**ABSTRACT**

In this paper a theoretical approach is presented to determine the traffic properties of concentration arrangements, with waiting, in common control systems. Artificial traffic measurements have been used to confirm that the analytical models developed, represent with good approximation actual practical systems.

In particular, for waiting systems, two analytical models are described which constitute limiting conditions of most realistic systems. The aim is to provide a basic approach for dimensioning consideration of common control circuits.

**INTRODUCTION**

In common control telephone switching systems, a concentration arrangement with waiting is widely used to connect incoming junctions and trunks etc., to common control circuits (c.o.c.). The factors which primarily influence the design of c.o.c. in these configurations are:

a) the ratio between the operating time of the controlling circuits and the holding time of incoming devices;

b) the inlet choice law which characterises the input process of call requests.

The problems involving the holding time ratio, in loss systems for two-stage configurations, have been tackled by Behlendorff, Eberhardt, Hagenhaus [11], whilst Gambe, Suzuki, Itoh [2] have considered two-stage link arrangements with waiting.

In this paper more generalized models are presented which enable a study of the above factors and provide an opportune background in view of dimensioning problems of c.o.c.

**SINGLE STAGE ARRANGEMENT**

The first arrangement of c.o.c. considered in the following figure is constituted by k sub-groups with n inlets and m outlets (c.o.c.) respectively, to which poissonian traffic A is offered.

Calls meeting congestion, queue for service on an inlet position and are served at random when a relative outlet becomes available. A call finding all the inlets engaged is rejected.

The holding times are assumed to be random variables with negative exponential distributions. The average operating time for outgoing circuits is denoted by \( t_s \) and the mean conversation time by \( t_c \).
The mean holding time of an inlet will then be composed by $t_c + t_s$.

The parameters characterising the traffic behaviour of the systems are:

- $P(\geq 0)$: call waiting probability
- $P(\geq t)$: probability of delay exceeding time $t$
- $P(t)$: time waiting probability
- $\bar{t}_w/t_s$: mean waiting time expressed in terms of $t_s$

Having fixed the traffic $A$ and the dimensions of the arrangement i.e. $n$, $m$ and $k$, the above parameters are influenced mainly by the following factors:

- a) the ratio between the holding time of the inlets and the operating time of the outlets denoted by:
  
  $\alpha = \frac{t_c + t_s}{t_s}$, \hspace{1cm} $\alpha \geq 1$

- b) the inlet - choice law for call requests.

It has been verified by traffic measurements that for most common inlet - choice procedures, the behaviour of the system is characterised by the inlet occupancy distribution considered independent of the outlets, when the delay probability is sufficiently small.

In other words factors a) and b) can be studied separately and if the interaction between inlets and outlets becomes significant, the $\alpha$ - transformation or factor a) may be taken into account by a step - by - step routine, which modifies the inlet distribution in accordance with the outlet occupancy.

### THE INFLUENCE OF THE $\alpha$ - TRANSFORMATION

Let $P(j)$, $(j=0,1,...,n)$, be the inlet occupancy distribution of a single sub-group relative to a given input process.

The inlet load is assumed to be defined by the first two moments, namely the mean value $R$ and the variance $V$ of the carried traffic.

It is assumed that $N_e$ equivalent independent sources generate the load (Bernoulli input process):

$$N_e = \frac{R}{1-V/R}$$

Traffic measurements have verified that the equivalent source model may be applied after the $\alpha$ - transformation, and the resulting first two moments of the service traffic offered to the outlets become:

$$R_s = R \alpha$$
$$V_s = R_s - \frac{(R-V)}{\alpha^2}$$

This model is valid within a range of values for the load variance $V$ resulting from a poissonian input process for the upper bound and a Bernoulli input process with $n$ independent sources for the lower bound.

It is now possible to evaluate the occupancy distribution on the $m$ outlets by offering these the equivalent source traffic $R_s$ obtained above.

The dependence of the inlet occupancy on the outlet distribution may be considered in terms of the probability ratio, for a call is delayed on an inlet when the relative outlets are found busy.

Although the inlet holding time is delayed by the waiting time $t_w$, traffic measurements have verified that $t_w < t_c + t_s$ if the delay probability $P(>0)$ is less than 10%.

However, if the waiting traffic becomes significant, then it can be taken into account by a step - by - step routine so that the equivalent sources $N_e$ generate a waiting traffic equal to the increment of the inlet load due to the waiting traffic.

