ANALYSIS OF CONGESTION IN SMALL P.A.B.X.'S

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1. INTRODUCTION

Small automatic telephone switching systems, such as P.A.B.X.'s (Private Automatic Branch Exchanges), R.A.X.'s (Rural Automatic Exchanges) and other similar trunking schemes, have received relatively little attention from traffic engineers and mathematicians interested in the application of probability theory to telephone problems. The usual approach has been to dimension each common circuit group separately, using standard traffic distribution models (e.g. Poisson, Binomial, or their truncated forms). For the sake of simplicity, the interaction of traffic in different circuit groups via the same limited pool of traffic sources has been neglected. Yet if the number of traffic sources is small and the traffic per source high, the neglect of this interaction can lead to significant errors in the estimates of congestion.

The mathematical models postulated in this paper embrace all trunk groups serving the traffic sources and thus avoid the errors mentioned above. Although confined to systems with no more than three groups of trunks, the models are sufficiently general and represent a large class of small automatic switching systems currently in use in all parts of the world.

The major part of the paper deals with the analysis of models representing typical trunking configurations of small P.A.B.X.'s and the development of expressions to determine congestion for different classes of calls. This is followed by discussion of a simple approximation to congestion probabilities, based on Engset Loss formula. Finally, comparison of analytical and approximate solutions with simulation results is given in the Appendix. A table of the Engset Loss Formula is provided as a supplement to the paper.

2. P.A.B.X. WITH N INTERNAL AND M EXTERNAL CIRCUITS.

2.1 List of symbols.

- \( A_i \) = mean traffic offered to internal circuits.
- \( A_e \) = mean traffic offered to external circuits
  = \( A_{ei} + A_{eo} \)
- \( A_{di} \) = incoming traffic offered to external circuits.
- \( A_{de} \) = outgoing traffic offered to external circuits.
- \( b_i \) = probability that a free source will originate an internal call.
- \( b_e \) = probability that a free source will originate an external call.
- \( d_i \) = probability that an internal call in progress will end.
- \( d_e \) = probability that an external call in progress will end.
- \( a_i = b_i/d_i \) = internal traffic offered per free source.
- \( a_e = b_e/d_e \) = external traffic offered per free source.
- \( \beta_i(n,m) \) = probability of blocking for an internal call, when the system is in state \((n,m)\).
- \( \beta_e(n,m) \) = probability of blocking for an external call, when the system is in state \((n,m)\).
- \( u_i(n,m) = 1 - \beta_i(n,m) \) = passage probability for an internal call, when the system is in state \((n,m)\).
- \( u_e(n,m) = 1 - \beta_e(n,m) \) = passage probability for an external call, when the system is in state \((n,m)\).

2.2 Description of the system.

Block diagram of the system investigated is shown in Figure 1. It could be a simple P.A.B.X., or a small rural exchange, with \( M \) bothway lines to parent exchange and \( N \) internal connecting circuits. The connecting network is link trunked and employs conditional selection. That is, no circuit can be occupied, unless a free path through the connecting network can be established. Also, to simplify the analysis, it is assumed that the called extension must also be free before a local or incoming circuit is occupied.

The internal congestion in this switching network is defined by the blocking factors \( \beta(n,m) \), or their complement, the...
passage factors $\mu(n,m)$, which are assumed to be given. The distribution of $\delta(n,m)$ and $\mu(n,m)$ depends on the trunking and the method of searching for a free path through the network.

2.3 Assumptions

(i) The P.A.B.X. operates as a busy signal system; calls, which cannot be immediately connected, are lost and are not re-offered.

(ii) All extensions (traffic sources) have the same mean traffic generation intensity.

(iii) All extensions, when they are free, can originate and receive both internal and external calls.

(iv) Individual extensions originate calls at random and quite independently of each other.

(v) Holding times of successful calls follow the negative exponential distribution.

(vi) Unsuccessful calls are cleared from the system and have zero holding times.

(vii) Setting-up and release of all calls is assumed to be instantaneous.

(viii) Call arrivals and departures are in statistical equilibrium.

2.4 Transition probabilities

The probability model used to represent the switching system is adequately defined by the set of assumptions listed above. To analyse it we use the tools of the classical method - state and transition probabilities, together with the principle of statistical equilibrium, first introduced by A. K. Erlang. However, the usual technique of setting up the equations of state and then solving the resulting system of differential equations does not appear very attractive in this case. Our approach is to develop simple recurrences from which the state probabilities, and eventually the congestion probabilities, for either group of circuits, could be computed.

