A METHOD TO FORMULATE BLOCKING FUNCTIONS OF NETWORK GRAPHS

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ABSTRACT

The traffic dimensioning technique of any telephone system, is based on an effort involving two main tools: (1) numerical analysis methods capable of evaluating loss and waiting probabilities and (2) simulation methods providing more realistic approximations than the calculation methods.

The main difficulty with both methodologies in the determination of the most appropriate law of traffic distribution for the sources in the case being studied. However, it is well known that numerical analysis is less advantageous than simulation, since this technique may take into account such factors as interdependence among successive stages in a network, call routing and hunting strategies, etc. All these when involved in the numerical analysis method and complicate it excessively from the mathematical aspect. We may go to such an extreme that, in relatively uncomplicated network, the problem and its solution lie out of reach. Nevertheless, the exact evaluation of service grade factors is not always necessary.

For example, in the design phase of a conversation or signalling network, the first evaluations are usually sufficient. At other times, the different characteristics of a system may be considered separately without imposing great mathematical complexity on the problem, as may occur during the dimensioning phase of a system. It is in these cases that numerical analysis is more efficient than simulation owing to the speed in obtaining results a very important factor in such phases. An economy also results in being able to provide more ample system utilization margins than with some scattered simulation points.

The objective of the present study is to present a new algorithm, which consists in associating some entities with the different links of the diverse stages of a graph. These entities will be described as distribution vectors, even though they do not constitute elements of vectorial space. Then we shall consider the application of the algorithm to a link system with concentration, with expansion, or with neither expansion nor concentration in the first stage. The corresponding formulas will be derived as well for call congestion as for time congestion. The advantages of the method are to be noted and the difficulties considered, which the proposed algorithm presents in some cases; and the manner in which these obstacles may be overcome will be explained.

Lastly, an example of calculation of networks with conditional selection will be presented.

INTRODUCTION

The gradually rising complexity in link systems has required traffic engineers to seek numerical analysis methods and algorithms different from those normally used. Until now there have been few advances in the incorporation of new hypothesis closer to reality than those commonly accepted. On the contrary, there have been numerous contributions to the creation of new algorithms which allow handling of complex graphs. Under any circumstances, in the opinion of the authors, it is more appropriate to develop algorithms than hypothesis at least for well defined objectives when we consider the current state of mathematical tools, the limited amount of traffic statistics and the urgency with which we occasionally must obtain results.

The algorithm considered in this study is based upon the analysis of the graph, which represents all the possible paths existing between a determined network inlet system and one or more network outlets (point-to-point and point-to-route graphs, respectively), and upon the following hypothesis:

a.1 The system will be considered as a "lost call cleared" system.

a.2 The selection of the links of each switching stage is carried out at random. This implies that all the links pertaining to each stage have the same average load.

a.3 The different stages which constitute the system will be considered independent. This hypothesis is only approximate but uses to give a blocking probability higher than the calculated probability, when considering the interdependence among stages.

a.4 The search for a free path is performed in the system through the conditional selection procedure.

It is evident that this group of hypotheses is used for the majority of current numerical methods.

Referring to the graphs, it is possible to distinguish two types which are totally differentiated.

A. Graphs of the series-parallel type in which two or more subgraphs may be defined. These subgraphs contain initial and final nodes such that the subgraphs do not contain any branches or common links (note that the combination of the subgraphs results in a complete graph).

B. Graphs of the interleaved type in which it is not possible to define subgraphs of the nature above described.
It is evident that graphs exist which pertain to the group B and have, among determined stages, type A structures.

As it may be seen below, the algorithm may be applied to any of the graphs described. However, the formulation of the expression which gives blocking of the graph considered, is simpler in the type A graphs than in those of type B, as it may be supposed. Even more, the previous classification, though too general, will be useful at the time of establishing the blocking formula. This follows naturally, since the geometry of the graph plays an important role in the algorithm considered.