At this stage it is worth considering the available waiting positions in each sub-group.

Since the occupancies refer to the carried traffic, it is intuitive to think that the waiting positions will be the difference between the inlet and outlet load $(n - m)$, and independent of the inlet load if $t_w < t_c + t_s$.

This fact has been verified by traffic measurements on a heavily loaded arrangement: $n = 10$; $m = 4$; $\alpha = 6$; and with an inlet load $P = 0.9$.

Thus, having determined the equivalent number of originating sources $N_e$, the waiting positions $(n-m)$, and given the serving devices $m$, the multi-channel waiting model $M/M/R/N - (W)$ in the annex may be applied to calculate the delay probability, mean waiting time, and the waiting time distribution of the system.

### THE INFLUENCE OF THE INLET - CHOICE LAW

The occupancy distribution $P(j)$ of a sub-group depends upon the inlet - choice law adopted for the input process.

Two laws have been considered because they are both compatible with the finite source model described previously, and comprise most practical cases.

**First law**: a sub-group is chosen at random and an available inlet is selected therein. If all the inlets are busy, the call is rejected (Jensen).

**Second law**: all the sub-groups are considered at each call request. An inlet is selected at random from a pool of available inlets. Again if all the inlets are busy, the call is rejected (Fortet).

### JENSEN'S LAW OF CHOICE

In this case, the traffic $A/k$ offered to each sub-group is poissonian and the occupancy $P(j)$ will be an Erlang distribution.

The curves shown in fig. 1 are a comparison between theoretical and measured values as $\alpha$, the holding time ratio, is varied for constant values of global service traffic $R_s$. It is interesting to note that the curves are bounded by an upper limit which denotes the delay probability referred to the traffic carried by a full - availability system with $m$ servers, and
Fig. 1 Waiting probability (Jensen’s law)

(n−m) waiting positions to which a traffic $R_n$ is offered. The lower limit denotes the delay probability of a system with n independent sources which offer a traffic $R_n$ to the m servants.

The validity of the theoretical model is also confirmed by the curves shown in fig. 2 of the mean waiting time $t_w$ and the curves in fig. 3 of the probability of delay exceeding $t$, $P(>t)$.

FORTET’S LAW OF CHOICE

In this case, the probability $P(i)$ of finding i, $(i = 0, \ldots, n-k)$, occupied inlets on all the k sub-groups is an Erlang distribution. Since the probability of finding j inlets occupied on a single sub-group is a hypergeometric distribution, it follows that the occupancy distribution $P(j)$ on a single sub-group will be given by:

$$P(j) = \sum_{i=0}^{n-k} \binom{n}{i} \binom{n-(k-1)}{j-i} \cdot P(i), j=0, \ldots, n$$

In fig. 4 the theoretical and measured values are compared, as $\alpha$, the holding time ratio, is varied for constant carried traffic $R_s$.

As $\alpha$, the inlet load, tends to one, both curves coincide. This is because the system operates as if n independent sources offer $R_n = n/\alpha$ to the m outlets.

Again it is apparent that the analytical model is extremely valid when considering both waiting-

Fig. 2 Mean waiting time (Jensen’s law)

time curves (fig. 5), and probability of delay exceeding curves (fig. 6).

TWO-STAGE LINK SYSTEM WITH WAITING

The equivalent source model described in the single-stage arrangement may also be used to study the traffic aspects of two-stage link systems with delay. Basically, the method consists in applying Jacobus’s theory [3] for link systems. It is possible to determine the congestion probability of the system when considering independent the occupancy distributions relative to the links and the outlets.

These distributions are determined by means of the equivalent source model since for low congestion probabilities the distributions of the inlets and of the outlets may be considered independent.

By this means the inlets, with their mean holding time given by $t_s + \alpha$, may be considered separately from the links and the outlets which have mean operating time $t_s$.