To start with, we assume that there is no incoming traffic - only internal and external outgoing calls are made. We shall commence by evolving the transition probabilities about the state $(n,m)$, signifying $n$ internal and $m$ external (outgoing) calls being in progress.

The probability that with $(n,m)$ calls in progress an external call will be released is, clearly,

$$m \cdot d_e \cdot P(n,m)$$

which is the transition probability of moving from state $(n,m)$ to state $(n,m-1)$. The probability of occupying another external circuit - that is moving from state $(n,m)$ to state $(n,m+1)$ - depends on the probability of a fresh external call being made and also on the conditional probability that it will not fail because of lack of a free circuit or internal congestion. The former probability is $(S-2n-m) \cdot b_e \cdot P(n,m)$, while the latter is specified as $\nu_e(n,m)$. The overall probability of transition from state $(n,m)$ to state $(n,m+1)$ is given by the product of the above two probabilities, namely

$$(S-2n-m) \cdot b_e \cdot P(n,m) \cdot \nu_e(n,m)$$

The probability of an internal call ending when there are $n$ local and $m$ external calls in progress - that is moving from state $(n,m)$ to state $(n-1,m)$ is

$$n \cdot d_i \cdot P(n,m)$$

The transition from state $(n,m)$ to state $(n+1,m)$ depends on the simultaneous occurrence of the following three events:

(i) a fresh internal call is originated.

(ii) the called party is free to receive the call, and

(iii) there is a free connecting path available between the two extensions.

The probabilities of the above events are, respectively:

$$(S-2n-m) \cdot b_i \cdot P(n,m); \quad \frac{(S-2n-m)}{S} \cdot \nu_i(n,m).$$

Assuming that these events are independent of each other, the transition probability is given by the product of these elemental probabilities.

By similar reasoning the other transition probabilities about the state $(n,m)$ can be developed. They are shown in Figure 2, which is a state diagram for $(n,m)$ and the adjacent states.

![Fig.2: Diagram of transition probabilities to and from state $(n,m)$](image-url)
2.5 Computation of state probabilities.

Now we postulate that the internal and the external traffics are both in statistical equilibrium. In other words, we assume that in the long run (the system remaining in statistical equilibrium) the number of transitions from any one state to its neighbouring state equals the number of transitions occurring in the opposite direction. E.g. there are as many transitions from (n,m) to (n,m+1) as there are from (n,m+1) to (n,m). Thus, we can equate the transition probabilities, depicted in Figure 2 as pairs of curved arrows pointing in opposite directions:

\[(S-2n-m)\cdot \beta_e(n,m) \cdot P(n,m) = (m+1)\cdot \delta_e \cdot P(n,m+1)\]

After rearrangement and substitution of \( \beta_e \) for \( \beta_e/d_e \) we get our first recurrence equation:

\[P(n,m+1) = \frac{\beta_e(S-2n-m)}{\delta_e} \cdot P(n,m) \quad \ldots \quad (1)\]

Similarly, by equating the transition probabilities between states \((n,m)\) and \((n+1,m)\), and setting \(b_1/d_1 = a_1\), we get the second recurrence:

\[P(n+1,m) = \frac{\alpha_e(S-2n-m)}{\delta_e} \cdot P(n,m) \quad \ldots \quad (2)\]

In order to explain the process of computing the values of all state probabilities, it is helpful to imagine them arranged in a matrix, as follows:

\[
\begin{array}{cccc}
P(0,0) & P(0,1) & P(0,2) & P(0,3) \\
P(1,0) & P(1,1) & P(1,2) & P(1,3) \\
P(2,0) & P(2,1) & P(2,2) & P(2,3) \\
P(3,0) & P(3,1) & P(3,2) & P(3,3) \\
\end{array}
\]

Starting with an arbitrarily chosen value for \(P(0,0)\), we proceed to calculate the relative values of the elements shown in the first column of the above matrix, using the first recurrence (1). Then, using the first column elements as starting points, we calculate the remaining elements - each row in turn, proceeding left to right - with the help of the second recurrence, (2). Alternatively, we could begin by calculating the elements of the first row, using recurrence (2), then proceed up the columns using recurrence (1). The final result would be exactly the same.

The conversion of the computed elements to state probabilities is accomplished by dividing each element by their sum; since the matrix covers all possible states of the system, the sum of their probabilities must equal unity.

