ABSTRACT

A method for the economic optimization of hierarchical organized telephone networks with alternative routing and non-coincident busy hours is described.

The proposed method uses the gradient iterative algorithm for the optimization process. It is shown in this paper that the computation of the gradient components can be done in a recurrent way by always using the same basic equation, which makes the procedure very suitable for a computerized calculation process. Although the procedure has not as yet been programmed, the gradient algorithm has shown to be a powerful instrument for resolving similar networks optimization problems.

It is important to note that two parameters, mean and variance, are used to define the traffic.

1. THE PROBLEM

The aim of this proposed method is the economic optimization of telephone trunk networks with alternative routes, hierarchical structure and non-coincident busy hours.

Before presenting the problem in a concrete way, the starting hypothesis and basic concepts referring to it, will be defined.

1.1. NETWORK DEFINITION.

Fig. 1 depicts a network with alternative routes and a three level hierarchical structure in a conventional area.

- represents a local exchange.
- a primary center.
- a secondary center.

The network trunks are all one-way.

1.2. ROUTING DISCIPLINE.

Fig. 2 shows a schematic representation of the hierarchical structure of the network in Fig. 1. The dotted vertical lines separate the exchanges which are located in the same primary zone. The solid vertical lines separate the exchanges which are located in the same secondary zone.

Let us consider any two local exchanges, number 3 and 13, for example. In Fig. 3a these two exchanges have been represented with all their centers of superior order (primary and secondary).
The drawing also shows the existing trunks in Fig. 1 between these two exchanges in the direction from 3 to 13, and their routing discipline (or overflow law).

Fig. 3b depicts all possible routes and their routing discipline between the exchanges in Fig. 3a, assuming that there is no restriction on the number of successive overflows that the switching equipment of an exchange can perform.

The network structure with its routing discipline, as seen in the above example, is the one we have considered because it is the most generally accepted. However, the proposed method is independent of the structure and serves for networks with different routing disciplines.

1.3. TRAFFIC

Traffic is defined in the proposed method by two parameters: mean and variance, for each considered hour.

The problem of calculating:
- The overflow traffic in each hour of any route.
- The number of circuits in final routes for a given offered traffic, satisfying the grade of service required under the worst conditions at any time,

can be made by using classical procedures.

1.4. NETWORK SERVICE QUALITY

Two criteria are generally used to define the service quality of a network. They are the "final trunk group blocking" and the "exchange to exchange blocking".

Either of these criteria can be used in the proposed method. Any final route will be dimensioned with a number of circuits to give a lost traffic, in its busy hour, that can be defined by either of the previous criteria. This will result in the route being overdimensioned for the hours other than its busy hour.

1.5. PROBLEM DEFINITION

After defining the hierarchical structure of a network and the traffic matrices at considered moments in time between exchanges of the lowest level of the network, the problem, therefore, is to determine the minimum cost network that satisfies the required grade of service.

The costs taken into account are those of the telephonic elements which come into play, transmission and switching. For the sake of simplicity, costs (switching and transmission) are defined in each route as linear functions of the number of circuits.

2. STUDY OF TWO INTRODUCTORY CASES.

2.1. GENERAL

A study of two simple cases that will serve as introduction for the derivation of the procedure for the general case study (section 3) follows. The method will consist of the application of the gradient algorithm to solve the optimization problem. The main steps in the optimization process will be:

- Preliminary step. The network will be dimensioned in order to satisfy the traffic requirements, although it will not be the optimum network.
- Calculation of the gradient vector of the economic function with respect to all the independent variables, i.e., the number of circuits in each one of the high usage routes. The main part of this paper is dedicated to finding the gradient calculation procedure, because the efficiency of the gradient algorithm depends directly on the simplicity or complexity of this calculation.
- The gradient vector will indicate the minimization direction for the independent variables of the problem. The distance to be moved in this direction will be determined by the Fibonacci algorithm.
- The process continues with the calculation of the gradient vector at the new point just obtained. The iterative procedure will continue until the minimum economic function is reached.

2.2. NETWORK WITH TWO ALTERNATIVES.

Fig. 4 depicts an elemental part of a two level hierarchical network with alternative routes in its easiest form, that is, with only two choices for carrying the traffic between two lower level switching points (direct route and final route).

