0 \), the arrival of a primary call does not affect the system state;
b) the arrival of a repeated call transforms the system into \((j, k-1)\) state if \( j < \nu \), if \( j = \nu \), the system state is not changed;
c) the rejection of an expected call transforms the system into \((j, k-1)\) state;
d) when a line is disengaged, it transforms the system into \((j-1, k)\).

Let us assume that all the intervals between any two occurrences of the same type are distributed exponentially, and the average time of line occupation equals to unity. Then the probabilities of respective occurrences (or events) are proportional to the following numbers:

a) arrival of a primary call \( \lambda \);
b) arrival of a repeated call \( \kappa (\nu) \);
c) rejection of a waiting call \( \kappa (\nu) \);
d) disengagement of the line - $\phi$ 

where $\lambda$, $\sigma$ and $\varsigma$ are given. Number $\lambda$ is primary call intensity, $\varsigma$ means the average number of repeated calls per unit of time from one waiting call, $\sigma$ characterizes the rate of rejection of a waiting call. The case $\sigma = 0$ means that all waiting calls were completed, because occurrences type c) are absent. It is convenient to give numbers $\tau = \frac{1}{\varsigma}$ and $\mu = \frac{\sigma}{\varsigma}$ instead of $\varsigma$ and $\sigma$.

Number $\tau$ is the average time between repeated calls from one source, $\mu$ may be considered as "measure of persistence" of a call ( $\mu$ equals the average number of repeated calls, which the waiting subscriber intends to make before abandoning).

Let $p(j, k)$ be the probability of the system being in $(j,k)$ state, provided the process is stationary. The probabilities $p(j, k)$ are derived from $(n+1)(\nu+1)$ algebraic equations:

\[
\begin{align*}
\left[\begin{array}{c}
\lambda + k (\nu + \varsigma) + j \\sum_{j=0}^{\nu} \sum_{k=0}^{\nu} p(j, k) = 1 \\
= \lambda (j+1, k) + (\nu+1) p(j+1, k) + (k+1) p(j, k+1) + \\
\nu + \varsigma (k+1) p(j, k+1) + \\
\nu (j+1, k+1) + \varsigma (j, k+1) p(j, k+1) + \\
(j+1, k+1) p(j, k+1)\end{array}\right]
\end{align*}
\]

\[\sum_{j=0}^{\nu} \sum_{k=0}^{\nu} p(j, k) = 1 \quad (2)\]

Probabilities $p(j, k)$ when $j$ and $k$ are not within range $0 \leq j \leq \nu$, $0 \leq k \leq \nu$ are assumed to be zero.

If $p(j, k)$ are known, the main probability characteristics of the system can be calculated by the following formulae.

Loss probability of a primary call (estimated as the relation of lost primary calls to the number of arrived primary calls; the call is lost, if in the arrival time $j = \nu$)

\[L = \frac{\lambda + K}{\lambda}\]

Average waiting time

\[R = \frac{\lambda}{\nu}\]

Total loss probability (estimated as the relation of all lost calls to all arrived calls)

\[J_l = \sum_{k=0}^{\nu} \sum_{j=0}^{\nu} p(j, k)\]

Average number of engaged lines

\[J = \sum_{j=0}^{\nu} \sum_{k=0}^{\nu} j p(j, k)\]

Average number of engaged waiting rooms

\[K = \sum_{j=0}^{\nu} \sum_{k=0}^{\nu} k p(j, k)\]

Average parameter of total call intensity

\[L = \nu + \frac{\lambda + K}{\lambda}\]

Average number of repeated calls per one primary call

\[M = \frac{K.\varsigma}{\nu}\]

Final loss probability (estimated as the relation of unserved primary calls to the number of arrived primary calls)

\[J_k = \frac{K.\varsigma}{\nu} + \frac{p(\nu, \nu)}{\nu}\]

Let us consider a limiting case, where $\nu \to \infty$; i.e. that the number of waiting calls is unlimited. In that case the number of equations (1) is infinite. However by calculating equations (1) with finite and great enough $\nu$, we may receive an approximate numerical result with a definite degree of accuracy. The criterion of accuracy of the result is the probability of state $(\nu, \nu)$; if $p(\nu, \nu)$ equals zero then with a definite degree of accuracy $\nu$ has been taken correctly.