2.6 Blocking coefficients.

As stated earlier, the blocking coefficients \(\beta(n,m)\) and their complements \(\mu(n,m)\) are assumed to be known or computed beforehand from the parameters of the connecting network. For link-trunked networks the blocking coefficients can be well approximated by the terms of either a geometric or a hypergeometric series. For simplicity (and for conformity with the A.P.O. practice in dimensioning link-trunked switching systems) we have used the geometric series in all our numerical computations. The series is specified by a single parameter \(p\), which is the point-to-point blocking between an inlet and an outlet of the connecting network. With the usual symmetrical arrangement and uniform loading of the links, \(p\) can be computed by combinatorial methods. Using the geometric approximation, the blocking coefficients are computed as follows:

\[\beta_e(n,m) = p^n \quad \text{for all values of} \ n; \]

\[\beta_e(n,m) = (1-p)^m \quad \text{for all values of} \ m.\]

In non-blocking link systems, where full availability access is provided to both groups of circuits, all blocking coefficients are equal to zero for \(n < N\), \(m < M\), while \(\beta_e(N,m) = 1\) for all \(m\), and \(\beta_e(n,M) = 1\) for all \(n\).

2.7 Traffics carried.

The distribution of traffic carried on the internal route is given by summing the state probabilities forming the columns of the previously discussed matrix.

Similarly, in summing state probabilities by rows, one obtains the distribution of traffic carried on the external circuits. First moments about the origin of these two resulting distribution give the mean intensities of the traffics carried by the two circuit groups.

(i) Mean internal traffic carried

\[Y_i = \frac{N}{n=0} \frac{M}{m=0} P(n,m) \quad \ldots \quad (3)\]

(ii) Mean external traffic carried

\[Y_e = \frac{N}{n=0} \frac{M}{m=0} P(n,m) \quad \ldots \quad (4)\]

Since an external call occupies one extension, while each internal call occupies two, the total traffic carried by the extensions (which is the mean number of busy extensions) is:

\[Y_S = Y_e + 2Y_i \quad \ldots \quad (5)\]

The usual data for any numerical calculation are the numbers of circuits in each route, \(N\) and \(M\), and the mean traffics offered to these routes, \(A_i\) and \(A_e\). The traffics per free source, \(a_1\) and \(a_e\), appearing in our recurrences (1) and (2), however, depend not only on \(A_i\), \(A_e\), and \(S\), but also on the total traffic carried by the extensions, namely

\[a_1 = \frac{A_i}{S - Y_S} \quad \ldots \quad (6)\]

\[a_e = \frac{A_e}{S - Y_S} \quad \ldots \quad (7)\]

Since \(Y_S\) itself is a function of the distribution of state probabilities, it is not known at the start of the calculation. Therefore, we begin with

\[A_e + 2A_i\]

as the first estimate of \(Y_S\) and iterate until \(a_1, a_e, Y_i, Y_e\) (and, therefore, \(Y_S\)) converge to their true values.

2.8 Congestion probabilities.

Having computed the state probabilities and knowing the blocking coefficients, time congestion probabilities can be easily computed from the following general expressions:

\[\]
The expression (12) is identical with equation (1), except that the traffic flow on each of the three groups of circuits is in statistical equilibrium. The resulting recurrences are given below.

(i) Internal calls:

\[ P(n,m,k) = \frac{a_i R (k-1) v_i (n,m,k)}{S (m+1)} \]  

(ii) Outgoing calls:

\[ P(n,m+1,k) = \frac{a_{eo} R (e_{out} - m) v_e (n,m,k)}{S (m+1)} \]
In the above,

\[ R = S - 2n - m - k. \]

Other symbols are the same as in the previous section, except that the third subscript, k, has been added.

To compute the state probabilities the same iterative process is employed as described in section 2. Starting with an arbitrary value for \( P(0,0,0) \) we compute all other \( P(n,m,k) \) with the help of the above recurrences and then employ the normalising equation.

\[ \sum_{n=0}^{N} \sum_{m=0}^{M} \sum_{k=0}^{K} P(n,m,k) = 1 \]  \( \quad \)  \( \ldots (21) \)

to convert the initially calculated values to state probabilities.

Next step is to compute the carried traffics on all 3 groups of circuits and obtain the mean number of busy traffic sources.

\[ Y_S = Y_{eo} + Y_{el} + 2Y_l \]  \( \quad \ldots (22) \)

Now we have a better estimate of the number of free sources, \( S - Y_S \), which is used to obtain improved estimates of \( \alpha_i \) and \( \alpha_{eo} \) (\( \beta_{el} \), not subject to iteration).

\[ \alpha_i = \frac{A_i}{S - Y_S} \]  \( \quad \ldots (23) \)
\[ \alpha_{eo} = \frac{A_{eo}}{S - Y_S} \]  \( \quad \ldots (24) \)

Next we go back and recompute the state probabilities. The iteration is continued until the desired degree of convergence in all variables is attained.

