DESIGN OF EFFICIENT LINK SYSTEMS WITH CONCENTRATION STAGES

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ABSTRACT

Results of earlier studies of efficient link system design are extended to include the effects of line concentration. A numerical method is given that permits a design engineer to obtain near-optimal switch sizes and concentration ratios, and to determine cost sensitivity to changes in these parameters. Cost profiles are presented for systems with two stage concentration, and there is evidence from these results that the choice of two concentration stages is economical for a broad range of applications.

1. INTRODUCTION AND REVIEW

This paper leans heavily on the results of two prior studies (Refs. 1, 2), both of which indicated progress in solving the problem of choosing basic parameters for a link system design. We demonstrate here that practical, efficient designs for switching networks in which terminal-to-link concentration is required can be achieved by concentration in the first two switching stages alone. In addition, a method is provided for selecting the switch sizes that will guarantee a minimum cost overall design.

Notation in the two prior works is largely consistent, and we will try to retain this uniformity except where new definitions are needed. Note 1 contains a complete list of symbols used and their meanings. The most basic of these are switch access size (k) in outlet terminals, number of stages (s), internal traffic load or "occupancy" per link (a). These are the parameters to be determined from assumed requirements for network size (N), level of first-trial matching loss or "point loss" (B), and from knowledge of a linkage-control price factor (n) which depends directly on the selected type of switching hardware and method of control access and operation.

In Ref. 1 the "SPD" (symmetrical pure distribution) network concept is defined and its cost (C) in crosspoints per erlang is expressed:

\[ C = \frac{a(k+n)}{s} \] (1)

In addition there is an access factor (A) which relates well, though not perfectly, to the required point loss B and may be determined from B numerically by charts (Figs. 3, 4 of Ref. 1).

\[ A = \frac{(k(1-a))^n}{n} \] (2)

For this somewhat idealized but basic link system form, we obtained a simple set of equations that minimize C, at least within the level of accuracy of the A vs. B approximate function.

\[ \ln[k(1-a)] = \frac{kn}{k} = \frac{a}{1-a} \] (3)

with

\[ s = \frac{\ln(N)}{\ln[k(1-a)]} \] (4)

These formulae imply that the choice of switch size and internal level of "operating load" should be made independent of both the network size and congestion objective; these should depend only on the linkage-control factor n. For the SPD form, the best way to grow (i.e., to increase N) or to lower the blocking level (to increase A) is to increase the number of switching stages.

Figures 1 and 2 are reproduced from Ref. 1, and give the solutions of equations (3,4) in graphical form, since we will refer to these often in the present work. Other charts in Ref. 1 show the sensitivity of C to actual commonly used growth strategies (increasing k, decreasing a) vs. the "optimal" plan of increasing s. These are not strictly relevant to the current study, but the reader may find them interesting reference material.

The key results of Ref. 1 are practical for some systems, since the range of "operating levels" of link load (.53 ≤ a ≤ .64 for 1 ≤ n ≤ 12) coincides well with the
expected levels of terminal load that would be seen during busy periods in a tandem or toll switching machine, with one terminal per trunk. (There is a simple relation between single-sided and two-sided expected levels of terminal load that would be seen during the present work.)

But what about basic line (subscriber, station) concentration and distribution systems, which are by a wide margin the "bulk of the market" for link systems? Assuming the two-sided network model, we will have terminal loads for practical systems in the range .05 to .20 erlang, and these differ from an efficient level of internal load for an SPD network (say, .6 erlang) by factors ranging from 3:1 to 12:1. The outlet terminal or trunk load can also be expected to be approximately .6 erlang, and the problem we now address is that of producing an efficient design for this inherently asymmetrical type of link structure.

