EXTENSION OF THE EQUIVALENT RANDOM METHOD TO SMOOTH TRAFFICS

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ABSTRACT

In the following we show that the equivalent random method can be extended in the desired direction by transforming the erlang formula in such a way that it may be expressed by the well known exponential integral. On the basis of a simple approximation formula for this integral we give an ALGOL-program for the calculation of the mean value and the variance coefficient for traffics overflowing from fictitious trunk groups with negative numbers of trunks.

On the basis of this formula we have calculated an additional working chart for the extension of the equivalent random method to negative numbers of trunks.

An example of the method is given and the results were compared with exact values and with the values obtained for an unlimited number of sources.

The results of the equivalent random method seem to be in rather good accordance with the exact values.

1. INTRODUCTION

As is well known in traffic theory, the equivalent random method is a valuable tool for handling overflow problems with rather good approximation to reality (see [1] to [4]). This method has, however, a certain limitation as the traffics offered must be of a pure poissonian type. Therefore it has not yet been possible to incorporate smooth traffics into this method.

The reason for this is the fact that the equivalent random method characterizes each traffic by a pair of variables, the mean value and the variance or the variance coefficient respectively of the number of occupied trunks, and replaces the given traffic by an equivalent random traffic having the same values for these variables, i.e. by a pure chance traffic overflowing from a fictitious number of trunks. As by the existing method this equivalent traffic can only be determined if its fictitious number of circuits is not negative, the variance of the given traffic must be greater or equal to its mean (see [1]) or its variance coefficient greater or equal to zero. On the other hand, the variances (or the variance coefficients) for truncated pure chance traffics or for traffics produced by a fixed number of sources violate this condition (see Appendix 3. and 4.). Therefore the existing equivalent random method must be enhanced if the types of traffics just mentioned are to be included. And they should be included to make the equivalent random method generally applicable to overflow problems.

2. THE EXTENDED EQUIVALENT RANDOM THEORY

In order to extend the equivalent random theory in the sense under discussion, it would be desirable to determine the pair of traffic variables for pure chance traffics overflowing from fictitious negative numbers of trunks too. For this it is sufficient to determine the values of the erlang formula for a negative number of trunks, for instance for minus one. Once this problem is solved the values for other numbers of trunks can be derived by a recursion formula (see Appendix 5.) and from these the values of the traffic coefficients themselves by other formulae (see Appendix 2.).

The erlang formula in its well known form

\[
\frac{(A^N/N!)}{0^{N}} \frac{(A^{F}/r!)}{0^{F}}
\]

cannot be used for \( N = -1 \), just as the recursion formula (see Appendix 5.) cannot be used in this case. But it is possible to transform the erlang formula for \( N = -1 \) in such a way that we obtain (see Appendix 6.)

\[
E_{1,-1}(A) = \frac{(A^{-1}e^{-A})}{\int_{-1}^{1}e^{-t}dt}
\]
This can be rewritten by means of the well-known exponential integral
\[ E_1(x) = - \int_{x}^{\infty} t^{-1} e^t dt \]

to
\[ E_{1,-1}(A) = (-E_1(-A) \cdot A)^{-1} \]

As for \(-E_1(-A)\) we have the series
\[ -E_1(-A) = C - \ln A + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{A^n}{n+n!} \]

with \(C = 0.577 215 665\) (Euler's constant) if \(A \neq 0\). It is easy to compute \(E_{1,-1}(A)\) to any required accuracy especially for small values of the offered traffic \(A\). Further for \(A \approx 1\) (see Appendix 7.) we have an approximation \(-E_1(-A)\) to \(-E_1(-A)\) whose relative error is not greater than \(2 \times 10^{-6}\). The introduction of this approximation into the formula for \(E_{1,-1}(A)\) results in a very simple formula, namely
\[ E_{1,-1}(A) \approx \frac{a_0 + a_1 A + a_2 A^2 + a_3 A^3 + a_4 A^4}{b_0 + b_1 A + b_2 A^2 + b_3 A^3 + b_4 A^4} \]

(for the coefficients \(a_k\) and \(b_k\) see Appendix 7.).

On the basis of this formula we have given in Appendix 8. an ALGOL-program for the calculation of the traffic values of traffic overflowing from negative numbers of trunks. Beyond this it is possible, of course, to develop a fully computerized procedure for the application of the equivalent random method in this field too.

3. DISCUSSION OF THE RESULTS

With the aid of the aforementioned ALGOL-program we have calculated an additional working chart for the equivalent random method for fictitious negative numbers of trunks (see fig. 1).