The cost of this elemental part of the network can be expressed as:

\[ C = c_1N_1 + c_2N_2 + c_3N_3 \]

where \( c_i \) is the unitary cost of route \( i \), and

\[ N_2 = \max \left( N_{2t} \right) \quad \text{and} \quad N_3 = \max \left( N_{3t} \right) \]

The condition that the network satisfies the quality of service required on each of the final routes under the worst conditions is expressed by these last two equations.

The number of trunks \( N_{2t} \) in final route 2 that satisfies the required grade of service at moment \( t \) can be written

\[ N_{2t} = N_2 + t \quad (M_2, V_2, M_3) \]
where

\[ \begin{align*}
M_1^t &= m_r + m_1^t \\
V_1^t &= v_r + v_1^t \\
\end{align*} \]

\( M_1^t \) and \( V_1^t \) are the mean and variance of the offered traffic to route 2.

An analogous expression can be written for route 3. \( m_j, v_j, m_1, v_1 \) define the traffic offered by the "rest of the network" to routes 2 and 3 (mean and variance). Suppose this traffic is constant. The problem is to determine the number of trunks in route 1 that keeps to a minimum the total cost of the elemental part of the network. In order to solve the problem, the gradient algorithm is suggested, as mentioned above.

This iterative algorithm needs to know, in each iteration step, the components of the gradient of the objective function with respect to the independent variables (a more detailed description is given in Section 4).

The component of the gradient vector of the network cost function, relative to \( N_1 \), can be expressed as follows:

\[ \frac{dC}{dN_1} = c_1 \frac{dN_2}{dN_1} + c_3 \frac{dN_3}{dN_1} \]  

(2.1)

where

\[ c_2 = c_2 \frac{d}{dN_1} \left[ \max (N_2^t) \right] \]

and

\[ c_3 = c_3 \frac{d}{dN_1} \left[ \max (N_3^t) \right] \]

The calculation of these two terms can be made as follows:

\[ c_2 = c_2 \frac{d}{dN_1} \left[ \max (N_2^t) \right] = c_2 \frac{d}{dN_1} \left[ \max \left( N_2^t \right) \right] \]

(2.2a)

\[ \Phi_{\mathbf{M}_1} = [v_{M_1}^t, v_{M_1}^{t-1}, ..., v_{M_1}^1] \]

(2.2b)

\[ \Phi_{\mathbf{V}_1} = [v_{V_1}^t, v_{V_1}^{t-1}, ..., v_{V_1}^1] \]

\[ \frac{d}{dN_1} \left[ \max \left( N_2^t \right) \right] \]

\[ \frac{d}{dN_1} \left[ \max \left( N_3^t \right) \right] \]

\[ \max \left( N_2^t \right) \]

\[ \max \left( N_3^t \right) \]

\[ \text{For } t^* \text{ being the moment for which } N_2^t \text{ is maximum.} \]

\[ \max \left( N_3^t \right) \]

\[ \text{For } t^* \text{ being (not necessarily the same as the previous one) the moment for which } N_3^t \text{ is maximum.} \]

Let's define the following matrices:

\[ \Phi_{\mathbf{M}_1} = [v_{M_1}^t, v_{M_1}^{t-1}, ..., v_{M_1}^1] \]

(2.2a)

\[ \Phi_{\mathbf{V}_1} = [v_{V_1}^t, v_{V_1}^{t-1}, ..., v_{V_1}^1] \]

(2.2b)

of which the elements are:

\[ \frac{d}{dN_1} \left[ \max \left( N_1^t \right) \right] \]

(2.2b)

\[ \frac{d}{dN_1} \left[ \max \left( N_2^t \right) \right] \]

(2.2b)

\[ \frac{d}{dN_1} \left[ \max \left( N_3^t \right) \right] \]

(2.2b)

\[ \text{That is, they are the derivatives of the cost of the final route } i \text{ with respect to the mean and variance of the offered traffic at the moment } t. \]

The elements of matrices \( \Phi_{\mathbf{M}_1} \) and \( \Phi_{\mathbf{V}_1} \) are null except the one corresponding to the moment \( t^* \) in which \( N_1^t \) is maximum.