2. NETWORKS WITH SINGLE STAGE CONCENTRATION: A NUMERICAL EXAMPLE

Figures 3 through 6 show analytically derived load-service curves (link occupancy vs. first-trial matching loss) for four-stage SPD type structures with k = 4, 8, 12 and 16 respectively. Each of these figures shows the "basic" SPD network loss function (no concentration or expansion) as well as a family of curves for 2:1, 4:1 and 8:1 concentration in the first switching stage, with $k_1 = k$ in each case. The generic formula used to produce these curves is derived from the basic "conditional probability" model of B. C. Molina (Ref. 3), also termed the "method of multiplication and summation of probabilities" (Ref. 4). All of the structures are 'full-spiderweb' type, with $N = k^3$ outlet terminals and $k^2$ paths in the linking or channel pattern; i.e., for the SPD networks:

$$A = \frac{(k^3)^5}{k} = k^5 \frac{5}{3}$$

Note 2 gives the channel pattern and blocking formula used for Figures 3 through 6, and remarks on its derivation. This approach has shown generally satisfactory agreement with both simulation results and observed load-service results for operating link systems. We have not, however, given up the search for better models of link system load vs. service performance, since the "conditional probability" approach (along with the others described in Ref. 4) is not uniformly good for all types of linking structures.

A numerical example will illustrate the relative magnitude of the costs encountered in SPD networks vs. those in networks with terminal concentration. Figure 5 shows the load-service curve for an SPD network that is optimal at $(B = .023, a = .608)$ for $n = 6.6, k = 12, N = (12) = 1728$ terminals per side. This structure has a cost, by equation (1), of

$$C = \frac{5(12+66.6)}{608} = 153 \text{ XPE (crosspoints per erlang)}$$

For direct comparison, consider the curve for $i_1 = 48$ in Figure 5. The network has 4:1 concentration and will operate at an efficiency $a = .524$ (for $B = .023$), with terminal load $a_1 = .131$ erlang. As a first estimate of a cost relative to that of the SPD network we have:

$$C = \frac{1}{n^2} \left( (a-1)(k+n) \right) = \frac{1}{a} + (a-1)(k+n)$$

This estimate is inaccurate for several reasons. The first is that the SPD structure is not precisely the same size as the 4:1 concentration. The SPD network switches only $(1728)(.608) = 1050$ erlangs, whereas the 4:1 network switches only $(1728)(.524) = 900$ erlangs. As shown in Ref. 1, however, this effect is small; it would be accounted for in an optimal manner by increasing the required number of stages only slightly - in this case by about 16 stages. (This factor is computed by hypothesizing that an "equivalent A" exists for the system with 4:1 concentration, for comparison with an SPD structure at the same size and same congestion level. We then have:

$$s^* = \frac{\ln(N_k/q)}{\ln(10)(5/24)}$$

The second effect is also small, and works in the opposite direction cost-wise. The 4:1 network will have a small amount of expansion to terminal-link expansion, viewed from the outlet side, in relation to the SPD network. The effect of this is that the cost of the last switching stage would be reduced from $\frac{km}{n} \ln(5/24)$, a reduction of $\frac{138}{274}$ km.

The combined effect of these two factors is to reduce the initial estimate of $C^*$ slightly, in this example. The new estimate becomes:

$$C^* = 246 + (1.085-.138)(246.6)(.524) = 244 \text{ XPE}$$

A third effect not accounted for in our estimate of $C^*$ is significant, and never negligible in systems with concentration. The broken line curve in Figure 5 just below the $i_1 = 48$ curve represents blocking probability of the concentrator stage alone, which we label $B_c$. Because the structures shown in Figure 5 are all efficient designs, the "blocking contribution" $B_c$ of the first stage is closely related to the cost of that stage. Roughly, $B = B_c + (1-B_c)B_r = B_c + B_r$, for service levels in the range of interest, and at $B = .023$ for the 4:1 network we have $B_c = .016, B_r = .007$. Since the concentrator stage(s) are critical in accessing common circuit groups (e.g., originating registers that accept and monitor a customer's initial service request), we normally require a $B_r$ much lower than the less restrictive "point blocking" objective B. Normally, $B < B_c < .005$ constitutes a reasonable range of service objectives for average busy hour traffic loads, with the actual level chosen depending on overload characteristics of the particular concentrator and network. In Figure 5, selection of $B_r = .003$ results in a marked reduction in the structure much further - all the way to $a = .425$ (actually, to $a = .434$ and 3.3 to 1 concentration, by interpolation between the 2:1 and 4:1 $B_c$ curves) instead of the $a = .524$ obtained directly for $B = .023$.