The use of this table may be demonstrated by an example. A traffic of \(R = 5\) erlang may be assumed, originated by \(M = 25\) sources of equal intensity. From the given formulae (see Appendix 4.) we derive the value of the variance coefficient in this case to be
\[ D = \frac{R^2}{M} = \frac{5 \times 5}{25} = -1. \]

In the chart (see fig. 1) we find for \(R = 5\) and \(D = -1\) the circled point and read that the equivalent random traffic \(A = 3.6\) erlang overflowing from the fictitious number of \(-1.65\) trunks has the given traffic values. If we want to determine the loss probability for the given traffic if it is offered to, say, \(N = 10\) trunks we must look for the overflow of a random traffic of \(A = 3.6\) erlang from \(10 - 1.65 = 8.35\) trunks. To gain this value we may use the customary charts for the equivalent random theory (see e.g. [3])

In Table 1 we give some more results for the same source traffic.

The results obtained by the equivalent random method seem to be in rather good accordance with the exact values.

APPENDIX

1. For a given traffic the probability for \(X\) occupied trunks may be \(W(X)\). Then the mean value \(R\) is defined by
\[ R = \sum X W(X) \]
the variance by
\[ \sigma^2 = \sum (X-R)^2 W(X) \]
and the variance coefficient by
\[ D = \frac{\sigma^2}{R} = \sum (X-R)^2 W(X) - R^2 \]

2. The values of the traffic overflowing from a full-access trunk group of \(N\) trunks can be derived as (see [1])
\[ R = A \cdot E_{1,N}(A) \text{ with } E_{1,N}(A) = \left( A^N / N! \right) / \left( \sum_{r=0}^{N} A^r / r! \right) \]
\[ \sigma^2 = R \left( \frac{A}{R+N+1} \right) + 1 - R \]
\[ D = \sigma^2 - R \]

3. For a truncated pure chance traffic we have
\[ W(X) = \frac{X}{X!} \cdot \left\{ \sum_{r=0}^{X} \frac{A^r}{r!} \right\}^{-1} \]
and therefore according to 1,
\[ R = \sum_{x=0}^{N} \frac{X}{X!} \cdot \left\{ \frac{A^r}{r!} \right\}^{-1} = A \left( 1 - E_{1,N}(A) \right) \]
\[ \sigma^2 = R - (A-R) \cdot (N-R) \]
\[ D = (A-R) \cdot (N-R) \]

4. For a traffic generated by a fixed number \(M\) of sources of equal intensity we obtain
\[ W(X) = \left( \frac{X}{X!} \right) \cdot a^X (1-a)^{M-X} \]
and therefore according to 1.
5. From the erlang formula

\[ E_{1,N}(A) = \frac{A^N}{N!} \left\{ \sum_{r=0}^{N} \frac{A^r}{r!} e^{-A} \right\} \]

it is easy to derive the recursion formula

\[ E_{1,N-1}(A) = \frac{N \cdot E_{1,N}(A)}{A (1 - E_{1,N}(A))} \]

6. On the other hand we can transform the erlang formula to

\[ E_{1,N}(A) = \left( A^N e^{-A} \right) / \left\{ \sum_{r=0}^{N} \frac{A^r}{r!} e^{-A} \right\} \]

and finally to

\[ E_{1,N}(A) = \left( A^N e^{-A} \right) / \int_A^\infty t^N e^{-t} dt \]

For \( N = -1 \) this gives

\[ E_{1,-1}(A) = \left( A^{-1} e^{-A} \right) / \int_A^\infty t^{-1} e^{-t} dt \]

The denominator of this expression is related to the exponential integral

\[ Ei(x) = - \int_x^\infty t^{-1} e^{-t} dt \]

a well known higher function in mathematics (see §53).

One can write

\[ \int_A^\infty t^{-1} e^{-t} dt = -Ei(-A) \]

and therefore finally obtain

\[ E_{1,-1}(A) = (-Ei(-A) \cdot A^{-1})^{-1} \]

7. For \(-Ei(-A)\) and for \( A \geq 1 \) we have the Chebyshev approximation (see §6)

\[ -Ei^n(-A) = e^{-A} \cdot A^{-1} \sum_{k=0}^{b} \frac{a_k A^k}{b_k A^k} \]

with

\[
\begin{align*}
    a_0 &= 0.2677737343 & b_0 &= 3.9584969228 \\
    a_1 &= 8.6347608925 & b_1 &= 21.0996530827 \\
    a_2 &= 18.0590167230 & b_2 &= 25.6329641486 \\
    a_3 &= 8.5733287401 & b_3 &= 9.5733223454 \\
    a_4 &= 1 & b_4 &= 1
\end{align*}
\]
Appendix B.