Once the state probabilities have converged to their true values, it is a simple matter to compute the congestion probabilities, which are defined below.

(i) Time congestion, internal circuits

\[ E_i = \sum_{n=0}^{N} \sum_{m=0}^{M} \sum_{k=0}^{K} \beta_{i}(n,m,k) \cdot P(n,m,k) \]  \( \quad \ldots (25) \)

(ii) Time congestion, outgoing circuits

\[ E_{eo} = \sum_{n=0}^{N} \sum_{m=0}^{M} \sum_{k=0}^{K} \beta_{eo}(n,m,k) \cdot P(n,m,k) \]  \( \quad \ldots (26) \)

(iii) Time congestion, incoming circuits

\[ E_{el} = \sum_{n=0}^{N} \sum_{m=0}^{M} \sum_{k=0}^{K} \beta_{el}(n,m,k) \cdot P(n,m,k) \]  \( \quad \ldots (27) \)

(iv) Call congestion, internal traffic

\[ B_i = \sum_{n=0}^{N} \sum_{m=0}^{M} \sum_{k=0}^{K} R \cdot \beta_{i}(n,m,k) \cdot P(n,m,k) \]  \( \quad \ldots (28) \)

(v) Call congestion, outgoing traffic

\[ B_{eo} = \sum_{n=0}^{N} \sum_{m=0}^{M} \sum_{k=0}^{K} R \cdot \beta_{eo}(n,m,k) \cdot P(n,m,k) \]  \( \quad \ldots (29) \)

(vi) Call congestion for incoming traffic is here equivalent to incoming time congestion, since the incoming traffic is assumed to be random and is given full availability access to all incoming trunks. That is,

\[ B_{ei} = E_{el} \]

As explained in section 2, the loss coefficients are determined by the trunking configuration and the method of path search. In most cases they can be well approximated by the terms of either Geometric or Hypergeometric series. Congestion on incoming calls also depends on whether reselection of another free circuit (or circuits) is possible after being blocked in the connecting network. The remarks made in this context about the loss coefficients \( \beta_{el} \) in section 2.9 apply also in this case.

4. SPECIAL CASES

4.1 Internal Traffic only.

In this case the model simplifies to a unidimensional one. The probability of transition from state \((n)\) to state \((n+1)\) is

\[ \frac{(S-2n) \cdot (S-2n-1) \cdot \mu_i(n) \cdot P(n)}{S} \]  \( \quad \ldots (30) \)

where \( \mu_i(n) \) and \( P(n) \) are written for \( \mu_i(0,0) \) and \( P(0,0) \), respectively. The transition probability for the move from state \((n+1)\) to state \((n)\) is

\[ (n+1) \cdot \beta_{ei} \cdot P(n+1) \]  \( \quad \ldots (31) \)

Equating the above two transition probabilities gives this recurrence:-

\[ \frac{S-2n}{S} \cdot \beta_{ei} \cdot P(n) = 1 \]  \( \quad \ldots (32) \)

If link congestion is zero, \( \beta_{ei} \) terms are equal to 1 for all \( j < N \), hence the products \( \beta_{ei} \) appearing in (20) will be replaced by 1. If we now substitute \( n = N \), we will get the probability of all internal circuits being busy, i.e. the time congestion.

(i) Time congestion, full availability access to all \( N \) circuits

\[ P(N) = \frac{(2n)! \cdot S^n \cdot N^n}{S^n \cdot N^n} \]  \( \ldots (33) \)

The general expression for call congestion can be simply deduced from equation (10):-

(ii) Call congestion, restricted access

\[ B_i = \frac{\alpha_i}{A_i} \cdot (S-2n) \cdot \beta_{i}(n) \cdot P(n) \]  \( \ldots (34) \)

(iii) Call congestion, full availability access

\[ B_i = \frac{\alpha_i}{A_i} \cdot P(n) \]  \( \ldots (35) \)
4.2 Outgoing traffic only

This case has been fully covered in the literature (e.g. see reference [2]). Using the method employed above, it is easy to derive a general expression for the probability of \( m \) circuits occupied:

\[
P(m) = \frac{1}{M} \sum_{j=0}^{M-1} \left( \sum_{x=0}^{1} \prod_{j=0}^{x-1} \alpha_{eo}(j) \right)
\]

In a system without link blocking, the products of passage probabilities are equal to unity. Setting \( m = M \), we get the time congestion probability:

(i) Time congestion, full availability access -

\[
P(M) = \frac{1}{M} \sum_{j=0}^{M-1} \left( \sum_{x=0}^{1} \prod_{j=0}^{x-1} \alpha_{eo}(j) \right)
\]

5. APPROXIMATE METHODS

Although the analytical approach presented in this paper is quite straightforward and the necessary computations can be easily programmed for an electronic computer, quick access to a computer is not always possible. There are many occasions, however, where a quick estimate of congestion for the different classes of calls in a P.A.B.X. (or another small switching system) is required. A simpler, even if not very accurate, method would, therefore, be quite useful.

