ABSTRACT

A full availability primary group, operated on a lost calls cleared basis and holding times negative exponentially distributed, is offered calls, 1) according to a Poisson process, or 2) by Engset sources. Discrepancies and similarities of the overflow call arrival process for the two cases are discussed in chapter 2.

In chapter 3 the problem of finding practically suitable approximations for the process is treated, and numerical examples are given.

Chapter 5 considers how the arrival process can be utilized to establish equivalent theories for overflow systems, using a particular representation for the overflow call process derived in Chapter 4.

The mathematical methods applied are outlined in Appendix 1 - 3.

1. INTRODUCTION

In telephone systems overflow arrangements are used to a very great extent, and much effort has been spent on the practical and theoretical investigations of such arrangements.

The theoretical investigations have apparently followed two different patterns, namely:

1. Study of systems in steady state, obtaining discrete distributions for the number of overflow calls permitted to exist due to certain criteriae. For references see Wilkinson (1), Chastang (6) and Burke (5).

2. Study of the time dependent overflow arrival process, originated by Palm (9), continued by Ekberg (8) and to some extent Benes (4) and (5). However, the intention has been drawn mainly to the case of the call input process being independent of the number of calls in service.

During the last decennium or so, it seems to the author that the main theoretical interest has been in favour of 1, which of course is due to the immediate success of the Wilkinson theory.

This paper, however, mainly follows pattern 2.

2. OVERFLOW CALL ARRIVAL PROCESS FROM FULL AVAILABILITY GROUP WITH ENGSET TRAFFIC SOURCES.

2.1 GENERAL PROBLEM.

As an Engset source can only make call attempts when it is free, the arrival process for the calls overflowing from a primary group (in the calculations supposed to possess n devices and N sources, each with calling intensity $\lambda$ when being free. Holding times negative exponentially distributed with mean value 1), will be dependent on the momentary number of calls being served. However, as this number will depend also on the arrival process for calls offered from other parts of the system, this means that one should have to consider the total part of the system which might interact directly or indirectly with the part of the system to which the calls offered to the specific primary group have access. This in practice means that an exact identification of the overflow arrival process in case of discrete sources, will be impossible, even for rather simple systems. This is a significant contrast to the case of Poisson offer, where the process is dependent only on the primary group. A further discrepancy is that in the Engset case the interarrival times for the overflow calls will not be independently distributed. In the following a simplified system with Engset sources, which allows an exact mathematical treatment, is to be studied.

2.2. SPECIAL CASE.

Assume that the secondary group consists of m devices, and that the interaction with other
parts of the system can be neglected. Then the overflow arrival process from the primary group will be exactly the same as the process for call arrivals to device no n+1 in a lost call cleared system with n+m devices and hunting in order, with home position at device no 1. Assume that the state distribution at time t=0 is known. Then, following Blordian (10), in order to register the appearance of the next overflow call, a special kind of states are introduced, into which the system will enter and remain (absorbing states). Until an absorbing state has been reached, this modified system will behave according to exactly the same stochastic law as the original one. Letting [p, q] (t) denote the probability at time t, that the state in the primary group will be p and in the secondary group q, one obtains the following state equations where (n+1, q) are the absorbing states.

\[
\frac{d}{dt}[p, q](t) + (L - L_{(n+1)})([(n+1 - q)p + q + 1])p, q(t) = 0.
\]

If \( p \not\in \{0, \ldots, n\} \) or \( q \not\in \{0, \ldots, m\} \), then \([p, q](t) = 0\).

The symbol \( \delta_{i,j} \) means the Kronecker delta.