Another pair of matrices are defined as:

\[ \Phi_{\mathbf{M}_1} = \begin{bmatrix} m_{M_1}^t \\ m_{M_1}^{t-1} \\ \vdots \\ m_{M_1}^1 \end{bmatrix} \]

(2.3a)

\[ \Phi_{\mathbf{V}_1} = \begin{bmatrix} v_{V_1}^t \\ v_{V_1}^{t-1} \\ \vdots \\ v_{V_1}^1 \end{bmatrix} \]

(2.3b)

of which the elements are:

\[ \frac{d}{dN_1} \left[ \max \left( N_1^t \right) \right] \]

(2.3b)

\[ \frac{d}{dN_1} \left[ \max \left( N_2^t \right) \right] \]

(2.3b)

\[ \frac{d}{dN_1} \left[ \max \left( N_3^t \right) \right] \]

(2.3b)

\[ \text{that is, these elements are the derivatives of the mean and variance of the overflow traffic with respect to the number of circuits of route } j, \text{ the offered traffic to this route being constant.} \]

Now, equation (2.1) can be written

\[ \frac{dC}{dN_1} = c_1 + \frac{3}{\sum_{i=2}^N \left[ \Phi_{\mathbf{M}_1} \cdot \Phi_{\mathbf{V}_1} \cdot \Phi_{\mathbf{V}_1} \right]} \]

(2.4)

which is the expression of the gradient vector of the cost function relative to the number of circuits in route 1. Once the gradient vector is known, the iterative minimizing algorithm, as explained in section 4, will be applied to obtain the optimum number of circuits in the direct route that will produce the minimum cost function.

2.3. NETWORK WITH MORE THAN TWO ALTERNATIVES.

The problem faced here is analogous to the one studied in the previous section with the difference that now more than two alternatives exist to carry the traffic between exchanges I and J.

Fig. 5 represents an elemental part of a two level network. Its cost can be expressed:

\[ C = c_1 N_1 + c_2 N_2 + c_3 N_3 + c_4 N_4 + c_5 N_5 \]

Number of trunks in routes 1 and 2 are independent variables.

The components of the gradient vector of function C (or of the cost function of the whole network) relative to \( N_1 \) and \( N_2 \) can be written:

\[ \frac{dC}{dN_1} = c_1 + \frac{3}{\sum_{i=2}^N \left[ \Phi_{\mathbf{M}_1} \cdot \Phi_{\mathbf{V}_1} \cdot \Phi_{\mathbf{V}_1} \right]} \]

(2.5)

\[ \frac{dC}{dN_2} = c_2 + \frac{3}{\sum_{i=2}^N \left[ \Phi_{\mathbf{M}_1} \cdot \Phi_{\mathbf{V}_1} \cdot \Phi_{\mathbf{V}_1} \right]} \]

(2.6)
The computation of the derivatives \( \frac{\partial \mathcal{N}_1}{\partial \mathcal{N}_2} \) is made as follows:

\[
\frac{\partial \mathcal{N}_1}{\partial \mathcal{N}_2} = \frac{2}{\mathcal{N}_2} \left[ \max(N_1^t) \right] - \frac{2}{\mathcal{M}_1^t} \left[ \max(N_1^t) \right] + \frac{2}{\mathcal{V}_1^t} \left[ \max(N_1^t) \right]
\]

where \( t^* \) is the moment \( t \) for which \( N_1^t \) is maximum.

Equation (2.6), taking into account the definition of matrices \( \mathcal{M}_1, \mathcal{M}_2, \mathcal{V}_1 \), and \( \mathcal{V}_2 \), equations (2.5) and (2.6), can be written.

\[
\Delta \frac{2}{\mathcal{N}_2} \mathcal{C} = \frac{\Delta \mathcal{N}_1}{\mathcal{N}_1} + \frac{\Delta \mathcal{M}_1}{\mathcal{M}_1} + \frac{\Delta \mathcal{V}_1}{\mathcal{V}_1} + \frac{\Delta \mathcal{V}_2}{\mathcal{V}_2} + \frac{\Delta \mathcal{M}_2}{\mathcal{M}_2} + \frac{\Delta \mathcal{V}_3}{\mathcal{V}_3} + \frac{\Delta \mathcal{V}_4}{\mathcal{V}_4}
\]

Let's now consider the computation of \( \frac{\partial \mathcal{N}_1}{\partial \mathcal{N}_2} \).