In order to expedite the explanation, the problem will be divided into two cases:

A. Graphs with a number of stages equal to or less than five.
B. Graphs with a number of stages greater than five.

This division will be justified subsequently.

1. ESTABLISHING THE BLOCKING EXPRESSION FOR CASE A

Let us consider a graph of either of the two types above; and, for a moment, let us make an abstraction of the complex interleaved structure which may exist between those links pertaining to a second stage and those pertaining to a fourth stage.

Let us consider as well, two groups C1 and C2 of abstract entities, which we are going to describe as vectors in spite of their not having vectorial characteristics, since these are not elements of vector space.

We shall assume that groups of links corresponding to the first and last stages are divided into certain numbers of parts, n1 and n2, which may be different. These magnitudes depend upon the geometrical characteristics which the graph presents and may be determined only after performing, a priori, a study of these characteristics. Under these conditions the number of "components" of an element of group C1, or of group C2, will be n1 or n2, respectively. The maximum value of each of the components will be given by the number of links which constitute each one of the parts into which the groups of links (pertaining to the initial and final stages) have been divided. The minimum value of each component will be zero and the remaining possible value will be integer numbers.

In this way, the i-th component of a vector x associated with the links of the first stage, will represent the number of free links in the i-th group of the division performed in this stage. And the j-th component of a vector y, associated with the links of the last stage will have the same meaning.

In the following, the distribution vectors, now defined, will be represented by "x" when we refer to vectors associated with the links of the first stage; and "y" will designate those associated with the links of the last stage. Accordingly, the general blocking function for the graph may be expressed in the following form:

\[ P = \sum_{x, y} W(x) \cdot W(y) \cdot J(x, y) \]  

(1.1)

where \( W(x) \), \( W(y) \) and \( J(x, y) \) are defined as follows:

\( W(x) \) is the probability that the state indicated by "x" will occur;
\( W(y) \) is the probability that the state indicated by "y" will occur; and
\( J(x, y) \) is the probability that, given the states indicated by "x" and "y", blocking exists.

In turn, \( W(x) \) may be expressed as the product of two probabilities: (1) the probability of having a certain number of links free in the first stage (this number will be equal to the sum of the components of the vector "x"); and, (2) the probability that, once this event has occurred, a distribution of free links is obtained in each group (equal to that indicated by the distribution vector "x").

The first factor will be determined in accordance with whichever law of probability distribution is most convenient for this first stage. This law may be any of those commonly employed: Engset's, Erlang's, Bernoulli's, etc., or any other which might be derived elsewhere.

The second factor, in accordance with hypothesis a.2 of paragraph 1 may be expressed in the following terms:

\[ P(x) = m_1 \sum_{\pi} \left( \frac{S_m}{S(x)} \right) \]  

(1.2)

where \( n \) = number of links in the first stage of the graph;
\( s_m \) = the number of links which constitute group \( m \);
\( a_m \) = the m-th component of the distribution vector \( x \); and
\( S(x) \) = the sum of the components of the distribution vector \( x \).

It is evident that, on obtaining \( P(x) \), we may consider, by means of an adequate weighting of the loads on the links, the type of strategy applied to seize a free link in the first stage. The strategy that, as is well known, affects decisively the value of the probability of blocking.

\( W(y) \) will be calculated in a manner analogous to that for \( W(x) \), under the same conditions, everything else being equal.

The most difficult factor to evaluate the formula (1.1) is \( J(x, y) \). The division of the links in the first and last stages into a reduced number of groups brings with it a consequent complication in this evaluation. This is owing to the greater number of links within the intermediate stages shared by different paths. Following this line of thought, the ideal solution would be to attain the highest possible number of groups in each of those stages. However, this solution, as we may see later, would result in an increase in execution time of the computer program prepared for the congestion analysis given in (1.1).

Taking all this into consideration, the minimum number of links shared by the different paths, as well as the speed of program
execution, there exist one or more compromise solutions which permit us not only to determine the optimum number of groups (in which the links pertaining to the first and last stages of the group should be divided) but the number of links which constitute each group, as well as the selection of these links.