If one should wish to determine the distribution function \( B_g(t) \) for the time between a randomly chosen overflow call and the next one, the following initial conditions should be applied:

\[
\text{If } m = 0, \quad [p, 0](0) = \delta_{p,0}.
\]

\[
\text{If } m > 0, \quad [p, q](0) = \begin{cases} 0 & \text{if } p < m \text{ or } q = 0, \\ [q, q] & \text{if } p = m \text{ and } q, \ldots, m-1, \\ [q, q + 1] & \text{if } p = m \text{ and } q = m. 
\end{cases}
\]

where \([q]\) denotes the state distribution met by the secondary group by randomly chosen calls overflowing the primary group, under the assumption of no absorbing states, and one obtains

\[
B_g(t) = \sum_{i=0}^{n} [n+1, q](t).
\]

Outline of the mathematical treatment of (1) is given in Appendix I. As only some properties of the Laplace transform of \( B_g(t) \), denoted \( \mathcal{L}[B_g] \), shall be studied here, neither \( B_g(t) \) nor \( \mathcal{L}[B_g] \) will be deduced explicitly, except for \( \mathcal{L}[B_g] \) in a special case.

From (A2), (A5), (A6), (A8) and (A9) in Appendix I, it is seen that \( \mathcal{L}[B_g] \) is a rational function of the complex variable s, the denominator being expressible as \( s \) multiplied by a product of polynomials \( \sigma_{n+1}(n+1, N) \) of degree \( n+1 \), which satisfy the following relation (A10):

\[
\sigma_s(i, n) = \frac{1}{s} \left( \frac{2^s}{(0)^s} \left( (i+s)^N \right)^{\frac{1}{s+2}} \right)_{x=0}.
\]

This polynomial is related to the Nyquist polynomial

\[
\sigma_s(i) = \frac{1}{s} \left( \frac{2^s}{(0)^s} \left( e^{\Lambda} (1-x)^{s-1} \right) \right)_{x=0},
\]

which played an important role in the development of the Wilkinson random theory. In fact, if \( N \to \infty \) and \( s \to 0 \) in such a way that \( N^2 \to \lambda \), then \( \sigma_s(n, N) \to \sigma_s(n) \). Palm found the zeroes of \( \sigma_s(n) \) to be distinct and in case \( A > 0 \) all negative and with intermediate distances larger than 1. In Appendix 2 is shown that if \( n \not\in N \) and \( A > 0 \), then the similar statements hold, but the intermediate distances will be larger than 1+s.

In the case of Poisson offer, the intermediate overflow times are independently distributed, and Palm (11) obtained

\[
\mathcal{L}[\sigma_s] = \frac{1}{s+1} \sigma_s(n, n+1) \mathcal{L}[\sigma_s(n, n+1)],
\]

For the system specified by (1), the similar result will be obtained by putting \( m = 0 \), \( [n, 0] (0) = 1 \), i.e.

\[
\mathcal{L}[\sigma_s] = \frac{1}{s+1} \sigma_s(n, n+1) \mathcal{L}[\sigma_s(n, n+1)],
\]

The polynomials \( \sigma_s \) are related to the special mathematical functions. For the Nyquist polynomials, Benes (4) states the following relation to the Poisson Charlier polynomial:

\[
\sigma_s(n) = (-1)^n \sqrt{n!} \Gamma_n (-s, A).
\]

As shown in Appendix 2, \( \sigma_s(n) \) is also expressible by the more commonly tabulated Laguerre function, in fact:

\[
\sigma_s(n) = (-1)^n \sqrt{n!} \Gamma_n (-s, A).
\]

Correspondingly, \( \sigma_s(n, N) \) is expressible by the Jacobi polynomial of 1. kind by

\[
\sigma_s(n, N) = (-1)^n \Gamma_n (-s-n, -n-1).
\]

As in principle the interarrival processes considered have been characterized by well tabulated mathematical functions, the exact treatment will be left, in favour of practical applications.

3. APPROXIMATE INTEROVERFLOW DISTRIBUTION FUNCTION IF OFFERED POISSON TRAFFIC.

From (6) and the properties of \( \sigma_s(n) \), it is clear that \( B(t) \) will possess \( n+1 \) exponential terms of form \( \exp (\lambda t) \) where \( \lambda \) is a zero of \( \sigma_s(n+1) \). For \( n > 3 \) these zeroes can generally not be determined by exact methods. However, a knowledge of \( B(t) \) or a suitable approximation would be applicable for instance in simulation, where under certain conditions the simulation of primary groups could be avoided, and the overflow traffic generated directly.