In case where $\nu = \infty$ all main probability characteristics of the system are connected by a range of simple correlations. If, for example, the average number of repeated calls per one primary call $M$ is known and the loss probability of a primary call is $J_l$, then the other characteristics are determined by the following formulae.
2. One-line and two-line systems

Equations system (1) has the following solution in case \( v = 1 \), \( n = \infty \) and \( \sigma = 0 \) [3]

\[
\begin{align*}
P(0, k) &= \frac{\lambda^K}{K!} \prod_{i=0}^{k-1} (\lambda + ic) \ P(0, 0) \\
P(1, k) &= \frac{\lambda^K}{K!} \prod_{i=0}^{k-1} (\lambda + ic) \ P(0, 0) \\
P(0, 0) &= \sum_{k=0}^{\infty} \frac{\lambda^K}{K!} \prod_{i=0}^{k-1} (\lambda + ic) \\
&+ \sum_{k=0}^{\infty} \frac{\lambda^K}{K!} \prod_{i=0}^{k-1} (\lambda + ic)
\end{align*}
\]

where

\[
\prod_{i=0}^{-1} (\lambda + ic) = 1
\]

The solution obtained provides possibilities to calculate main characteristics of a two-line system, but no answers were found to simplify its analytical expressions.

3. Method for solving the equations system

This method, gives possibility to receive numerical results in a considerably more rational manner as compared to the method of successive approximation [2, 5] and other universal methods. The algorithm is based on solving the above equations by dividing system (1) into subsystems consisting of \( v \) equations, each being solved separately.
Let us note by \( p(\mathbf{v}, \mathbf{n}) \) that equation of system (1), the left part of which contains probability \( p(\mathbf{v}, \mathbf{n}) \). Solution begins by resolving the sub-system of equations

\[
[0, n], [1, n], \ldots, [v-1, n]
\]  

(4)

System (4) has \( \mathbf{v} \) equations and \( (\mathbf{v} + 1) \) unknowns. Giving one of unknown an arbitrary value, differing from zero, for example assuming

\[
p(\mathbf{v}, \mathbf{n}) = 1
\]

(5)

we receive a system with \( \mathbf{v} \) equations and \( \mathbf{v} \) unknowns, which has a definite solution. Now let us assume, that the values of probabilities \( p(\mathbf{v}, \mathbf{n}) \) are determined for all \( \mathbf{v} = 0, 1, \ldots, \mathbf{v} = \mathbf{v} - 1, \mathbf{n} \), and let us consider the sub-system of equations

\[
[0, \mathbf{k}], [1, \mathbf{k}], \ldots, [\mathbf{v}-1, \mathbf{k}]
\]

(6)

The system (6) has \( \mathbf{v} \) equations with \( (\mathbf{v} + 1) \) unknowns \( p(0, \mathbf{k}), p(1, \mathbf{k}), \ldots, p(\mathbf{v}, \mathbf{k}) \) because probabilities \( p(\mathbf{v}, \mathbf{k} + 1), p(1, \mathbf{k} + 1), \ldots \) are considered to be known. The value \( p(\mathbf{v}, \mathbf{k}) \) is determined by equation \( [\mathbf{v}, \mathbf{k} + 1] \), which contains it as the only unknown. Now system (6) has \( \mathbf{v} \) equations with \( \mathbf{v} \) unknowns and has only one solution. Giving the value of \( \mathbf{k} \) in succession \( \mathbf{n} - 1, \mathbf{n} - 2, \ldots, 0 \), we receive the solution of all the system (1). Note that system (1) determines values \( p(\mathbf{j}, \mathbf{k}) \) with precision up to the constant multiplier and that system definition is obtained by introducing an additional condition (5). Substituting (5) by condition (2), and this is obtained by dividing the obtained values \( p(\mathbf{j}, \mathbf{k}) \) by their sum, we receive true values of probabilities \( p(\mathbf{j}, \mathbf{k}) \).