Therefore \( \mathcal{W}(x,y) \) will be determining factor for the optimum distribution of groups in the first and last stages. Its calculation may be performed by a Bernoulli distribution law, since the use of another type of distribution in the intermediate stages of the graph would increase considerably the complexity of calculation of this factor. This is most important if we wish to consider the link seizure strategy, which also may be taken into account in the evaluation of \( \mathcal{W}(x,y) \). The calculation may be carried out through a proper load assignment on the intermediate links of the graph.

Referring to the types of distribution used normally for the calculation of \( W(x) \), those of Engset and Bernoulli may be used, (or even a distribution derived from Bernoulli's law), in the links of the first stage, according to the ratio between the number of sources, \( N \), and the number of outlets, \( n \). With regard to \( W(y) \), the Erlang and Bernoulli distributions are more utilized. An important point in the use of these distributions is that, while the Engset and Erlang distributions are based on considerations of traffic offered, the Bernoulli distribution is used with carried traffic. The combined use of these different distributions obliges to use iteration processes for determination of blocking, but this may be avoided, when the blocking is not excessively high. This fact has been taken into account in the general calculation program.

2. ORGANIZATION OF AND GENERAL CONSIDERATIONS IN CALCULATION PROGRAM

2.1 Block Organization

The program is organized by blocks as indicated in the diagram. The processes indicated are as follows:

1. Read-out of data which form the network and define the parameters necessary for calculation of the graph.
2. Calculation of combinatorial numbers, factorials and other functions of repetitive use which depend upon the data read out in (1). The values so obtained are stored for use in the memory.
3. Read out of traffic data necessary for the calculation of blocking.
4. Calculation of the chosen probability distribution values whose results also will be stored in the memory.
5. Initiating the expression for blocking.
6. Generation of "x" vectors and calculation of \( W(x) \).
7. Generation of "y" vectors and calculation of \( W(y) \).
8. End of "y" vector generation.
9. Calculation of the blocking term expression for the vector pair "x" and "y"; these values being accumulated successively.
10. End of "x" vector generation.
11. Degree of precision desired for the blocking probability.
12. Print-out of results.

As may be understood, the variable blocks for the users are 4, 7, 9. Block 7 will be modified in accordance with the type of distribution utilized; block 9 will correspond with the interconnection of the exchange links of the graph.

In agreement with what has been pointed out until now, it may be advised that, in the case of graphs which are quite complex and in which the number of distribution vectors "x" and "y" is very high, three basic problems are presented in the utilization of the present algorithm. Such problems are concerned with:

- Speed of execution
- Errors committed in the determination of blocking, and
- Memory required.

Obviously, these problems are aggravated in case graphs are considered which have numbers of stages between five and eight.

2.2 Speed of Execution

The speed of execution is a function of the number of terms to be calculated as well as the number of operations to be performed on each term. Therefore, the task is to find an optimum solution in the sense that both of these factors are to be reduced.

The number of terms to be calculated may be reduced by the joint effect of the following steps:

a) Whenever the division carried out upon the group of links in the first or last stage produces equal groups within that stage, is the case normally encountered in practice, this may be accomplished in one of the stages, with a representative of each one of equivalence classes defined by the following equivalence relation rather than with all the distribution vectors. This relation is:
The first type of error may be reduced through which provides the number of terms neglected whose value is less than one which has been. Then the problem arises to guarantee that the number of neglected terms is such that the disadvantage in execution time would be greater than the advantages sought.

The number of operations to be accomplished on each term may rise to a high magnitude if they are calculated by means of the program in its normal calculating method sequence. Through this step it is interesting to perform the evaluation and memory storage of all those partial results which may be useful in a high number of cases, before entering into the vector generation phases.

These simplifications may be of greater or lesser interest depending upon the situation. It may occur that some of these simplifications, because of their implementation, would introduce such complexity into the program that the disadvantage in execution time would be greater than the advantages sought.