Note that at $(B_c = .003, a = .434)$ in Figure 5, for 3.3 to 1 concentration, we obtain $B = .009 (B_r = .001)$. Clearly we now have a number of possible alternatives to evaluate in retaining $B_r = .003$, but permitting $B = .023$. The difference in network load (.608 to .434) is large enough to lead one to expect that the small k = 12 switch size $k = 12$ would be justified economically. This was tested for a number of examples, and only very small reduction in the computed C appears possible. The alternative of retaining $k = 12$, estimating $a$ for an increase in network size, and employing an expansion switch in the last stage is as good as any other. (This is just a repeat of the previous computations, except that $a$ and the ratio of $a$ to the terminal load are no longer negligible factors.)

For this redefined network we compute $s^* = .204$ and obtain:

$$C^* = \frac{1}{.80} + (3.49) (2.5) + (7.42)$$

This new cost estimate is a bit high, since the expansion rate .434 to .608 in the last switching stage is enough to cause lower $B_r$ than .023 because of "sure idle" effects in the last (internal) linking stage. Nonetheless, there is a difference of $(274-153) = 121$ XPE between the "exact" optimum SPD structure and a "pseudo-optimal" structure with single-stage concentration which must operate at a much lower internal link occupancy to insure low $B_r$.

The marginal cost of concentration is a very significant factor in the design of a link system, and by the numerical example we have tried to establish a motive for
examining structures with more than one stage of concentration. In a network with single stage concentration, especially high costs are incurred because concentrator congestion prevents the network from carrying the objective load of an SPF structure. From this we are led to conclude that a reduced cost (if not truly optimal) system can be achieved with two stages of concentration, if the SPF network load \( a \) is attained in its second linking stage.

In the next section we develop an approximate method for designing two-stage concentrators as the "front end" of an arbitrary SPF network. For the example here, the derived switch access sizes at \( a = 131, a = 608 \) are \( (k_1 = 6, k_2 = 12) \) in the first and second stages, and the marginal cost is only 65 XPE. This is much less than the marginal cost for the same network with single stage concentration.

3. NETWORKS WITH TWO-STAGE CONCENTRATION: DERIVATION

A structure with two-stage concentration has three basic parameters to be determined, of the five that characterize it. We choose to direct attention to \( k_1, a_2, \) and \( k_2 \) with \( a_1 \) and \( q_2 \) determined from \( a_1 = k_1 a_2 \) and \( q_2 = k_2 a_2 \) (cf. Eqn 1). For the remainder of this section we will assume \( (B_2 = 0.003, B_2 = 0.004) \), though the procedure to be given applies equally well for any choice of these objectives. The key parameter is \( a_2 \), the traffic load per inet terminal.

With \( B_2 = 0.003 \) we readily construct the chart shown as Figure 7, directly from binomial (or Enset) tables. The binomial assumption was used in this case, and Figure 7 simply relates \( a_1, a_2 \) for \( k_1 \) in the range of interest.

