BOUNDS FOR THE GROWTH OF THE NUMBER OF CONTACTS IN CONNECTING NETWORKS

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ABSTRACT

This paper reports the application of new techniques to the problem of bounding the number of contacts required for a switching network, given the number of inputs and the required blocking probability. The approach is similar to that used by Claude Shannon in 1947 to prove his channel capacity theorem. One of Shannon's great contributions was utilizing the observation that, if an ensemble of systems has a certain average performance, at least one member of the ensemble must have performance equal to the average of the ensemble. A similar strategy is used in this paper to show that there must exist classes of networks whose number of contacts increases as O(n log n) + O(n log 1/e).

Thus for fixed ε the number of contacts increases as n log n ∼ log n!, the informational minimum. For fixed n the number of contacts increases as log 1/ε. These are actual bounds which are found by a sequence of bounding approximations as opposed to the approximate models usually used in studying real systems. Thus, although these results do not directly lead to practical systems, they do give definite limits with which to compare the behavior of proposed networks.

The basic ensemble used to prove the above bound is an ensemble of networks described by four parameters. The members of the ensemble are all networks with these four parameters, which meet certain structural constants. A bound is found for the average blocking probability over the ensemble at one stage as a function of the average at the previous state. Now, given a bound for the average blocking probability at the first stage, a bound for the overall average blocking probability is found; then it is possible to find that the number of contacts as a function of n and ε is O(n log n + log 1/ε).

A second ensemble with five parameters is then defined. All but the first and last stages form a network identical with those in the previous ensemble. Then, using the Chernoff bound, the final bound given at the beginning is found.

The significance of these results is twofold. First, the bound gives an objective with which to compare network designs. Second, the results show the utility of some techniques used in information theory to teletraffic theory.

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1. INTRODUCTION

The purpose of this paper is to derive bounds for the growth of the number of contacts in a connecting network as a function of the number of inputs and the desired blocking probability (congestion). The results are strictly of a theoretical nature and lead to no explicit constructions for connecting network. They are solely existence proofs. However, these results provide a benchmark against which network designs and growth plans can be compared.

Section 2 discusses the previous results in the area of growth of network complexity. The key concept to the new results, ensembles of networks, is introduced in section 3. Then, in sections 4 and 5, two theorems are proved using ensembles.

2. BACKGROUND

Since the pioneering work of A. K. Erlang, many useful models have been developed to study blocking probability (congestion) in connecting networks. Early systems utilized direct control of the hardware by the digits of the call. Thus, they could be modeled as a sequence of obstacles through which a call must pass to be completed. In many cases, independence assumptions were completely justified. The introduction of common control and crossbar networks in the 1930's created a new class of problems. This type of system was more efficient in utilizing hardware, but the old type of independence assumptions was no longer valid. For now it was possible for a call to be blocked even though links were available along its whole path - a phenomenon called link mismatch.

Approximations have been developed to estimate the blocking probabilities in crossbar-like networks. Most results in this area are reviewed in the comprehensive work of Syski (1). However, none of these models are both simple enough to be really tractable and on firm mathematical ground. For example, C. Y. Lee (2) and P. Lefkii (3) have developed a model which is very simple to deal with and is reasonably accurate in many cases. However, Beneš (4) has recently shown that it is on very shaky theoretical ground. The model of C. Jacobaeus (3) is more detailed but it still is based on assumed a priori distributions.

One of the most successful approaches to calculating blocking probabilities has been the NEASIM Program of Grantgees and Sinowitz (6). This is a hybrid approach combining both simulation and mathematical modeling. However, it does not give much insight into the general problem.

Shannon has shown (7) that a connecting network with n inputs must have at least log n! - n log n coils to perform all permutations. This result is called the informational bound and is closely related to information theory.

There have been two basic papers in the area of limits for contact growth in connecting networks. Ikeno (8) has shown that networks can be built with X(n) contacts where

\[ X(n) < 10.9 A \log A \]

where A is the total carried load in erlangs, or

\[ A = n a(1-A) \]

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where \( a \) is the offered load per input line and \( B \) is the blocking probability. This result gives us no direct relationship between blocking probabilities and network growth because it only talks about the total amount of traffic offered and may lead to networks having expandors or concentrators so that the inputs to the network may carry more or less traffic than the original lines we are interested in.