Assuming valid the equivalent source model, the traffic offered to the links and to the outlets may be taken to be generated by an equivalent number of independent sources which take into account the smoothing effects due to the inlet load as $\alpha$, the transformation ratio, varies.
Fig. 3 Waiting time distribution (Jensen's law)

Fig. 4 Waiting probability (Fortet's law)

Fig. 5 Mean waiting time (Fortet's law)

Fig. 6 Waiting time distribution (Fortet's law)
The two-stage link system may be represented as shown in the following:

The following assumptions are made:
1) the system operates in a loss - delay mode.
2) the delayed calls are chosen at random in each A column.
3) two-step random hunting for free links and free outlets is considered.
4) the inlets have a mean holding time $t_c + t_s$ whereas the links and the outlets are occupied for $t_s$.
5) the holding times are exponentially distributed.
6) two inlet choice laws (i.e. Jensen, Fortet) are considered.

CONGESTION PROBABILITY

In accordance with the theory of Jacobaeus, let $B(p)$ be the probability of $p$ busy devices in a column B.

$C(p)$ be the probability of $p$ busy devices in the $g$ columns of stage C.

By considering $H(m-p)$ to be the conditional probability that all the outlets, seen by the available $(m-p)$ links, are busy when there are $p$ busy links in the B column, then the total time congestion probability:

$$P(\geq 0) = \sum_{p=0}^{m} B(p) \cdot H(m-p),$$

where

$$H(m-p) = \sum_{j=m-p}^{m} C(mg-1+j) \cdot \frac{1}{m} \cdot \frac{1}{m-p} \cdot 0 < p < m$$

$H(0) = 1$

It has been verified that the above expressions are also formally valid for call distributions instead of time distributions, i.e. by substituting in the above expressions:

$B_c(p)$ the conditional call pr. of $p$ busy links in a B column and

$C_c(p)$ the conditional call pr. of $p$ busy devices in the $g$ columns of C.

The distributions $B(p)$ and $B_c(p)$ may be determined by considering $k$ inlet sub-groups as in the previous single-stage arrangement, for both inlet choice laws denoted by Jensen and Fortet.

It is, therefore, possible to determine the following parameters:

- $N_1$ number of equivalent sources
- $R_1$ service traffic carried by each sub-group
- $S_1 = m$ number of servers
- $W_1 = (n-m)$ number of waiting positions.

In determining the distributions $C(p)$ and $C_c(p)$ the two inlet-choice laws are considered separately, for an equivalent system composed of $(n-k)$ inlets and $(g-m)$ outlets:

Jensen's law of choice: the $k$ traffic streams, offered to the two-stage arrangement are considered independent.

With good approximation the correlation introduced by blocking may be neglected; the occupancy distribution $P(i)$ of the $(n-k)$ inlets is characterised by the first two moments:

- mean value $R* = k \cdot R$
- variance $V* = k \cdot V$

where $R$ and $V$ denote the mean and variance respectively of an Erlang distribution of the inlets of a single sub-group due to traffic $A/k$.

Fortet's law of choice: in this case, the distribution on the $(n-k)$ inlets will be Erlangian with mean $R*$ and variance $V*$.

It is, therefore, possible to determine the distributions $C(p)$ and $C_c(p)$ by applying the equivalent source model with:

$$N_2 = \frac{R*}{1-V*/R*}$$

$$R_2 = \frac{R*}{R}$$

$$S_2 = (m-g)$$

$$W_2 = (n-k)-(m-g)$$

WAITING TIME

A call waits on an inlet position in an A column when there are $p$ links busy on the relative B column and the $(m-p)g$ particular outlets on the C stage are busy.

A waiting call may be served when either of the following disjoint independent events occur:

$E_1$: a particular outlet connected to one of the $p$ links busy becomes free. A waiting call in the relative A column is then served at random.

$E_2$: a particular outlet among the $(m-p)g$ busy ones becomes free. A waiting call accessible from the liberated outlet is served at random.
Fig. 7 Waiting probability and mean waiting time (Jensen's law)

Fig. 8 Waiting probability and mean waiting time (Fortet's law)

Fig. 9 Waiting time distribution (Jensen's law)

Fig. 10 Waiting time distribution (Fortet's law)
Let \( V_{1p}(t) \) be the distribution of the waiting time exceeding \( t \) in the state \( p \) - links busy in column \( B^* \) relative to event \( E_1 \), and \( V_{2p}(t) \) be that distribution relative to event \( E_2 \). Then with good approximation the composite distribution relative to state \( p \) is:

\[
V_p(t) = V_{1p}(t) \cdot V_{2p}(t)
\]

The resultant distribution of the waiting times is then obtained by composing \( V_p(t) \) with the blocking probability for state \( p \), and summing over each \( p \).