4004/005 ALGOL (VER007) 1-081071-7) "TRAFF" SOURCE PROGRAM

LINE NO. SOURCE TEXT

00000* 'BEGIN' 'COMMENT'
00001* CALCULATION OF APPROXIMATIONS FOR THE MEAN VALUES R
00002* AND VARIANCE COEFFICIENTS D FOR PURE CHANCE TRAFFICS,
00003* A OFFERED TO FICTITIOUS FULL AVAILABLE TRUNC GROUPS WITH
00004* NEGATIVE NUMBERS N OF CIRCUITS.
00005* (A MUST BE NOT SMALLER THAN ONE);
00006* 'REAL' 'AA', 'AD', 'AE', 'R', 'D';
00007* 'INTEGER' 'N', 'NE';
00008* READ ('AA', 'AD', 'AE', 'NE');
00009* 'FOR' 'A' := 'AA' 'STEP' 'AD' 'UNTIL' 'AE' 'DO'
00010* 'BEGIN'
00011* 'N' := -1;
00012* 'R' := (((('AA' + 'AD' + 'AD' + 'AD' + 'AD') + 'AA')
00013* + 'AD' + 'AD' + 'AD' + 'AD' + 'AD') + 'AA')
00014* (((('AA' + 'AD' + 'AD' + 'AD' + 'AD') + 'AA')
00015* + 'AD' + 'AD' + 'AD' + 'AD' + 'AD') + 'AA');
00016* 'D' := ('AA' - ('A' + 'R' + 'A' + 'A'));
00017* PRINT ('A', 'R', 'D');
00018* 'FOR' 'N' := -2 'STEP' 1 'UNTIL' 'NE' 'DO'
00019* 'BEGIN'
00020* 'R' := (((('AA' + 'AD' + 'AD' + 'AD' + 'AD') + 'AA')
00021* + 'AD' + 'AD' + 'AD' + 'AD' + 'AD') + 'AA')
00022* (((('AA' + 'AD' + 'AD' + 'AD' + 'AD') + 'AA')
00023* + 'AD' + 'AD' + 'AD' + 'AD' + 'AD') + 'AA');
00024* PRINT ('A', 'R', 'D');
00025* 'END';
00026* 'END';
00027* ** ALGOL END
APPENDIX 9.

The exact calculation of the variance coefficient of general Poisson overflow traffics

Let us consider an arbitrary switching network K to which m Poisson traffics are offered. We wish to calculate the traffic intensity and variance coefficient of the overflow traffic consisting of those calls which contribute to any 1 of the offered Poisson traffics (1 ≤ i ≤ m) and which flow over from the given switching network owing to a shortage of trunks. This type of traffic will be referred to as general Poisson overflow traffic.

Let us assume that the switching network has N+1 different occupancy states. If these states are numbered in some sequence from 0 to N, and \([k,r]\) indicates the limiting value of the state probability of simultaneously finding the state \([k]\) in the switching network and \(r\) trunks busy with the chosen overflow traffic in a subsequent unlimited trunk group, the following set of linear equations can be formulated for these state probabilities:

\[
f(k,r) = \sum_{i=0}^{N} p_k^i [i,r] + d_k [k,r-1] - d_k [k,r] + (r+1) [k,r+1] - r [k,r] = 0
\]

where

\[
0 \leq k \leq N ; \quad 0 \leq r < \infty
\]

and

\[
p_k^i = \sum_{j \neq k} p_j^k
\]

Introducing generating functions of the following form:

\[
F(k,y) = \sum_{r=0}^{\infty} f(k,r) y^r, \quad 0 \leq k \leq N,
\]

enables (1) to be written in the form

\[
F(k,y) = \sum_{r=0}^{\infty} \sum_{i=0}^{N} p_k^i [i,r] y^r + \sum_{r=0}^{\infty} d_k [k,r-1] y^{r-1} - \sum_{r=0}^{\infty} d_k [k,r] y^r + \sum_{r=0}^{\infty} (r+1) [k,r+1] y^r - \sum_{r=0}^{\infty} r [k,r] y^{r-1} \cdot y \quad \equiv 0.
\]

This can be rewritten as

\[
F(k,y) = \sum_{i=0}^{N} p_k^i \left( \sum_{r=0}^{\infty} [i,r] y^r \right) + (y-1)d_k \left( \sum_{r=0}^{\infty} [k,r] y^r \right) - (y-1) \sum_{r=0}^{\infty} r [k,r] y^{r-1} \equiv 0
\]

where

\[
0 \leq k \leq N.
\]

If we put \(y = 1\) and introduce the state probabilities for the switching network alone, \([i] = \sum_{r=0}^{\infty} [i,r]\), equation (2) yields
\[ P(k;1) = \sum_{i=0}^{N} p_i^k \quad [i] = 0, \quad 0 \leq k \leq N, \] (3)

i.e. the set of equations for calculating these state probabilities \([i]\).