2.3 Errors Committed in Blocking Determination

Errors may occur principally for two reasons:

a) An error committed in the mathematical operations to be performed.
b) An error committed in neglecting terms below a set limit.

The first type of error may be reduced through a proper study performed a priori, of the formula to be programmed and through an examination of the operations to be carried out. This should be done in such a manner that the programming minimizes error sources to the greatest extent possible.

Even though the number of possible states given by the graph might be very high, it is obvious that not all the terms which constitute the blocking expression contribute in the same way to its final value. Therefore it may be useful to effect a rise in the speed of execution by neglecting all those terms whose value is less than one which has been predetermined.

Then the problem arises to guarantee that the number of neglected terms is such that the error committed is of a convenient order of magnitude when compared with the results desired. This problem has been resolved in the program by the addition of a counter, which provides the number of terms neglected. From this may be derived a limit on the error committed for the reasons mentioned.

2.4 Memory Required

Even though the required memory is not a very important factor to be considered at the time of writing the program, it is nevertheless interesting to dwell on it. On some occasions the memory may limit the speed of execution when not all the intermediate results desired may be stored. It may be understood that sometimes the variation of a parameter does not affect a given portion of the blocking expression, since the part which remains unalterable during the variation may indeed be stored in the memory. The calculation program, as it is conceived now, requires a total memory capacity of about 64K bytes.

3. VARIANT OF SERIES-PARALLEL TYPE GRAPHS

By means of the techniques explained above, execution times may vary from a few seconds up to about five minutes, depending upon the complexity of the graph, when a 128Kb computer, model IBM 360/40, is used. A drastic time reduction may be achieved for series-parallel type graphs by use of the variant described below.

Let us consider the subgraphs $S_i$, defined in paragraph 1 and associate with each subgraph a function $F_i(l)$, depending on the number $l$ of the free links of the last stage of $S_i$. (These subgraphs determine the division into groups, as specified in par. 1). In turn, $F_i(l)$ represents the loss probability of the subgraph on the assumption that there are $l$ free links in the last stage. It is evident that it will be necessary to construct as many functions of this type as there are distinct subgraphs. $F_i(l)$ is given by the following expression:

$$ F_i(l) = \sum_{j=0}^{m_i} X_i(j) Y_i(j,\l) $$

where $m_i$ = the number of links in the first stage of subgraph $S_i$;

$X_i(j)$ = the probability of $j$ free links in the first stage of $S_i$; and

$Y_i(j,\l)$ = the probability of blocking in $S_i$, assuming $\l$ and $j$ free links in the first and last stages of $S_i$.

The value of $X_i(j)$ will be determined by the following formula:

$$ X_i(j) = \sum_{K=j}^{n} Z(K) U_i(j/K) $$

where $n$ = number of links in the first stage of the graph;

$Z$ = the probability that there are $K$ free links in the first stage of the graph;

This probability is to be calculated in accordance with the most appropriate distribution for the case under study.

$U_i(j/K)$ = the probability that, if there are $K$ free links in the first stage of the graph, then there are $j$ free links in the first stage of the subgraph $S_i$.

Therefore, in accordance with the above, the probability of blocking is given by:

$$ P = \sum_{i=1}^{m} W(y) \cdot \prod_{i=1}^{n} F_i(l) $$

$W(y)$ and $F_i(l)$ are the functions of the graph and the subgraph, respectively.
where y, W(y) and n, have the same meaning as before, and b, represents the i-th component of the vector y. Obviously, all the simplifications previously described may be applied in this case.

4. GRAPHS WITH MORE THAN FIVE STAGES

Throughout this paper it has remained clear that the objective sought has been to produce a means permitting the calculation of \( \Pi(x,y) \) in simple form. Therefore the first part of this study was dedicated to graphs with a number of stages in order that \( \Pi(x,y) \) could be calculated easily in the majority of cases. Simultaneously the interleaving which really exists, in the switching system under study, has remained unaltered.