Then:

\[
V(t) = \sum_{p \in P} V_p(t) \cdot B(p) \cdot H_0(m-p)
\]

The distribution \( V_{1p}(t) \) and \( V_{2p}(t) \) may both be determined by defining the opportune parameters of the equivalent source model.

For the distribution \( V_{1p}(t) \):

- \( N = N_1 \)
- \( W = (n-m) \)
- \( S = p \)
- \( A_{N_1} = (p \cdot R_1) / m \)

For the distribution \( V_{2p}(t) \):

- \( N = N_2 \)
- \( W = (n \cdot k - m - g) \cdot (m-p) / m \)
- \( S = (m-p) g \)
- \( A_{N_2} = (m-p) R_2 / m \)

Thus the mean waiting time is given by:

\[
\bar{W}_w = \int_0^\infty V(t) \, dt
\]

COMPARISON BETWEEN MEASURED AND THEORETICAL VALUES

On fig. 7 are shown the theoretical and measured values relative a two-stage arrangement with \( n=10, k=10, m=5 \) and \( g=5 \).

The input process offering traffic to the system is characterised by Jensen's inlet law of choice, and fig. 8 shows the results relative to the same system with Fortet's law of choice.

In both curves the call waiting probability \( P_0(\infty) \) and mean waiting time \( \bar{W}_w/\bar{W}_s \) are plotted as functions of with constant total service traffic \( R_s \).

In both cases, the curves are obtained within two bounds. The upper limit is obtained by offering directly to the system the service traffic \( R_s \) (see [2]), and the lower limit is obtained when considering the inlets as independent sources; i.e. the parameters of the equivalent source model being:

\[
N_1 = n \quad \text{and} \quad N_2 = n-k
\]

It is interesting to note that with Fortet's law (fig. 8) the plots for the two-stage system are more distant from the lower limit than in the case of the single-stage arrangement. This phenomenon is due to the fact that the parameters \( \delta(0) \) and \( \bar{W}_w/\bar{W}_s \) depend upon the interaction between the occupancy distribution of the outlets of a sub-group \( A \), and the outlets' distribution of the two-stage arrangement. In the case of Fortet, the carried traffic streams of the \( A \) sub-group are correlated, and the resulting traffic offered to the outlets will have a greater variance.

In other words, in this example, the congestion characteristics, mainly depend on the outlets whereas in the case of Jensen's law of choice, they depend essentially on the links.

On figs. 9 and 10 the delay distribution with a service traffic \( R_s = 14 \text{ erl} \), and an inlet load \( g = 0.7 \text{ erl} \), for Jensen and Fortet respectively, are plotted.

The validity of the theoretical model is confirmed by the good agreement between theoretical and measured values.

CONCLUSIONS

In this paper single and two-stage concentration arrangements, of common control circuits with waiting, have been studied.

It has been shown that the traffic properties of these systems depend in a significant way on two factors:

a) the difference between the holding times of the inlets and outlets,

b) the inlet law of choice for call requests.

In particular, two inlet choice laws representative of extreme situations comprising most practical cases, were considered.

The analytical approach for both single and two-stage arrangements was based on a finite source model which enabled us to calculate the delay probability, the waiting distribution and the mean waiting time. The model was perfected and verified by means of artificial measurements with a simulation program written in Simula. The algorithm proposed here can be used in dimensioning problems involving common control circuits, and is representative of most practical systems.

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REFERENCES


FINITE - SOURCE WAITING MODEL \( M/M/S/N - (W) \)

The equivalent traffic model consists of \( N \) independent sources offering traffic to \( S \) channels. A call request encountering congestion on the channels can wait on one of the \( W - W \) waiting positions. When an occupied channel becomes free a waiting call is chosen and served at random. An originating source generates poissonian traffic, when it is free, with an interarrival rate \( \lambda \).