Since \( \mathcal{L}[B] \) determines \( B(t) \) uniquely, it can be used as a basis for making approximations for \( B(t) \). However, the inverse Laplace transform is not generally stable, which means that the inverse transformation of a good approximation of \( \mathcal{L}[B] \) does not have to be a good approximation of \( B(t) \). For various cases so called Tauberian theorems assigning this quality, can be established. For such theorems is referred to (7).
Approximations for \( \{B\} \) can be divided into two kinds, local or global, according to whether the approximation is designed to fit well in distinct points, or in the total range of definition. In the latter case, Tauberian theorems are very difficult to establish. Here only local approximations of \( \{B\} \) will be considered, and the points \( s=0 \) and \( s=\infty \) are chosen, because in these points the Tauberian results are most easy to obtain. In our case it is evident that a good approximation of \( \{B\} \) at \( s=\infty \) will give a good approximation for \( B(t) \) at \( t=0 \), and respectively \( s=0 \) and \( t=\infty \). Since \( B(t) \) is a sum of exponentials, its approximations will also be chosen to have this form, but generally a less number of terms, in the treated case two. This means
\[
\sigma(t) = \left[ 1 - \sigma \right] e^{-\sigma t} - \sigma \}
\]
with the requirement that \( \{B^n\} \) has a Laurent development at \( s=0 \) and \( \infty \) which are approximations to \( \{B\} \). To assure \( \{B^n\} \) to be a distribution function, assume \( \beta_1, \beta_2, \ldots, \sigma, \sigma_1, \sigma_2, \ldots \). Hence will be developed two approximations \( \{B^n\} \) and \( \{B^n\} \). \( \{B^n\} \) will have the first coefficients at \( s=0 \) and the first three coefficients at \( s=\infty \) identical with the respective ones for \( \{B\} \). \( \{B^n\} \) will have two coefficients common with \( \{B\} \) at each of the points \( s=0 \) and \( s=\infty \).

The Laurent developments at these points are
\[
\{B^n\} = \left[ 1 - \left( \frac{n}{n} + \frac{s}{n} \right) + \left( \frac{s}{n} + \frac{s}{n} \right) \right] s = \cdots \quad \text{(11)}
\]
\[
\{B^n\} = \left[ \frac{n}{n} + \frac{s}{n} + \frac{s}{n} \right] s = \cdots \quad \text{(12)}
\]
\[
\{B^n\} = \left[ 1 - \left( \frac{n}{n} \right) \right] s^2 = \cdots \quad \text{(13)}
\]
\[
\{B^n\} = \left[ 1 - \left( \frac{n}{n} \right) \right] s^3 = \cdots \quad \text{(14)}
\]
Comparison of these expressions with \( \{A^n\} \) and \( \{A^n\} \) gives:
\[
\alpha_2 = \frac{1}{2} \left( 1 + \frac{n}{n+1} \right) \quad \text{(15)}
\]
\[
\beta_2 = \frac{1}{2} \left( n + 2A \right) \left( n + 1 \right) \quad \text{(16)}
\]
Both \( B^n \) and \( B^n \) will be asymptotically equal to \( B(t) \) at \( t=0 \) and \( t=\infty \), \( B^n \) yielding the closest approximation at \( t=0 \), \( B^n \) at \( t=\infty \). For \( n=0 \) and \( n=1 \) of course these distributions functions are exact. In diagram 1 and 2 \( B^n \) and \( B^n \) can be compared with the "true" distribution function \( B(t) \), obtained by simulation, and with a negative exponential distribution function with the same mean value (i.e. corresponding Poisson traffic). A=24 erlang in both diagrams, with \( n=20 \) in the first, \( n=10 \) in the latter. As might be expected the best agreement with \( B(t) \) is observed for the case with higher congestion and the more random overflow traffic. It is also seen that the application of \( B^n \) instead of \( B^n \) will give a higher congestion in the secondary group, the opposite being the case for \( B^n \).