The addition \( \mathcal{C}_1 \) analogously to the previous paragraph, can be written:

\[
\mathcal{C}_1 = \frac{\partial \mathcal{N}_1}{\partial \mathcal{N}_2} = \frac{\mathcal{C}_1}{\mathcal{N}_1} \left[ \max(N_1^t) \right] - \frac{\mathcal{C}_1}{\mathcal{M}_1^t} \left[ \max(N_1^t) \right] + \frac{\mathcal{C}_1}{\mathcal{V}_1^t} \left[ \max(N_1^t) \right]
\]

The computation of \( \frac{\partial \mathcal{N}_1}{\partial \mathcal{N}_2} \) is a little more complex but can be handled as follows:

\[
\frac{\partial \mathcal{N}_1}{\partial \mathcal{N}_2} = \frac{\mathcal{C}_1}{\mathcal{N}_1} \left[ \max(N_1^t) \right] - \frac{\mathcal{C}_1}{\mathcal{M}_1^t} \left[ \max(N_1^t) \right] + \frac{\mathcal{C}_1}{\mathcal{V}_1^t} \left[ \max(N_1^t) \right]
\]

\[
\frac{\partial \mathcal{N}_1}{\partial \mathcal{N}_2} = \frac{\mathcal{C}_1}{\mathcal{N}_1} \left[ \max(N_1^t) \right] - \frac{\mathcal{C}_1}{\mathcal{M}_1^t} \left[ \max(N_1^t) \right] + \frac{\mathcal{C}_1}{\mathcal{V}_1^t} \left[ \max(N_1^t) \right]
\]

Let's define the marginal mean and variance of the overflow traffic with respect to the mean and variance of the offered traffic in route 2:

\[
\begin{align*}
\mathcal{M}_2^t &= \mathcal{M}_2 \mathcal{M}_1^t \mathcal{V}_1^t \\
\mathcal{V}_2^t &= \mathcal{V}_2 \mathcal{M}_1^t \mathcal{V}_1^t
\end{align*}
\]

By substitution of this variable in the above equation:

\[
\frac{2}{\mathcal{N}_2} \mathcal{C} = \frac{\mathcal{C}_1}{\mathcal{N}_1} \left[ \mathcal{M}_2^t \mathcal{M}_1^t \mathcal{V}_1^t \right] - \frac{\mathcal{C}_1}{\mathcal{M}_1^t} \left[ \mathcal{M}_2^t \mathcal{M}_1^t \mathcal{V}_1^t \right] + \frac{\mathcal{C}_1}{\mathcal{V}_1^t} \left[ \mathcal{M}_2^t \mathcal{M}_1^t \mathcal{V}_1^t \right]
\]

Equation (2.5) becomes

\[
\Delta \frac{2}{\mathcal{N}_2} \mathcal{C} = \frac{2}{\mathcal{N}_1} \left[ \mathcal{M}_1^t \mathcal{M}_2^t \mathcal{V}_1^t \mathcal{V}_2^t \right] - \frac{2}{\mathcal{M}_1^t} \left[ \mathcal{M}_1^t \mathcal{M}_2^t \mathcal{V}_1^t \mathcal{V}_2^t \right] + \frac{2}{\mathcal{V}_1^t} \left[ \mathcal{M}_1^t \mathcal{M}_2^t \mathcal{V}_1^t \mathcal{V}_2^t \right]
\]

Note that \( t^* \) is in general different for \( i=4 \) and \( i=5 \).