Differentiating the generating functions \(P(k;y)\) twice with respect to \(y\) and putting \(y = 1\) again, we have

\[ P_y(k;1) = \sum_{i=0}^{N} \frac{d}{dy} p_i^k (\sum_{r=0}^{\infty} r \cdot [i,r]) + d_k \sum_{r=0}^{\infty} r \cdot [k,r] - \sum_{r=0}^{\infty} r \cdot [k,r] = 0 \] (4)

and

\[ P_{yy}(k;1) = \sum_{i=0}^{N} \frac{d^2}{dy^2} p_i^k (\sum_{r=0}^{\infty} r(r-1) \cdot [i,r]) + 2 \cdot d_k \sum_{r=0}^{\infty} r \cdot [k,r] - 2 \cdot \sum_{r=0}^{\infty} r(r-1) \cdot [k,r] = 0 \] (5).

From (1a) it follows that

\[ \sum_{k=0}^{N} \sum_{i=0}^{N} p_i^k \sum_{r=0}^{\infty} \frac{d}{dy} p_i^k \sum_{r=0}^{\infty} \frac{d}{dy} r \cdot [i,r] = 0. \]

Since

\[ \sum_{r=0}^{\infty} [i,r] = [i] \]

and the state probabilities of the 2nd type are given by

\[ \sum_{r=0}^{\infty} r \cdot [i,r] = [i] \]

we get from (4) and (5)

\[ \sum_{k=0}^{N} d_k [k] = \sum_{k=0}^{N} \sum_{r=0}^{\infty} r \cdot [k,r] = \sum_{r=0}^{\infty} r \left( \sum_{k=0}^{N} [k,r] \right) \] (6)

and

\[ \sum_{k=0}^{N} d_k [k] = \sum_{r=0}^{\infty} r(r-1) \left( \sum_{k=0}^{N} [k,r] \right) \] (7).

By definition, the traffic intensity \(R\) and the variance coefficient \(D\) of the overflow traffic are given

\[ R = \sum_{r=0}^{\infty} r \left( \sum_{k=0}^{N} [k,r] \right) \]

\[ D = \sum_{r=0}^{\infty} (r-R)^2 \left( \sum_{k=0}^{N} [k,r] \right) - R = \sum_{r=0}^{\infty} r(r-1) \left( \sum_{k=0}^{N} [k,r] \right) - R^2. \]

Using (6) and (7) these can therefore be written in the form

\[ R = \sum_{k=0}^{N} d_k [k] \quad \text{and} \quad D = \sum_{k=0}^{N} d_k [k] - R^2 \] (8) (9).

To be able to determine \(R\) and \(D\) from these formulae, the sets of linear equations for the \([k]\) and \([k]\) must be solved.
As is well known, the set of equations for the calculation of the $[k]$ is given by (3).

The set of equations for the calculation of the $[k]$ can be derived from (4). The result is as follows:

$$\sum_{i=0}^{N} \left( p_i^k - \delta_i^k \right) \xi_i = -d_k [k]$$ \hspace{1cm} (10)

with

$$\delta_i^k = \begin{cases} 1 & \text{for } i = k \\ 0 & \text{for } i \neq k \end{cases}$$

This set of equations (10) for the state probabilities of the 2nd type is very similar to the well-known set of equations (3) for the usual state probabilities of the 1st type. It is however inhomogeneous, since some of the state probabilities of the 1st type appear on the right-hand side, and in addition the absolute value of all the elements along the main diagonal of the matrix on the left-hand side of the set of equations is increased by 1.

The set of equations for the state probabilities of the 2nd type can only be solved therefore if the state probabilities of the 1st type are known. The more dominant contribution of the main-diagonal elements means, however, that this set of equations converges better than the first set of equations when the usual iteration methods are used.
<table>
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<th>Number of Trunks</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>exact figure according to Rubas</strong></td>
<td>5.260 %</td>
<td>2.327 %</td>
<td>0.877 %</td>
<td>0.280 %</td>
</tr>
<tr>
<td><strong>Equivalent Random Method</strong></td>
<td>5.06 %</td>
<td>2.40 %</td>
<td>1.02 %</td>
<td>0.39 %</td>
</tr>
<tr>
<td><strong>unlimited number of Sources</strong></td>
<td>7.00 %</td>
<td>3.74 %</td>
<td>1.84 %</td>
<td>0.83 %</td>
</tr>
</tbody>
</table>

**Table 1**
Loss probability for traffic of 5 erlang, originated by 25 equivalent sources offered to N trunks

**Fig. 1**
Additional working table for the equivalent random method:
Negative number of fictitious trunks
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