A more direct approach to the problem of network size with blocking is the work of Lotze (9) which was recently extended by Feiner and Kappel\(^*\) (10). They define an access factor \( \gamma \) as follows

\[
\gamma = \frac{k^2(1-a^2)}{n} \quad 2.3
\]

where the networks are composed of \( k \times k \) crossbar switches. Then the number of contacts can be minimized by taking derivatives and setting them equal to zero. Feiner and Kappel have shown empirically that this access factor is highly correlated with blocking for several different networks. The minimum number of contacts is

\[
X(n) = 4(n \log n + n \log e) \quad 2.4
\]

The biggest problem with this approach is that the relationship between \( A \) and the blocking probability is not rigorously proven. Indeed, the authors give plots of blocking obtained by simulation as a function of \( A \) for several networks and sometimes the curves are convex \( U \) and in some cases they are convex \( \bar{U} \).

A similar approach to minimisation is possible with the probability graph model of LeGall and Lee. Even though this model has been helpful in engineering applications, there is no reason to believe it will yield absolute bounds.

It is interesting to note that the study of networks without blocking has been more successful due to the fact that this is strictly a combinatorial problem without any stochastic aspects. For example, Cantor (11) has shown that it is possible to construct nonblocking networks with

\[
X(n) < 8 n \log^2 n.
\]

Then

\[
X(n) = 0(n \log^2 n). \quad 2.5
\]

It can be shown (12) that rearrangeably nonblocking networks can be built with \( n \log n - n + 1 \) so-called \( S \) elements which are equivalent to \( 2 \times 2 \) switches. Thus, these networks achieve the informational bound given by Shannon.

3. ENSEMBLES OF NETWORKS

In this section, the concept of an ensemble of connecting networks is defined. All networks in a given ensemble have a fixed number of inputs, stages, and number of contacts, but different connection patterns. The rest of the paper is devoted to studying the properties of two ensembles. A bound is computed for the average blocking probability for an ensemble with an arbitrary number of inputs and cross points. Such a result is then turned around to give an upper bound for the minimum number of contacts required to achieve an arbitrary blocking probability given the number of inputs.

The inspiration for the approach used comes from Shannon's classic 1948 paper (13) which introduced a new approach to bounding problems: the average performance of an ensemble of systems. In describing his proof of the Coding Theorem, he states:

"The method of proving the first part of this Theorem is not by exhibiting a coding method having the desired properties, but by showing that such a code must exist in a certain group of codes. In fact we will average the frequency of errors over this group and show that this average can be made less than \( e \). If the average of a set of numbers is less than \( e \), there must exist at least one in this set less than \( e \). This will establish the desired result."

Thus, one can bound the average error probability for an ensemble of systems even though the probability of a member of the ensemble cannot be explicitly calculated. The same approach will be applied below to connecting networks.

The first ensemble of networks that will be analyzed is an ensemble of homogeneously structured switching networks and a typical member network shown in Figure 1. In this diagram, inputs, outputs, and intermediate points are shown as nodes. Contacts are shown as edges connecting nodes and are assumed to be normally open. These networks can be classified with four parameters:

- A network has \( n \) inputs and \( s \) stages.
- \( k \) is called the expansion factor. All stages except stage \( s \) and stage \( s-1 \) have \( k \) nodes. \( k \) does not have to be integral as long as the product \( kn \) is.
- \( c \) is called the fanout. All nodes except those in stage \( 0 \) are connected to nodes in the next stage to the right by \( c \) contacts. The contacts in stage 1 and stage \( s-1 \) are placed without replacement so that each node of stage 1 and stage \( s-1 \) are connected to \( c \) distinct nodes in stage 0.
- The contacts in all the other stages are placed with replacement so it is possible that two or more parallel contacts might join two nodes. This may seem somewhat inefficient but it simplifies the mathematics later. The blocking for such a network must be larger than if only placing contacts without replacement was used, and if \( c < k \) the fraction of the contacts that are involved in such parallel links is negligible.

Now an ensemble of such networks can be defined. Definition 3.1: \( N(n,s,c,k) \) is the ensemble of all homogeneous randomly structured switching networks with the same value for parameters \( n, s, c, \) and \( k \), as described above, each with equal probability.