4. TIME DISTRIBUTION FUNCTION BETWEEN ARBITRARY INSTANT AND THE NEXT OVERFLOW CALL.

Consider the system specified in 2.2, but with \( m=0 \), i.e. no secondary group. Assume that observation for overflow calls is started at an instant, say \( t=0 \), when statistical equilibrium can be assumed. The time distribution function \( F(t) \) for the first registration is sought. Of course, this can be found by solving equation (1) with initial conditions being the Engset stationary state distribution in the primary group.

In doing this, one would obtain:
\[
\{F\} = \left[ (n-n) \right] T_n(\beta, N) \quad \text{(17)}
\]
\[
\{F\} = \left[ (n-n) \right] T_n(\beta, N) \quad \text{(18)}
\]
where \( T_n(\beta, N) \) is the stationary state probability for all \( n \) devices busy. By using easily obtainable relations for \( \sigma \), it can be shown that
\[
\{F\} = \left[ (n-n) \right] T_n(\beta, N) \quad \text{(19)}
\]
However, this simple expression suggests more straightforward approaches to be likely, and one such will be demonstrated:

Let \( \tau \). Define the following probability for infinitesimal \( \tau \):
\[
F(\tau) = \text{Prob} \{ \text{an overflow call arrives during time interval } (-\tau, -\tau) \} \quad \text{(20)}
\]
This implies
\[
F(\tau) = \left[ \frac{n}{n} \right] T_n(\beta, N) \quad \text{(21)}
\]
The Poisson case is trivially obtained by
\[ \left\{ F \right\} = A E_{m,n}(A) \frac{G(n)}{G(n+1)} , \quad (20) \]
\[ F(t) = A E_{m,n}(A) \int_0^t (1 - G(t)) \, dt , \quad (21) \]

Remark that in this derivation of (17) and (21) no assumptions concerning the holding time distribution have to be made.

The Maclaurin coefficients \( b_k \) of \( F \) are given in terms of the respective parameters \( \nu \) of \( F \) by
\[ b_k = \left\{ \left( \frac{n}{m} \right) \right\} t \frac{1}{k!} , \quad (22) \]
and the moments are related by
\[ M_k = (n-m) \frac{T_n(a,n)}{T_{n-1}(a,n-1)} , \quad (23) \]
The modifications for the Poisson case are obvious.

5. EQUIVALENT THEORIES BASED ON \( F(t) \).

Assume that primary groups are offered Poisson traffic and possess a common secondary group. The purpose of the equivalent theories is then to substitute the \( g \) primary groups by some full availability group offered a certain Poisson traffic, in such a way that, due to some fixed criteria, the overflow traffic from the two systems can be considered equivalent. The study of such theories will here be based on the fact that the complement of \( F(t) \) is multiplicative over the primary groups. To see this, assume statistical equilibrium at \( t=0 \). Then the state distribution in each primary group will be Erlang. Denoting the time distribution function until the next arrival of any overflow call from the total arrangement of \( g \) primary groups by \( F(t) \), one obtains
\[ F(t) = \sum_{k=1}^{\infty} \frac{n}{k!} \int_0^t (1 - G(t)) \, dt , \quad (24) \]
where \( F_k(t) \) denotes the \( F \) distribution function for group \( k \). By means of this relation, the overflow arrival process is totally identified, because to \( F(t) \) there exists a unique distribution function \( B(t) \) for the interarrival times of the overflow calls. In full analogy with the previous chapter one namely obtains
\[ F(t) = \frac{m}{n} \int_0^t (1 - G(t)) \, dt , \quad (25) \]
where \( m \) is the mean value of the traffic overflowing the \( g \) groups; the uniqueness of \( B(t) \) being assured by this relation. From a perfect equivalent theory it would now be reasonable to require that the equivalent group should reproduce \( F(t) \) exactly. Except for the trivial case \( g=1 \), however, this will not be possible, what can be demonstrated by putting the Maclaurian development of \( F_k(t) \) into right side of (24) and observing the first three coefficients. Not even if the equivalent group, theoretically, is allowed to possess a noninteger number of devices, will it be able to reproduce the actual overflow arrival process exactly. This means that only partial properties of the overflow traffic (or equivalently \( F(t) \)) should constitute criteria for the construction of the equivalent group. Let these properties be characterized by approximations \( F_k(t) \) of \( F(t) \), in such a way that each set of properties imply a certain set of approximations \( A \), which preferably should satisfy:

a) Each \( \text{E} \) defines a full availability group and its offered Poisson traffic uniquely.

b) If \( \text{E}_1 \) and \( \text{E}_2 \), then \( /-\text{E}_1 \), \( \text{E}_2 \), the latter statement merely requiring that the complements of the approximations, too, should be multiplicative over the primary groups.

In the following it will be outlined how this technique might be used to deduce various equivalent theories.

1. Berkeley’s equivalent theory.

Might be considered as a direct extension of the Berkley theory, because the Maclaurian series in \( A \) are restricted to have negative first coefficient with absolute value not less than the square of the first coefficient. Then the requirement about the equality of the first two Maclaurin coefficients for \( F \) and \( F_k(t) \) is made, where by both a) and b) will hold.

2. Ekberg’s equivalent theory.

Might be considered as a direct extension of the Ekberg theory, because the Maclaurian series in \( A \) are restricted to have negative first coefficient with absolute value not less than the square of the first coefficient. Then the requirement about the equality of the first two Maclaurin coefficients for \( F \) and \( F_k(t) \) is made, where by both a) and b) will hold.

3. Wilkinson’s equivalent theory.

As the mean value \( M \) and variance \( V \) of the overflow traffic are additive over the primary groups, then because of b) the set should contain approximations \( F_{M,V}(t) \) which satisfy
\[ \frac{F_{E_k}}{F_k} = \frac{F_{E} + (1 - F_{E_k})}{F_k} \]
To obtain this, one might apply approximations of the form
\[ F_{M,V} = \int_0^t \frac{G(t)}{H(t)} \, dt , \quad (27) \]
where \( G(t) \) and \( H(t) \) are independent of the size of the full availability group and its offered Poisson traffic. Since \( M \) and \( V \) define the equivalent group and its offered traffic uniquely, a) is also fulfilled.

Equivalent theories different from the ones mentioned here might also be constructed, using a) and b). However, this will not be the scope of this paper.

Appendix 1. Treatment of the state equation (1).

Introduce the Laplace transform
\[ \left\{ \{ r \}, \{ s \} \right\} = Q_{r,s}(s) , \quad (A1) \]
then
\[ \left\{ \{ Q \} \right\} = \sum_{\{ q \}} Q_{r,s}(s) , \quad (A2) \]
The solution of this system could be obtained in the following way:

1) From (A3) calculate \( Q_{p,q}(s) \) for \( p=0,1,\ldots,n; q=0,1,\ldots,m \) in terms of \( Q_{0,k}(s) \). This can be done using ordinary generating function methods. The result will be

\[
Q_{p,q}(s) = \sum_{k=s}^{n+p} \left( \sum_{h=0}^{p+q-k} \binom{p+q-k}{h} Q_{p+q-k,h}(s) \right),
\]

(A6)

where \( \mathcal{A} \) is the minimum operator and

\[
\mathcal{G}_s^j(j,N) = \left( \sum_{k=s}^{n+m-j} \left( \sum_{h=0}^{p+q-k} \binom{p+q-k}{h} Q_{p+q-k,h}(s) \right) \right),
\]

(A7)

which is a polynomial in \( s \) of degree \( j \).