Thus, when the system is in state \( i \) the interarrival rate of birth, \( \lambda_i \), is assumed to be:

\[
\lambda_i = \lambda_i (N - i)
\]

The operating time of the channels has a negative exponential distribution with mean value \( 1/\mu \); the death rate in state \( \mu_j \) will be:

\[
\mu_j = \mu_j , \quad 0 \leq j < S
\]

and

\[
\mu_j = S/\mu , \quad S \leq j \leq S + W
\]

By solving the Birth and Death equation the state probability expression \( P(j) \) is obtained:

\[
P(j) = P(0) \left( \frac{\lambda}{\mu} \right) \left( \frac{n}{N} \right) , \quad 0 \leq j < S
\]

\[
P(j) = P(0) \left( \frac{\lambda}{\mu} \right) \left( \frac{n}{N} \right) S^{-j} j! , \quad S \leq j \leq S + W
\]

where \( P(0) \) is determined from the condition

\[
\sum_{j=0}^{S+W} P(j) = 1
\]

By substituting in 1) \((N-1)\) for \( N \) and \((W-1)\) for \( W \), the conditional call probability for those calls accepted by the system \( P_c(j) \) is determined.

By substituting \((N-1)\) for \( N \), the conditional call probability of all the calls offered to the system \( P_0(j) \), is determined; since the call congestion \( B \) is referred to all the calls, then:

\[
B = P_0(S + W)
\]

The probability of delay corresponding to time congestion is given by:

\[
P(\gt 0) = \sum_{j=S}^{S+W} P(j)
\]

and the probability corresponding to the call congestion for the accepted calls by:

\[
P_c(\gt) = \sum_{j=S}^{S+W-1} P_c(j)
\]

If \( A_N \) is the traffic offered by the \( N \) sources, the interarrival rate is:

\[
\lambda = \frac{A_N}{N - E(Y(t))}
\]

where \( E(Y(t)) \) is the average value of the process of source congestion. When the lost traffic is negligible with respect to \( A_N \), it follows that

\[ E(Y(t)) \approx A_N + A_W \]

where \( A_W \) is the waiting traffic, defined as:

\[
A_W = \sum_{j=S}^{S+W} (j - S) P(j)
\]

Since the probabilities \( P(j) \) depend on \( \lambda \), by 1) and \( \lambda \) depend on \( P(j) \), by 2) in order to evaluate the interarrival rate corresponding to an offered traffic \( A_N \), it is necessary to iterate.

Since the existing calls are served at random, the distribution of delay exceeding \( t \), \( P(t) \) can be approximately calculated in practice, by mixing delay distributions corresponding to a first-come-first-served basis, FIFO.

Let \( u = S + t/T_0 \) and \( P_p(>u) \) be the delay distribution in the case of FIFO service; then:

\[
P_p(>u) = \frac{1}{c_0} \sum_{j=1}^{\infty} a_1 \sum_{j=0}^{\infty} e^{-u}
\]

where

\[
c_0 = \sum_{j=0}^{N-S+1} \left( \frac{\lambda}{\mu S} \right)^j
\]

\[
a_1 = \left( \frac{\lambda}{\mu S} \right)^N
\]

The object is to approximate the distribution \( P_p(>u) \) with a composition of type:

\[
P_p(>u) \approx \sum_{i=1}^{\infty} A_i P_p(>Y_i u)
\]

Where \( A_i \) and \( Y_i \) are constants independent of \( u \) which are to be determined so that the first \( k \) moments of both sides of the expressions are the same. If \( N_r \) is the \( r \)th moment of \( P_p(>u) \), \( N_r \) is given by:

\[
N_r = \sum_{i=1}^{\infty} A_i Y_i^r
\]

In the case of FIFO, the moments may be calculated by 4) by integrating, while in the case of random service the procedure is indicated in Segal [4]. This method, however, is rather involved and costly; approximate formulae for the coefficients in 5) are used, by truncating when \( \omega = 2 \):

\[
A_1 = \frac{1}{Y_1} Y_1 1, \quad A_2 = \frac{1}{Y_2} Y_2 2, \quad A_3 = \frac{1}{Y_3} Y_3 3, \quad A_4 = \frac{1}{Y_4} Y_4 4, \quad \text{etc.}
\]

where

\[
Y_1 = \frac{A_1}{\left( A_2 \left( 3 - 1 \right) \right)^{1/2}}
\]

It may be noticed that for \( N = \infty \) and \( W = \infty \), the coefficients are the same as those obtained by Ricordan [3], for the waiting system \( M/M/S \) with random service.

The mean waiting time of all the calls in, therefore:

\[
T_w = P_c(>0) \frac{T_0 S^{-}}{c_0} \sum_{j=0}^{\infty} a_1 (j + 1)
\]