4. ANALYSIS OF THE ENSEMBLE \( N(n,s,c,k) \)

In this section a bound will be derived for the average blocking probability of the ensemble \( N(n,s,c,k) \). The blocking probability is the probability that an input contact pair cannot be connected because of a path, given that the end points are idle. In general, this probability is a function of the traffic offered to the network and the resulting probability distribution of network states. It is virtually impossible to directly calculate or even bound the network state distribution or the blocking probability for a particular network. However, it is possible to bound the average blocking probability over the ensemble of networks in a relatively straightforward manner. This then shows that at least one network in the ensemble has the average blocking probability. In particular, the following theorem will be proved.

Theorem 4.1: For any blocking probability \( e \) there is a \( c \) such that for any number \( n \) of inputs there exists at least one connecting network with \( n \) inputs and blocking probability \( c \) which has \( c < n \log n \) contacts.

For such networks the requirement of finite blocking probability \( c \) results in networks whose number of contacts is a multiple of the informational minimum. It will be shown that the constant \( c \) is \( 0(\log 1/e) \).

If the blocking probability of a network is known given that \( q_{ij} \) paths are connected, the overall blocking can be computed as follows,

\[
Pr(\text{blocking}) = \Sigma_{q_{ij}=0}^{n-1} Pr(q_{ij}) Pr(\text{blocking} | q_{ij}) \quad 4.1
\]
As stated before, it is difficult to calculate \( Pr(q) \). However, if \( Pr(\text{blocking}|q=q_0) \) is upper bounded for all \( q \), then an upper bound for the overall blocking can be computed as follows:

\[
Pr(\text{blocking}) \leq \max \{Pr(\text{blocking}|q=q_0)\}.
\]

So bounding the blocking probability of a network is reduced to bounding the probability of blocking given that a certain number of paths are connected.

Now consider the ensemble of all networks of the form \( N(n,J,c,k) \), each of which is in all possible states in which \( q_0 \) paths are connected. The next step is to calculate the blocking probability from an idle point in stage \( j \) to an idle point in stage 0 averaged over this ensemble. This probability is called \( P_j(q_0) \) and can be shown to be given by

\[
P_j(q_0) = \left[1-\left(\frac{q_0}{kn}\right)(1-P_{j+1}(q_0))\right]^c
\]

4.2

The (1 - \( \frac{q_0}{kn} \)) term is the probability that the point in stage \( j-1 \) that a contact goes to is idle, assuming that all routing patterns are equally likely. The \((1-P_{j+1}(q_0))\) term is the probability that given a point in stage \( j-1 \) is reached, a connection can be made to the desired output point. The exponent \( c \) is used because there are \( c \) contacts leaving every node. Equation 4.2 is monotonic increasing as a function of \( q_0 \) so it can be upper bounded by setting \( q = n-1 \).

The value for \( P_1(q_0) \) is independent of \( q_0 \) since other paths do not interfere with the possibility of connecting from stage 1 to stage 0.

Then

\[
P_1 = 1 - \frac{q_0}{n}.
\]

4.3

Now the problem of bounding the blocking probability of the ensemble is reduced to bounding the iterations obtained from Equations 4.2 and 4.3.

It can be shown that

\[
P_j < \left[1 - \left(\frac{k-1}{k}\right)(1-P_{j-1})\right]^c.
\]

4.4

The nature of this relationship is shown in Figure 2. Here \( P_j \) is plotted as a function of \( P_{j-1} \). As can be shown by the derivative, the curve is strictly convex \( U \). In the limit as \( c \) and \( k \) become large, the curve becomes shaped like a backward "L".

 Bounds for \( P_j \) given \( P_1 \) can be found graphically as in Figure 3. Using this technique, we can find

\[
\overline{P}_j > P_j \text{ for all } j > 1.
\]

The blocking does not decrease without limit for as \( j \to \infty, P_j \to P_m \).