And now by defining

\[
\begin{bmatrix}
\mathcal{M}_2^t \\
\mathcal{V}_2^t
\end{bmatrix} = \begin{bmatrix}
\mathcal{M}_1^t \\
\mathcal{V}_1^t
\end{bmatrix}
\]

we obtain

\[
\Delta \mathcal{C} = \frac{2}{\mathcal{N}_1} \left( \mathcal{C}_1 \mathcal{M}_1^t \mathcal{M}_2^t \mathcal{V}_1^t \mathcal{V}_2^t \right) - \frac{2}{\mathcal{M}_1^t} \left( \mathcal{C}_1 \mathcal{M}_1^t \mathcal{M}_2^t \mathcal{V}_1^t \mathcal{V}_2^t \right) + \frac{2}{\mathcal{V}_1^t} \left( \mathcal{C}_1 \mathcal{M}_1^t \mathcal{M}_2^t \mathcal{V}_1^t \mathcal{V}_2^t \right)
\]

It is interesting to see that \( \mathcal{M}_1^t \) and \( \mathcal{V}_1^t \) are the derivatives of the cost of the network, \( \mathcal{N}_2 \) being constant, with respect to the mean and variance of the offered traffic to route 2 at the moment \( t \).

Equations (2.7) and (2.9) have the same form.

In these equations two kinds of parameters are involved: parameters \( \mathcal{M}_1 \) and \( \mathcal{V}_1 \), that are characteristic of the routes over which the considered route directly overflows, and parameters \( \mathcal{M}_2 \) and \( \mathcal{V}_2 \), that are characteristic of the considered route. The fact that both equations have the same form is very important from the point of view of computerizing the model.

Parameters \( \mathcal{M}_1 \) and \( \mathcal{V}_1 \) are obtained in the high usage routes with the aid of equation (2.8), from the values of these parameters in the routes over which the considered route overflows.

3. GENERAL EQUATIONS.

Let's now obtain the general form of the components of the gradient vector of the cost function with respect to the number of circuits in any high usage route.

In a network with hierarchical structure, let's consider any route, for instance, route 1 in Fig. 6.

![Fig. 6 - General Case Network.](image)

Traffic offered to this route has as its destination those exchanges whose higher order exchange is B (exchanges D and D' in Fig. 6).

The component of the gradient vector relative to the number of trunks in route 1 is

\[
\Delta \mathcal{C} = \frac{2}{\mathcal{N}_1} \mathcal{C}_1 \mathcal{N}_1 + \mathcal{C}'
\]

where \( \mathcal{C} \) is the cost of the network. It can be expressed as follows:

\[
\mathcal{C} = \mathcal{C}_1 \mathcal{N}_1 + \mathcal{C}'
\]

\( \mathcal{C}_1 \mathcal{N}_1 \) being the cost of route 1 and \( \mathcal{C}' \) the cost of the remaining network. Cost \( \mathcal{C}' \) depends on \( \mathcal{N}_1 \) through the overflow traffic (mean and variance) at each moment \( t \), \( (m_1^t, V_1^t) \).
So, it can be written:

\[
\frac{2C'}{2N_i} = c_i + \frac{\partial C'}{2N_i} \frac{2m^t_i}{2N_i} + \frac{2C'}{2N_i} \frac{2v^t_i}{2N_i} \quad (3.2)
\]

\[
2C' = 2m^t_i + 2v^t_i + 2M^t_i + 2V^t_i - \frac{2M^t_i}{2m^t_i} - \frac{2V^t_i}{2v^t_i} \quad (3.3)
\]

where index \( i \) extends to all destination exchanges that receive traffic offered to the route under consideration. \( M^t_i \) and \( V^t_i \) are the mean and variance of the offered traffic to the route being considered whose destination is exchange \( i \). Introducing these matrices, the previous equation becomes:

\[
2C' = \sum_{j} \left[ \mu_j \phi_1 \left( \sum_{i} m^t_i \phi_{M_j} \right) + \phi_1 \left( \sum_{i} v^t_i \phi_{V_j} \right) \right] \quad (3.4)
\]

where index \( j \) extends, for each destination \( i \), to all the routes to which the overflow traffic, or the traffic carried, of the route under consideration is offered as a first choice. Parameter \( \mu_j \) is \(+1\) or \(-1\), depending on whether route \( j \) receives traffic overflowed or carried by the considered route.