We formulate the cost function for a concentrator operating as the "front end" of an SPF structure as follows:

\[
C = \frac{1}{a_2} + \frac{k_1}{a_2} + C(s^*)
\]

where

\[
C(s^*) = \ln \frac{k_1}{a_2} + \frac{k_2}{a_2} + \frac{B}{a} + C(s^*)
\]

The expression \( k_2 a^2 \) represents the "effective accessibility" of the first switching stage, with \( \frac{k_2}{a_2} \) (that is, \( a_2 > a_2 \)) for a concentrator switch. Properly there should also be a term \( k_2 a^2 \) instead of \( k_2 a \), to account for the concentration switches in the second stage. However the requirement \( B_2 = B_1 > 0.001 \) implies that there is a very small probability of all links busy in the second linking stage. Because of this the effect of the second stage link distribution function on \( B_2 \) is judged negligible. (In the SPF form, all links are accessible in a derived channel pattern.) The term \( k_2 a^2 \) in equation (7) is to account for switching stages - usually, a fraction of one stage - to be added to or subtracted from the structure in order that its access factor be equal to that of the SPF network, since in general \( (k_1 a_2 + k_2 a_2) \neq (k_2 a^2) \). We obtain immediately, from equations (7) and (3):

\[
C(s^*) = \ln \frac{k_1}{a_2} + \frac{k_2 a_2}{a}
\]

Determination of \( s^* \) was attempted by several methods, and the one decided upon for use is a simple variant of a method suggested by W. S. Hayward as a "shorthand" Equivalent Random method. For a server group \( c \) with offered load \( a \), peakness \( v/a \), estimate the blocking or delay probability \( F(c_2) \) - for nonrandom input load \( a \) - by \( F \left[ \frac{v}{c} \right] = 1 + \frac{v}{c} \), where \( F \) is any of the standard loss or delay functions for random input. This is virtually equivalent to the assumption that calls arrive in "batches" of size \( v/a \) to the c servers. Applying this method we obtain \( q_2 = \frac{a^2}{q_1} \). The derivation of this and a checking procedure are given in Note 3.

One more approximation is needed to permit the numerical determination of \( k_1, k_2, a_2 \) and \( k_2 \). We must choose \( (k_1 a_2, k_2) \) so that an overall concentrator blocking \( B_2 \) is obtained, for a given \( n \) (which determines \( a_2 \), the occupancy of the second linking stage, and of the remaining symmetrical network).

If the \( q_2 \) approximation is reasonably valid, then \( (k_1 a_2) \) should be a good "equivalent group size" parameter for the second stage links. At \( a < 0.5 \) (for \( n = 0 \) it is then easy to find a constant \( D \) such that \( k_1 q_2 k_2 = D \) should be acceptable.) For this \( q_2 \) may be made \( D = 0.003 \), and \( B_2 = 0.004 \), where \( D = 0.001 \) and \( B(\cdot) \) are the Poisson and Erlang-B functions respectively. For \( a > 0.5 \) and \( a > 0.5 \), the group size \( (k_1 a_2) \) must be increased to maintain an objective level of service. Strictly by experiment, using common "conditional probability" methods for evaluating the concentrator blocking (Ref. 4, 5), we obtain \( k_1 a_2 = 0.003 \) as an approximation that works well in maintaining a consistent level of \( B_2 \). At \( (D = 26, a = 0.5) \), analytic estimates of \( B_2 \) vary only from \( 0.001 \) to \( 0.0005 \), at \( k_1 = 4 \) to \( k_1 = 16 \) (with \( q_2 = q_2 a_2 \) and \( k_2 \) determined directly from this and Figure 7, for any \( a_2 \) and \( B_2 = 0.003 \). There is a very slight bias in the direction of increasing \( B_2 \) as \( n \) and \( a_2 \) increase, and we obtain estimates of \( B_2 \) in the range \( 0.0014 \) to \( 0.0008 \), at \( k_1 = 4 \) and \( k_2 = 16 \) respectively, for \( (a = 0.537, a = 16) \). This small bias is judged insignificant, since it appears to occur mainly where the \( (k_1, k_2) \) pair determined is a long way from any (heuristically) reasonable design choice.

With the approximations made we restate the problem of optimizing the two-stage concentrator design as the "front end" of an SPF structure in the following terms:

Minimize \[ C = \frac{k_1}{a_1} + \frac{k_2 a_2}{a_2} + B + C(s^*) \]

with \( B_2 = 0.003 \); i.e., \( a_2 = f(a_1) \) determined from Figure 7, and \( B_2 = 0.001 \); i.e., \( k_1 a_2 = 26.0 \pm 0.5 \).