There is no closed form solution for \( P_m \) in general; however,

\[
\left(\frac{1}{k}\right)^c < P_m < e^{-c(k-1)}.
\]

4.5

The expression for \( P_j \), Equation 4.4, does not lend itself directly to further bounding so, in the following section, a two piece linear approximation for the \( P_j \) curve is used. This is shown in Figure 4. Since these lines lie above the curve, any value for \( P_j \) found by using them must also bound \( P_j \). Thus, if \( P_j' > P_j \) is the value for \( P_j \) obtained by this approximation,

\[
P_j' < P_1 - (P_2' - P_1) \frac{j-1}{j-1}
\]

Now

\[
P_j' > P(0) + \frac{m_j}{c}
\]

4.6

for

\[
P_j' > P(n).
\]

Now taking

\[
P_1 = 1 - \frac{q_0}{n}
\]

it can be shown that

\[
P_j' < 1 - \frac{q_0}{n}m^{j-1}
\]

4.7

Setting \( \delta = \frac{1}{2} \) for convenience

\[
m_1 = 2\left[1 - \left(\frac{k+1}{2k}\right)^c\right]
\]

4.8

Then

\[
P_j < \left[2\left(1 - \left(\frac{k+1}{2k}\right)^c\right)\right]^{j-1}
\]

4.9

Let us define \( j_1 \) as the value of \( j \) such that

\[
P_{j_1+1} < P_{j_1}.
\]

Then it can be shown that

\[
J_1 < \frac{1}{2}\log \frac{m_1}{\log 2(1 - \left(\frac{k+1}{2k}\right)^c)} + 2
\]

4.10

Thus for arbitrary \( k \) and \( c \), the blocking probability can be reduced to \( \delta \), in this case \( \delta = \frac{1}{2} \), with a number of stages which is \( O(\log n) \).

A similar approach is used to find \( j_2 \), the number of stages required to get the blocking probability to a desired \( P_{f} \).

The strategy is to find values of the parameters such that \( P_f = 2P_m \). It can be shown that

\[
m_2 < 2\left(\frac{k+1}{2k}\right)^c
\]

4.11

Then

\[
P_{f} < 6.4\left(\frac{1}{k}\right)^c, \quad c \geq 3, \quad k \geq 2.
\]

4.12

This leads to

\[
\log 12.8 + \log \frac{1}{c} = \frac{1}{\log k}
\]

4.13

Then using a geometric sum as in Equation 4.6

\[
P_j < P_m + \frac{1}{2} m_j^{j-3}
\]

4.14

Now we have already set \( P_{f} = P_f/2 \) so to get the sum equal to \( P_f \) we set the two terms equal

\[
6.4\left(\frac{1}{k}\right)^c = \frac{1}{2} m_j^{j-3}
\]

for \( c \geq 3 \) and \( k \geq 2 \) as assumed earlier. This leads to

\[
j_2 < 4.
\]

This means that if \( k \) and \( c \) are chosen in accordance with Equation 4.12 only four stages are required to reduce the blocking from \( \frac{1}{2} \) to \( P_f \) regardless of the value chosen for
Thus, the value of $P_\text{f}$ only affects the number of $f$ contacts by controlling the values of $k$ and $c$, not the number of stages.

The total number of contacts now required for the network is
\[
x = n(j_1 + j_2)ck = n \frac{\log \frac{n}{k}}{\log \left[2 \left(1 - \left(\frac{k+1}{k}\right)^c\right)\right]} + 4ck
\]
where
\[
\log 12.8 + \log \frac{\lambda}{f} + \log k
\]
Thus
\[
x = O(n \log n \cdot \log \frac{\lambda}{f})
\]
which proves Theorem 4.1.

5. A SECOND ENSEMBLE

The ensemble of networks defined in the previous section has constants $c$ and $k$ throughout the network. It will be shown in this section that a somewhat different ensemble can have a slower order of growth. The following theorem will be proven.

Theorem 5.1: There is a constant $c_0$ such that, given any probability of blocking $\varepsilon > 0$, there exists a value $n_0$ such that for any $n > n_0$, a network can be constructed with $< c_0 \log n$ contacts and blocking probability $< \varepsilon$.

Conversely, a network cannot achieve blocking probability $\varepsilon < 1$ for large $n$ with less than $c_0 \log n$ contacts.

In particular, it will be shown that networks exist for any $n$ and $\varepsilon$ which have $O(n \log n) + O(n \log 1/\varepsilon)$ contacts. Alternatively, this result can be viewed as meaning that for any $\varepsilon$ there exists a class of networks such that for large $n$ the ratio of the number of contacts in the network to log $n$ approaches a constant. The converse says that it is impossible to achieve a lower order of growth.