Equation (3.4) gives the component of the gradient vector of the economic function with respect to any high usage route, as a function of parameters \( \phi_M \) and \( \phi_V \), corresponding to the routes to which the overflow or carried traffic by the route under consideration is offered as first choice. Computation of the elements of these matrices can be accomplished for any high usage route as follows:

\[
\begin{align*}
\phi_M &= \sum_{j} \mu_j \phi_{M_j} \sum_{i} m^t_i \phi_1 \left( \sum_{i} v^t_i \phi_{V_j} \right) \\
\phi_V &= \sum_{j} \mu_j \phi_{V_j} \sum_{i} v^t_i \phi_1 \left( \sum_{i} m^t_i \phi_{M_j} \right)
\end{align*}
\]

The terms \( \phi_M \) and \( \phi_V \) have been calculated in equation (3.3). The general expression of parameters \( \phi_M \) and \( \phi_V \), corresponding to route \( 1 \) can be expressed as follows, bearing in mind the two preceding equations:

\[
\phi_{M_1} = \sum_{j} \mu_j \phi_{M_j} \sum_{i} m^t_i \\
\phi_{V_1} = \sum_{j} \mu_j \phi_{V_j} \sum_{i} v^t_i
\]

This set of equations provides a recurrence procedure for the computation of matrices \( \phi_M \) and \( \phi_V \) for any high usage route, when the value of these parameters is known in the routes over which the considered route overflows or offers its carried traffic. Thus, the recurrent method of calculation for these parameters would be to start computing the values of \( \phi_M \) and \( \phi_V \) in the final routes, according to equations (2.2b).

Then, by using the above set of equations in a successive way, the computation of \( \phi_M \) and \( \phi_V \) in all the routes can be accomplished. Equation (3.4) gives the component of the gradient vector relative to the number of circuits in any high usage route, once values of \( \phi_M \) and \( \phi_V \) are known.

4. OPTIMIZATION ALGORITHM.

In the previous paragraphs the method of calculating the gradient vector has been studied in
order to apply the iterative gradient algorithm for the optimization process. The algorithm is defined by the equation:

\[ N^* = N - K \cdot \text{Grad } C \]  

(4.1)

where \( N \) is the vector whose components are the number of junctions in the high usage routes in any iteration and \( N^* \) is the resulting vector of the iterative step. \( K \) is a parameter that gives the extent of the iterative step and needs to be adjusted in each iteration. \( \text{Grad } C \) is the gradient vector that has been calculated in the previous paragraph.

Fig. 7 gives a geometrical representation of the algorithm when applied to a function of two variables.

Thus, the optimization algorithm would consist of the following steps:

1. Estimation of the number of circuits in all the routes of the network, and the offered and overflow traffic in each one of them, for every moment in time.

2. Calculation of parameters \( \phi_m \) and \( \phi_v \) in all the final routes, with the aid of equations (2.2b).

3. Calculation of parameters \( \phi_m \) and \( \phi_v \) in the high usage routes by making a "sweep" of the network from the final to the direct routes, by using reiteratively equations (3.5). The components of the gradient vector relative to the number of circuits' in each route will be made simultaneously with the aid of equation (3.4).

4. By using equation (4.1) a new number of circuits in each high usage route will be obtained. The adjustment of parameter \( K \) can be made by using the Fibonacci algorithm (Ref. 2).

5. To see if the result obtained is the same as the previous one. If it is, a stationary point of the economic function has been reached, and the algorithm process is stopped. If different, goes to point 2.

5. CONCLUSIONS.

The method described is especially suitable for a computer treatment. The reason is that equations defining the components of the gradient vector always have the same form, independently of the route. So, the minimization process is completely repetitive.

The method can be adopted to any routing discipline.

6. NOTATIONS

- \( M_i^t, V_i^t \): Mean and Variance of the offered traffic to route \( i \), at moment \( t \).
- \( m_i^t, v_i^t \): Mean and Variance of the overflow traffic from route \( i \), at moment \( t \).
- \( A_i^t \): Offered pure random traffic mean to route \( i \), at moment \( t \).
- \( c_i \): Cost per trunk in route \( i \).
- \( N_i \): Number of trunks in route \( i \).
- \( N_i^f \): Number of trunks in final route \( i \), that satisfies the grade of service conditions at moment \( t \).
- \( q_{ij} \): Lost traffic mean admitted on final route \( i \). (It is a function of the grade of service required).

7. REFERENCES