2) The unknown functions \( Q_{p,q}(s) \) can be found by substituting (A6) into (A4) and use of the relation

\[
Q_{p,q}(s) = \left( \sum_{h=0}^{p+q-k} \binom{p+q-k}{h} Q_{p+q-k,h}(s) \right) + \left( \sum_{h=0}^{p+q-k} \binom{p+q-k}{h} Q_{p+q-k,h}(s) \right),
\]

(A8)

This expression recursively determines \( Q_{p,q}(s) \) starting with

\[
Q_{0,m}(s) = \left( \sum_{h=0}^{p+q-k} \binom{p+q-k}{h} Q_{p+q-k,h}(s) \right),
\]

(A9)

The unknown functions \( Q_{p,q}(s) \) for \( p=0,1,\ldots,n; q=0,1,\ldots,m \) will thus be rational functions of \( s \), with the denominator expressible by a product of polynomials \( \sigma_{s,j}(n+1,N-j) \).

3) \( Q_{n+1,q}(s) \) can now be found by means of (A5).

Appendix 2. Properties of \( \mathcal{G}_s^j(j,N) \).

1. Zeroes of \( \mathcal{G}_s^j(j,N) \) (or \( J \)).

These will be shown to be distinct and negative with intermediate distances greater than \( 1+\beta \) (in case \( \beta>0 \)). Taking into account that \( \mathcal{G}_s^j(1,N)=s+N \), this will follow from the following theorem.

Theorem.

Let \( s_k,j,N \) be the zeroes of \( \mathcal{G}_s^j(j,N) \), ordered with respect to decreasing values. Then for \( 0<j \leq N \):

\[
-N(1+\beta) < s_{k+1,j,N} - s_{k,j,N-1} - \beta \quad (1 \leq k < j-1), \quad \text{(A10)}
\]

Moreover

\[
s_{k,j,N} < s_{k+1,j,N+1} \quad (1 \leq k < j-1), \quad \text{(A11)}
\]

with the stated equalities only in the trivial case \( \beta=0 \).

Proof. If \( \beta=0 \), then

\[
\sigma_0^j(j,N) = \frac{\sigma_0^j(1,N)}{j!} = \frac{1}{j!}, \quad \text{(A12)}
\]

and the theorem is fulfilled.

Now let \( \beta>0 \). It is obvious that (A11) implies (A10). What needs to be proved, is that (A10) implies (A11), because then the theorem follows by induction. The starting point will be the relation

\[
(j+1)(j-1) \sigma_0^j(j+1,N+j) = \sigma_0^j(j,N+1) + j \cdot \sigma_0^{j+1}(j,N+1), \quad \text{(A13)}
\]

Taking into account that

\[
\sigma_0^j(j,N) = \left[ \frac{\beta^j}{j!} \right] > 0, \quad \text{(A14)}
\]

(A10) implies

\[
(-1)^k \sigma_0^j(j+1,N+j) < 0, \quad \text{(A15)}
\]

\[
(-1)^k \sigma_0^j(j+1,N+j) > 0, \quad \text{(A16)}
\]

\[
(-1)^k \sigma_0^j(j+1,N+j) > 0, \quad \text{(A17)}
\]

and (A11) follows.

Since (A10) is valid for \( j=2 \), the theorem is proved.

A corollary of this theorem will be that neither in (7) nor (16) do the numerator and the denominator have common zeroes.

2. Relations between \( \sigma_0^j(j,N) \) and the special mathematical functions.

The nomenclature used here corresponds to (1), where further references on the subject, both for theory and numerical tables might be found. A hypergeometric function can be represented by its Gauss series

\[
\mathcal{F}(a,b;\zeta) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \zeta^n, \quad \text{(A18)}
\]

where

\[
(x-\beta) = \sum_{j=0}^{n} \frac{(x+j)}{(x+j)} \quad \text{for } x-1 \alpha N. \quad \text{(A19)}
\]