To get approximate estimates of congestion in a small local switching system of the type analysed in this paper only a table of the Engset loss formula is required if full (or nearly full) availability access is given to all trunk groups. Estimates of outgoing and incoming congestion can be obtained directly from the table by entering it with the appropriate traffics, the numbers of trunks, and the full number of traffic sources (or sinks, in the case of incoming traffic). An estimate of congestion on internal calls can be obtained from the same table, but it must be entered with half the actual number of traffic sources (number of extensions in a P.A.B.X.).

The above procedure gives a very good estimate of congestion on outgoing calls. Congestion estimates for incoming and local calls tend to be too high, particularly when the number of sources per trunk is small. The reason for the discrepancy is the smoothing effect that busy sources have on the terminating traffic. Further work is needed to develop a better approximation for the cases where \( S/N \) does not exceed about 10. A suitable approximation for systems with link blocking also remains to be found.

6. CONCLUSION

The analysis of small model switching systems has shown how more accurate estimates of congestion in the common trunk groups can be obtained. The results of the analysis have justified the method of employing channel equations in multi-dimensional probability systems, and pave the way to simpler solution of similar problems.

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8. REFERENCES


APPENDIX : COMPARISON WITH SIMULATION

A computer simulation program has been prepared to check the traffic capacity of link-trunked P.A.B.X.'s. This program was also used to check the applicability of the theoretical model analysed in this paper to practical P.A.B.X.'s.

The simulator is of the Markov chain type and has been designed to imitate the traffic flow in a 3-stage link-trunked switching system with one local and three external groups of circuits - incoming, outgoing and bothway. All system parameters are specified by data, so that a wide range of three- and two-stage systems can be simulated. The program has been written in FORTRAN and all tests to date have been run on CDC 3200/3600 computers.

A comparison between mean and carried traffic and probabilities of call congestion obtained by theoretical methods and simulation tests are shown in Table 1. A comparison of carried traffic (occupancy) distributions for one set of data is provided in Table 2.

The operation of the simulator departs in some respects from the assumptions underlying the analytical model. For example, all circuit groups and the links in the connecting network are hunted sequentially from a fixed home position. This results in non-uniform loading of the links, which means that the geometric series is not a very accurate model for the distribution of loss coefficients \( \delta(n,m,k) \). Also, in some simulation tests a fourth (bothway) group of circuits was provided, which carried calls flowing from the incoming and outgoing trunk groups. In other words, a 3-dimensional analytical model was compared with a 4-dimensional simulation one. However, the proportion of total traffic carried by the bothway circuits was low and did not seriously affect the comparison.

In spite of the above differences, quite good agreement has been obtained between the results obtained by simulation and those derived analytically. As expected, agreement is better for non-blocking systems, than for those exhibiting link congestion, but even in the latter case only two congestion estimates are outside 95% confidence limits.

Approximate estimates of congestion are given only for cases where full availability access to all trunk groups has been provided. Agreement with simulation results is very good for \( B_{eo} \), reasonable for \( B_{in} \), but only fair for \( B_{out} \).
For test Nos. 13, 14, and 15 the internal loss coefficients \( \beta_i(n,m,k) \) were computed as

\[
2p_i(0.5(1-2p_i))^{N_0-N-1}
\]

for all \( n < N \). In all other tests l.c. were computed from

\[
\beta_i(n,m,k) = (2p_i)^N
\]

In all cases \( \beta_i(N,m,k) = 1 \) for all \( n,m \).

For outgoing calls the loss coefficients were computed from

\[
\beta_{\text{ei}}(n,m,k) = \frac{p}{P(0)}
\]

for all \( n,m,k \). On incoming calls, because no retest was allowed, all \( k < K \) terms of the \( \beta_{\text{ei}}(n,m,k) \) distribution were made equal to the average link blocking probability \( p_i \); again, \( \beta_{\text{ei}}(n,m,k) = 1 \) for all \( n,m \).