Since \( n \) (and implicitly \( k \) and \( a_1 \) are the parameters assumed known, the bracketed \( \{ \} \) portion of the cost function is independent of the particular selection of \( (k_1, k_2) \). We will refer to this as "base cost" and simply note its amount on the charts of results that follow.

\[ C_{12} = \frac{k_1 k_2 a_2}{a_1 a_2} \]

is the "variable cost" to be minimized.

4. RESULTS AND REMARKS

Figures 8 through 11 are charts of \( C_{12} vs. k_1 \) with \( k_2 \) and \( a_2 \) determined from the constraint functions. Figure 8 is for SPF network load \( a \) as minimized (\( n = 0 \), \( k = 2e = 5.44 \), \( a = 0.5 \)), whereas Figures 9 through 11 cover a range of SPF network parameters \( (k = 8, n = 2.1), (k = 12, n = 6.6), (k = 16, n = 12.1) \) respectively. In each chart \( C_{12} \) is shown for the realistic range of line terminal loads normally encountered in large scale local
switching networks; i.e., \( a_1 = 0.05 \) through \( 0.20 \) erlangs (1.8 through 7.2 CCS/terminal). \( C_{12} \) is expressed in XPT instead of XPE on each chart, to avoid the dimensioning problem (XPE \( \neq a_1 \neq 0 \)).

Integer constraints were not observed in computing the cost profiles of Figures 8 through 11, but the round-off error is generally small, so that costs determined by observing these constraints are not significantly different. Table I gives a few sample switch size dimensions at near-minimum cost for \( a_1 = 0.15 \), for the range of SPD network parameters indicated in the four charts.

Note from the four charts that the concentrator cost is not extremely sensitive to the choice of \( k_1 \), for \( n > 2 \) and \( k_1 > k \). In fact, \( k_1 = 6 \) appears to be a fairly good "general purpose" selection for any level of \( a_1 \) or \( n \). However, the required \( k_2 \) at \( k_1 = 6 \) varies somewhat with \( a_1 \) and to a greater extent with the SPD parameters. Because of the fixed \( b_1 = 0.003 \) assumed here, \( i_1 \) also varies widely with \( a_1 \); or if \( i_1 \) is fixed, \( i_2 \) must vary over a broad range to accommodate various levels of terminal load.

Variable \( i_2 \) can be achieved by multiplying of the second stage links, as in the Bell System No. 1 ECS network (Ref. 5). Variable \( i_1 \) is generally obtained by the use of "buildout" switches, but a promising alternative to this is to provide a small set of limited access multiple port switches (Ref. 7). Here a sequence of different "effective \( k_1 \)" access sizes would be attained, according to the \( i_1/k_1 \) ratio and pattern. Further study will be needed to extend the methods of this paper to cover systems with graded or random slip multiples. In practice of course the designer cannot provide perfectly flexible switch options that permit \( i_1, k_1, i_2 \) and \( k_2 \) to vary as needed to achieve design economy for the full range of terminal loads expected. Nor does this work provide him with all of the proper tools for selecting the few integer switch sizes that will best serve an expected range of terminal loads. However, a method is given here that should enable the engineer to arrive more quickly than he could before at the switch sizes that will achieve reasonable design economy. Also, the simple algebraic cost formulae enable one to compare cost profiles quickly and to test a large number of design alternatives.

Selecting \( k_1 \) at minimum cost from Figures 9 through 11, at \( a_1 = 0.15 \), we observe from Table I that \( k_2 \) should increase very slowly with increasing \( n \). That is, \( k_2 \) increases much more slowly than \( k \) (\( k_2 < k \) at \( n = 6.6 \) and \( n = 12.1 \)). Moreover, the base concentrator cost noted on each figure increases rapidly with \( n \). This means that for an economical design at large \( n \), much of the cost of concentration is to be absorbed by the remaining SPD network stages, just as it would be in a design with single stage concentration. This leads us to expect that a single stage design would be more economical than a two-stage design at large \( n \). A likely tradeoff point is at \( i_2 = k \), (where \( i_2 = k = \sqrt{a_1} \)), that is, where the second stage is equal in cost to each remaining stage.