Now let us consider in more detail the solution of sub-system (6) (sub-system (4) is essentially similar to (6)). Let us introduce the following values:

\[
x_j = p(\mathbf{j}, \mathbf{k}) \quad (j = 0, 1, \ldots, \mathbf{v} - 1)
\]

\[
\ell = p(\mathbf{v}, \mathbf{k})
\]

\[
\lambda = \lambda + k (n + 1)
\]

\[
t_j = (k + 1) \left( \frac{p(\mathbf{j}, \mathbf{k} + 1)}{p(\mathbf{j}, \mathbf{k} + 1)} + \frac{\ell p(\mathbf{j}, \mathbf{k} + 1)}{p(\mathbf{j}, \mathbf{k} + 1)} \right)
\]

\[
(j = 0, 1, \ldots, \mathbf{v} - 1)
\]

Now system (6) has the following appearance

\[
\begin{align*}
x_{v} - x_{v} &= t_{0} \\
-\lambda x_{v} + (\ell + 1) x_{v} - 2 x_{v} &= t_{1} \\
&\vdots \\
-\lambda x_{v-3} + (\ell + v - 1) x_{v-2} + v x_{v-1} &= t_{v-1}
\end{align*}
\]

(7)

Note that values \( p(\mathbf{j}, \mathbf{k} + 1) \) \((j = 0, 1, \ldots, \mathbf{v})\) and hence \( t_{j} \) are considered to be known, whereas \( \frac{\ell}{\lambda} p(0, \mathbf{k}) \) is calculated by equation \( \frac{\ell}{\lambda} p(0, \mathbf{k}) \). For solving system (7) multiply the first equation by \( \lambda \) and add the result to the second equation. After that the second equation is multiplied by \( \lambda / \ell \), where \( \ell \) is the coefficient with \( x_{v} \) of the second equation and is added to the third equation and so on. As a result system (8) obtains the following appearance

\[
E_{0} x_{0} - x_{1} = E_{v}
\]

\[
E_{1} x_{1} - x_{2} = E_{v-1}
\]

\[
\vdots
\]

\[
E_{v} x_{v} = E_{v-1}
\]

(8)

where \( E_{j} \) and \( F_{j} \) \((j = 0, 1, \ldots, \mathbf{v} - 1)\) are determined by the following recurrent correlations

\[
E_{0} \ell - \ell ; \quad E_{j} = \ell + j - \frac{\lambda j}{E_{j-1}}; \quad (j = 1, 2, \ldots, \mathbf{v})
\]

\[
F_{0} = t_{0}; \quad F_{j} = t_{j} + \lambda F_{j-1}; \quad (j = 1, 2, \ldots, \mathbf{v})
\]

The values \( x_{0}, x_{1}, x_{2}, \ldots, x_{v} \) are determined by correlations

\[
x_{v-1} = \frac{F_{v-1} + \ell}{E_{v-1}}
\]

\[
x_{j} = \frac{F_{j} + (j + 1) x_{j+1}}{E_{j}}
\]

\((j = v-2, v-3, \ldots, 0)\)

4. Calculation algorithm

Let us consider an algorithm for numerical calculation of a fully available system with repeated calls. The algorithm is based on the method described in the preceding chapter. For writing down the algorithm we shall use algorithmic language "A". Language "A" employs generally accepted mathematical symbols with letter notation (indexes are allowable) for variable, and with usual numerals and symbols for arithmetical operations. Algorithmic entry consists of commands, which are divided by symbol ; (semicolon). Some commands are preceded by a mark, which represents a number separated from the command by a right bracket. There are two main types of commands: assignment and comparison commands. Assignment commands are divided by symbol := (assignment symbol) into two parts; when the command is being executed the right part is calculated and the result is assigned to the variable (or variables.
if there are several), which are found in the left part. After that the next command is executed. Comparison commands represent correlation (condition) after which follows a mark in brackets. Comparison command orders to pass over to the execution of the command with the indicated mark, if the condition is met with, otherwise the next command is executed. Absence of the condition (absolute passage to the left part). After that the correlation (condition) after which follows a mark with the indicated mark, if the condition is executed.