However, on increasing the number of stages, the objective becomes proportionally more difficult, to the extreme that formula (1.1) becomes unusable. Following the original line of thought, it is logical to develop new generations of vectors either (1) in the stages adjacent to the terminals, or (2) in the intermediate stages. However, in order that the number of terms to be calculated will be lower, it is more advisable to follow up the first option. In this case the blocking expression will take the following form:

\[
P = \sum_{x} W(x) \sum_{y} W(x') \Pi(s, x', y') \sum_{y'} W(y') \Pi(s, x', y')
\]

where the meanings of the functions are analogous to those of paragraph 1.

5. PRACTICAL APPLICATIONS

By way of example, two graphs of distinct characteristics will be computed by use of the algorithm presented.

Graph no. 1 represents a five-stage network, with concentration in the first stage, and all the possible paths from a subscriber to a determined outlet point. In this case the expressions for \( W(x) \), \( W(y) \) and \( \Pi(x,y) \) are the following:

\[
W(x) = \frac{1}{S(x)} E(N, S, a, S(x))
\]

\[
W(y) = \sum_{i=1}^{16} B(S(y), d, 16)
\]

\[
\Pi(x,y) = \frac{1}{i=1, j=1} \left[ 1 - (1 - b_{ai})(1 - c_{bj}) \right]
\]

where \( b \) and \( c \) are the average loads per link in the second and third stages.

As it may be seen from the figure, the number of groups, in which the link groups of the first and last stages have been divided, is eight. Each group of the first stage is formed by one link, while each group of the last is constituted by two.

The second graph shows a network of four stages, with expansion in the first stage, and represents the possible paths between a determined inlet up to a route of 16 trunks with commoning by two in the last stage. In this case the expressions for \( W(x) \), \( W(y) \) and \( \Pi(x,y) \) are:
\[
W(x) = \left( \sum_{i=1}^{4} \frac{1}{i} \right) \cdot \frac{S(x)^2}{S(x)} - 5 \cdot S(x) - (1-n) \cdot S(x) - 5
\]
(5.4)

where \(0 \leq a, \leq 2\), and \(5 \leq S(x) \leq 8\), provided that the first stage has four inlets and eight outlets. The average load per inlet is

\[
W(y) = \left( \sum_{i=1}^{c} \frac{b_i}{S(y)} \right) \cdot E(S(y),16,A)
\]
(5.5)

where \(E(S(y),16,A)\) is the probability of state of \(S(y)\) free links, with 16-\(S(y)\) links busy, as given by Erlang's formula for 16 trunks and traffic \(A\). \(0 \leq b_i \leq 8\) are the components of vector \(y\).

\[
\Pi(x,y) = \left[ 1 - (1-b)(1-c) \right] \cdot <x', y'>
\]
(5.6)

where \(<x', y'>\) is the scalar product of \(x'\) and \(y\), and \(x' = (a_1', a_2')\), is a vector of two components and is obtained in the following manner: \(a_1' = a_1 + a_2\), \(a_2' = a_1 + a_4\); \(b\) and \(c\) are the loads on the links in the second and third stages.

In this case, it may be seen that the number of subgraphs, into which the link groups are divided, in the first and last stages, will be unequal: four for the first, and only two for the last.

6. CONCLUSIONS

1. The method described here reduces the complication which, in the calculation of the probability of blocking, introduces the interleaved complex among the intermediate stages of the graph.

2. It permits the application of any distribution to the links of the first and last states, without much difficulty.

3. Its formulation may be simple and easy to program.

4. The method permits the development of a program in blocks, which provides adaptability for application to distinct hypotheses or graphs.

5. Its disadvantages are the consequence of the high magnitude of terms which is introduced. Therefore special care is necessary in the treatment of the factors, such as execution time, accuracy and memory requirements.

This method continues under investigation in order to improve and broaden it; the object of this effort has been to develop a general graph calculation program, or at the very least to define interchangeable blocks to be applied to different graphs and traffic distributions.

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