Conversely, for \( n = 0 \) and for very small \( n \), \( C(s^*) \) is negative and \( k_2 \) is larger than \( k_1 \). This indicates that the second switching stage is bearing "more than its share" of overall network cost, and along with Ref. 2 we interpret this to mean that more than two stages of concentration are warranted. However, both the economical prove-in range and the gains achievable by using more than two stages appear to be very small, especially when the practical limits on choices of switch size are considered. Investigation of systems with more than two stages of concentration is probably not justified.

Though the results here are all derived on the basis of SPD network design parameters, the technique is clearly not dependent on this. One can readily assume that the design level of link occupancy for the internal linking stages is, say, 0.5 or 0.45 even at \( n = 6 \) and compute concentrator costs based on these levels of \( a_1 \). Generally, the SPD networks have steeper load-service curves than most link systems in use today, so that in using the SPD parameters directly one should set "high day" or "average of several high days" service objectives to correspond to the derived load levels, both for the concentrator and for the remaining SPD network form.

5. SUMMARY

With a concentrator viewed as the "front end" of an SPD network, there is evidence that two stage concentration is an economical design choice for a broad range of applications. A numerical method is given for deriving switch sizes that assure a near minimum-cost concentrator with specified constraints on the grade of service. This should prove a useful tool to engineers trying to locate the small set of design choices that will best fit a large range of switching network applications.

NOTE 1: DEFINITIONS AND NOTATION

Symmetrical Pure Distribution (SPD) Network: a link system form "idealized" in the sense that there are no concentration or expansion switching stages, all switch sizes are equal, and the carried load per link (occupancy) is therefore the same as its terminal load. In symbols:

\[ \begin{align*}
&k: \text{base switch access size (} k \text{ crosspoints)} \\
&s: \text{number of switching stages}
\end{align*} \]

\[ a(q = 1 - a): \text{average link load, in erlangs (occupancy)} \]

\[ n: \text{cost per link attributable to the wiring, selection and control process. (In Ref. 1 we make the assumption that one of the "external" linking stages, but not both, should be counted in estimating an average for an SPD structure. This results in the simple cost formula } C = a(k+n), \text{ with } a \text{ in crosspoints per erlang. } n \text{ is then properly expressed as "crosspoints per switch outlet". Sensitivity of the derived results to this assumption is shown in Ref. 1.)} \]

A network with two stages of concentration, but with an SPD form in all other stages, is expressed as follows:

\[ \begin{align*}
&k_1 > k_2 \\\text{and } k_3 > k_4 \\
&i_1 \quad i_2 \quad i_3 \quad i_4 \quad s
\end{align*} \]

\[ a(s^*_1 = 1 - a_1): \text{average traffic load per inlet terminal (erlang). (Throughout this work we implicitly assume equal calling rates or } \text{PCT's} \text{ type input on the first stage switches. In practice unequal call rates decrease average congestion, whereas unbalanced loads among switches increase congestion, and the two effects tend to offset each other.)} \]

\[ i_1: \text{number of inlet terminals per first stage switch. (For computations we assume all switches 1005 equipped, though this "terminal fill count" effect is easily adjusted in computing } i_1 \text{ from the derived results.)} \]

\[ k_1: \text{first stage switch access size, or number of outlet terminals per first stage switch.} \]

\[ i_2: \text{number of inlet terminals per second stage switch.} \]
\( k_2 \): second stage switch access size, or number of outlet terminals per second-stage switch.

\( a_2(q_2 = 1-a_2) \): average load per link (occupancy) between the first and second linking stages.