The algorithm work begins with the execution of the command with mark 1) and ends with the command STOP (stop command). The initial data are \( v, \pi, \lambda, T, U \), results are \( A, K, M \) and \( p = p(v, \pi) \). In algorithms work there are used two masses of variables \( A_1, A_2, \ldots, A_v \) and \( B_1, B_2, \ldots, B_v \), and a number of auxiliary variables. Between variables \( A_j \) and \( B_k \) (after the execution of command \( B_j = b \cdot c \) between marks (4) and 5) on the one hand and \( E_j, F_j \) on the other hand there is correlation

\[
A_j = \frac{F_j}{E_j} \quad B_j = \frac{j+1}{E_j}
\]

The algorithm is written down in language "A" as follows:

1) \( \tau := \frac{1}{T} \); \( \varnothing := \frac{U}{T} \);

\[
A_0, A_1, \ldots, A_v := 0 ;
\]

\[
f := \frac{1}{\lambda} \quad \tau, \rho, t := 1 ;
\]

\[
q, K, T := 0 ; \quad k := n+1 ; \quad j := 0 ;
\]

2) \( k := k-1, d := k \cdot p, m := m+d \cdot \tau(n+1) ;
\]

\[
a := 0 ; \quad t := t ; \quad c := \frac{1}{m} \quad (4) ;
\]

3) \( c := \frac{(m + b - B_j)}{j} ; \quad j := j + 1 ; \quad b := b \cdot p ;
\]

4) \( a := \frac{(a + A_j \cdot \tau)}{c} \); \( A_{j+1} := a ;
\]

\[
B_{j+1} = b \cdot c ; \quad j \neq n-1 \quad (3) ;
\]

\[
a := a + B_j \cdot f ; \quad s := s ; \quad \tau := \tau + f ;
\]

\[
f := f(q + b + d \cdot c) - a - A_{j+1} \cdot \tau ;
\]

\[
A_{j+1} := d - s - \tau ;
\]

5) \( s := s + a ; \quad b := a \cdot d ; \quad A_{j+s} := A_{j+s} + b \cdot c ;
\]

\[
A_{j+1} := b \cdot c ; \quad j := 0 \quad (6) ;
\]

\[
j := j + 1 ; \quad a := a + B_j + A_{j+1} \quad (5) ;
\]

6) \( \tau := \frac{1}{(s + q)} ; \quad q := 1 ; \quad K := k \cdot s \cdot 2 ;
\]

\[
f := f \cdot \tau ; \quad p := p \cdot \tau ; \quad K := K \cdot 3 ; \quad \tau = \tau ;
\]

\[
k \neq 0 \quad (2) ; \quad M := K \cdot \lambda \quad \text{STOP} ;
\]

According to this algorithm BESM-2 and BESM-4 computers were used at the Latvian State University to calculate and print tables, in which values \( \mu \) and \( T \) have the following parameters

\[
\mu = 0.1, 0.2, 0.3, 0.5 ; \quad T = 0.01, 0.02, 0.05, 0.1, 0.2, 0.5, 1, 2, 5 ;
\]

\[
1 \leq \sqrt{\xi} \leq 50, \quad 0.05 \leq c \leq 1 \quad \text{with} \quad 0.05 \quad \text{step.}
\]

There were also made numerical calculations for some examples of limited availability networks with repeated calls, with the use of successive approximation method (5). The results obtained allow to assume that the principles of constructing optimum limited availability networks (6) hold true for cases with repeated calls.

REFERENCES