Using the rule for differentiation of a product of functions, one obtains
\[ 1. \text{Laurent development at } s=s_0. \]

\[ I(s) = \frac{A C(s_0)}{(m+1)C(s_0)} = \sum_{k=1}^{\infty} \frac{a_k}{s^k}. \]  

The coefficients \( a_k \) might be found recursively from

\[ a_k = \lim_{s \to s_0} s^k \left( \frac{A C(s_0)}{(m+1)C(s_0)} - \frac{1}{k} \right). \]  

Using (A28) one obtains

\[ (n+1) \sum_{j=1}^{\infty} \frac{a_j}{s^{j+1}}, n_1, n_2 = A C(s_0)^{-1} s_0^{n+1}, \]  

where \( v \) denotes the maximum operator.

Taking into account that

\[ \delta_n^{(n)} = \frac{n}{2} \frac{1}{4^v} \]  

it can be shown that

\[ \alpha_2 = A, \alpha_3 = -A(n+1), \alpha_4 = A(n+1)^2. \]  

From the Laurent development of \( I(s) \) at \( s=\infty \)

could easily be derived the Maclaurin development of \( B(t) \) at \( t=0 \). It is given by

\[ B(t) = \sum_{k=0}^{\infty} \frac{a_{k+1}}{k!} t^k. \]  

2. Laurent development at \( s=0. \)

\[ I(s) = \frac{A C(s_0)}{(m+1)C(s_0)} = \frac{b_0}{s} + \sum_{k=1}^{\infty} b_k s^k, \]  

\[ b_k = \lim_{s \to 0} s^{-k} \left( \frac{A C(s_0)}{(m+1)C(s_0)} - \frac{k!}{k} \right). \]  

which gives

\[ (n+1) \sum_{j=1}^{\infty} b_j \frac{a_j}{s^{j+1}}, n_1, n_2 = A C(s_0)^{-1} s_0^{n+1}. \]  

Using that \( a_0 = 0, a_1 = (n+1)! \), the three first coefficients are easily found to be

\[ b_1 = 1, b_0 = -\frac{i}{AE_C(A)}, b_1 = \frac{n!}{A_{\text{max}}} \sum_{j=0}^{\infty} \frac{A_{\text{max}}}{E_C(A)}. \]  

Let \( \mu_j \) be the \( j \) th moment of \( B(t) \) about zero. Then

\[ \mu_j = (-i)^j j! b_{j+1}, \]  

which together with (A39) might be used to evaluate the moments.

Denote the mean and variance of the interarrival times by \( \mu \) and \( v \). Then obtain

\[ \mu = \mu, = \frac{1}{AE_C(A)}. \]  

\[ \text{Appendix 3. Properties of } I(s). \]

These properties can be investigated through the Laurent series at the corresponding points. For theory is referred to (2).

The following expression for \( C_s(n) \) will be useful:

\[ C_s(n) = \sum_{m=0}^{n} C_m, n S^{(m)} \]  

where

\[ C_m, n = \frac{1}{m!} \sum_{j=m}^{n} \left( \frac{1}{j!} \right) \frac{A^n}{m^n} S_j^{(m)} \]  

\( S^{(m)} \) denote the Stirling numbers of the 1. kind and could be defined by

\[ \frac{1}{j!} (x-m) = \sum_{m=0}^{\infty} S_j^{(m)} \frac{x^m}{m!}. \]
In particular

\[ \eta_n(A) = \frac{V}{m_1} = 1 + \frac{1}{A} \sum_{j=0}^{n-1} \frac{E_{n-j}(A)}{E_n(A)} \]  

(A44)

\[ \lim_{A \to \infty} \eta_n(A) = 1 \]  

(A45)

which is reasonable, since for \( n=0 \), \( B(t) \) is, and when \( A \to \infty \), \( B(t) \) tends towards a negative exponential distribution function.

However, one may also show that

\[ \lim_{A \to 0} \eta_n(A) = 1 \]  

(A46)

and that \( \eta_n(A) \) is bounded for every \( n \).

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