\( B_c \) (or \( \hat{B}_c \)): average probability of blocking (call congestion) of a concentrator; that is, the probability that a free inlet terminal will see no idle path. Generally we use \( B_c \) as the service objective for a concentrator, \( \hat{B}_c \) to indicate an analytic estimate of blocking probability.

\( B_1 \) (or \( \hat{B}_1 \)): analogous to \( B_c \) for a first stage switch with \( i_1 > k_1 \). \( \hat{B}_c = B_1 \) in a system with single stage concentration.

\( B_2 \) (or \( \hat{B}_2 \)): conditional blocking objective (or estimate), with at least one idle link to the second switching stage. For small objective \( B_2 \), \( B_0 \approx B_1 \) + \( B_2 \).

\( B_N \) (or \( \hat{B}_N \)): conditional network blocking probability, with at least one idle link out of the concentrator.

\( B \) (or \( \hat{B} \)): overall blocking probability, objective or estimated, considered as "point loss" or "first trial matching loss," i.e., the likelihood that no path exists between a free inlet terminal and a free outlet terminal. Again, for low \( B_0 \), \( B \approx B_0 + B_N \).

\( N \): network size; total inlet or outlet terminals, for an SPD structure; total outlet terminals, for a two-sided structure with (line) terminal concentration. (In a single-sided or 'folded' structure, \( N \) is the number of links in any internal linking stage with the derived "optimal" occupancy \( \hat{\alpha} \).

\( A \): the access factor, which in Ref. 1 is shown to relate reasonably well to analytically estimated \( B \) for SPD structures. For a network with two-stage concentration, \( A = B_0 \times \frac{k-a_k q(kq)}{N} \).

\( a_0(q_0 = 1-a_0) \): an "equivalent" link occupancy for \( a_0 \) discussed in detail in Note 3.

\( C(s^*) \): incremental (or decremental) cost of the remainder of an (SPD) structure, with two-stage concentration as its "front end," because of the (usually fractional) switching stage \( s^* \) needed to insure \( A \) or \( B \) at a given size \( N \). \( C(s^*) \) is considered part of the basic concentrator cost.

\textbf{NOTE 2: SPD NETWORK FORM FOR FIGURES 3 THROUGH 6}

Each of the network forms is a "full spiderweb" type, so called because there are exactly \( k^2 \) internal linking paths, \( k_1 \times k_2 \), and there is a channel from every link in the first linking stage to every link in the last (in this case, fourth) internal linking stage. This we may also refer to as a "minimal" spiderweb structure, since generally an objective \( B \) for given \( n \) and \( N \) will result in even more than \( k^2 \) paths in the channel pattern. This may be checked readily from Figures 1 and 2, for the derived optimal SPD structures of Ref. 1. (Here and elsewhere we refer to SPD networks with parameters chosen according to the results of Ref. 1 as "optimal". The model is only an approximate one, and true optimization is not attained; 'optimal' means in this usage 'minimum cost under the assumptions made'.)

All four of the charts are therefore derived from the same basic channel pattern:

Assuming independent Bernoulli load distributions on all linking stages except the first, it is straightforward to obtain:

\[
\hat{B} = \sum_{x=0}^{k-1} \frac{k-1}{x} \cdot \left( \frac{1}{x} \right)^x \cdot \left( \frac{x-1}{x} \right)^{x-1} = \frac{k-1}{k} \cdot \frac{k-1}{x} \cdot \left( \frac{1}{x} \right)^x \cdot \left( \frac{x-1}{x} \right)^{x-1}
\]

The factor \( \frac{k-1}{k} \), used to adjust the computed average occupancy per link for the effect of interdependence between successive linking stages, was suggested by E. C. Molina on the basis of the same type of conditional probability analysis used to derive both this formula and analogous formulae for other pattern structures. Improved methods of accounting for interstage dependence are being derived (e.g., Ref. 6).

In this work we are only marginally concerned with whether or not the formula for \( B \) provides the "best" generic estimator of blocking probability for the structures of Figures 3 through 6. (To produce a conservative estimator we should omit one or even two of the factors \( \frac{k-1}{k} \) ; i.e., ignore the interdependence effects in part.)

What we want here is to illustrate that the effect of terminal concentration is a substantial one, and this is shown clearly in the charts. More important, the use of an internally consistent formulation such as this enables us to compute "equivalent" access factors for comparing systems with varying degrees of concentration with their base SPD structures.
NOTE 3: THE APPROXIMATION \( q^*_2 = \frac{q^2}{q_1} \)

Consider a \((k \times k)\) switch to be characterized by a binomial distribution with parameters \(k\) and \(q_1\) with mean \(ka\) and variance \(kq_1\). This is an approximation implicit in the switch access factor \(k_a\) for the first stage and for all other stages. Actually, the certainty of an idle link is \((k_1 \times k_1)\) first stage switch may lead one to conclude that the "true" accessibility is \((k_1-1)q_2\) instead of \(k_2q_2\) in that stage, but the access factor vs. blocking relation holds true reasonably well for SPD structures in spite of (or possibly, in part because of) this built-in "sure idle link" effect in both the first and last linking stages.

A switch with \(ck_1 = i_1\) inlets and \(k_1\) outlets \((c > 1)\) should then be characterized by a mean offered load \(ck_1q_1 = k_2q_2\), and by variance \(ck_1q_1q_2\). If we can assume that, at least to a good approximation, a "sure idle link" exists in the first switching stage. With \(B_2 = .003\) this is, if we make the first reservation that, in any comparison of a structure with concentration to an SPD network, \(B\) for the SPD network must be equal to \(B_2\) for the network with concentration \((ck_1q_1)\) switch, the peakedness ratio is simply the variance ratio

\[ k_1 \frac{q_2}{q_1} = \frac{q_2}{q_1} \]

The accessibility of the first stage switch is

\[ k_1^* = \frac{q_2}{q_1} \]

Here in effect we have chosen to define the "true" of Section (3) as an access function \((k_1q_2)\) instead of a blocking or delay function, and the "random" load distribution is the distribution generated by the inlets of \(k_1\) by \(k_1\) first stage switch. However, the approach works well for full access trunking systems apparently only because the derived server group size has an effect equivalent to that of a true representation of the load distribution on the servers (in our case, the first stage switch outlets). The process here is identical. Even though there is no "blocking" in the first stage switch, its carried load distribution generates mismatch congestion in the network as a whole. We attempt to approximate the effect of a concentrator switch's outlet load distribution by deriving an equivalent outlet group size \(k_1q_2\).

A brief check of this analytic method was made by deriving \(q_2\) numerically, at the \(B_2 = B_1 = .003\) blocking level, for each of the networks with single-stage concentration in Figures 3 through 6. This is done by calculating \(S_2 = B_2 - B_1\), then equating access factors for each network at each concentration ratio with the access factor for the corresponding SPD structure. Good agreement was obtained at all concentration ratios for \(k = k_0 = 6\) and 12 (Figures 4 and 5). Moderate differences at \(k_1 = k = 4\) and 16 (in opposite directions in each case), are explained by the fact that the network access factors beyond the first stage are far from being comparable.

REFERENCES


<table>
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<tr>
<td>SWITCH SIZES FOR TWO-STAGE CONCENTRATORS</td>
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<tr>
<td>Parameters:</td>
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<tr>
<td>( B_1 = 0.03 ), ( B_0 = 0.004 ), ( \alpha_1 = 0.15 )</td>
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DESIGN PARAMETERS FOR SPD NETWORKS

<table>
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<th>FIGURE 1</th>
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<tbody>
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<td>EQU. CROSSPOINTS PER SWITCH OUTLET</td>
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<tr>
<td>OPTIMUM SWITCH SIZE (S) AND INTERNAL OCCUPANCY (A) ARE INDEPENDENT OF THE NETWORK SIZE AND SERVICE OBJECTIVE.</td>
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