It should be observed that this bound is not useful for values of $P_\text{f}$ less than $e^{-n}$ since the second term then becomes $O(p^2)$ and networks with no blocking can be built with only $n^2$ contacts, i.e., the square crosspoint nets. Also, the proof using randomly structured networks breaks down at this point. Because of the nonblocking square networks, the bound still holds. Actually, such low values of blocking are not of practical interest since even for $n = 5000$, $e^{-n}$ is so small that blocking would never occur during the physical life of the system. The type of network used in this section is called a nonhomogeneously structured connecting network and is illustrated in Figure 5. It is fundamentally the same as the networks discussed in the previous sections with the following changes: The contacts between the two end stages and the stages adjacent to them are placed differently than the other contacts. In particular, every node in stage $l-1$ has $c_1$ contacts connecting it to nodes in stage 1-2. Every node in stage 0 has $c_1$ contacts connecting it to nodes in stage 1. Thus, the contacts are placed from left to right everywhere in the network except between stage 0 and stage 1. The rest of the network has fanout $c_0$ placed as before.

Definition 5.1: $N(n, j_1, c_0, c_1, k)$ is the ensemble of all nonhomogeneous randomly structured switching networks with parameters $n$, $c_0$, $c_1$, and $k$ as described above.

It has been shown in the previous section that the end to end blocking of the center part of the network can be $P_\text{o}$ if the network has $O(n \log n \log 1/P_\text{f})$ contacts. Now if each of the overall inputs and outputs of the network can reach $m$ inputs and outputs of the center network, the overall blocking can be bounded by
\[
P_\text{f} < (P_\text{o})^m.
\]
This can be shown as follows. Pick one of the $m$ available inputs and one of the $m$ available outputs. By definition, the probability of blocking between these points is $P_\text{o}$. Now, if the connection between these points is blocked, the conditional probability of blocking between any other pair of points is less than $P_\text{o}$. The conditional blocking can be viewed as the blocking in an ensemble of homogeneous networks with, at most, $n-2$ calls and $k-1$ nodes in each stage. The original center network had at most $n-1$ call in $k$ nodes. Since
\[
R-1 > R-k(n-1)\text{ for }k > 1
\]
the blocking in this new network must be less than the blocking in the center network. Similarly, it can be shown that the probability of blocking between the ith input-output pair, given that 1-1 input-output pairs are blocked, is also less than $P_\text{o}$. Equation 5.1 then follows from the fact that there are $m$ input-output pairs which can be used.

Actually, this is a very conservative argument. There are $n^2$ possible paths which can be used. It seems likely that the probability of each of these being usable must be in the order of $P_\text{o}$, so that the overall probability would be in the order of $(P_\text{o})^m$. However, this is not readily shown and the bound of Equation 5.1 is sufficient for the proof of Theorem 5.1. Then, by the union bound, it follows that
\[
P_\text{f} \leq P_\text{beg} + (P_\text{o})^m + P_\text{end}
\]
where
\[
P_\text{f} = \text{overall blocking probability}
P_\text{beg} = P_\text{f}[\text{an input can reach less than }m \text{ idle center inputs}]
P_\text{end} = P_\text{f}[\text{an output can reach less than }m \text{ idle center outputs}]
\]
Now, given certain value of $P_\text{f}$, we find values of $m$ network parameters which make each term in 5.3 $\leq P_\text{f}$. This leads to
\[
\log \frac{P_\text{f}/3}{\log P_\text{o}} = \frac{m}{k}
\]
Using arguments similar to those in the previous section, it can be found that
\[
J_1 = \frac{\log kn}{\log[2(1 - (\frac{k+1}{k})^c)]} + 1
\]
To find the appropriate value of $c_1$, the Chernoff bound (14) is used.

This gives
\[
P_\text{beg} = P_\text{end} = e^{-sm} \frac{1}{k} + \frac{k+1}{k} e^{q_1^0}, s < 0
\]
Then

\[ c_1 < \frac{\log P + sm}{\log \left( \frac{1}{k} + \frac{Kn}{k} \right)} \]

5.7

Taking \( s = -\log k \) (any negative value of \( s \) will satisfy the conditions of the bound) leads to

\[ c_1 < \frac{\log \frac{P}{k} + m \left( \log k - \log 2 \right)}{\log \frac{2}{P}} \]

5.8

The total number of contacts required for the network of Figure 5 is

\[ n \left[ 2c_1 + j_0 k \right] \]

5.9

Letting the overall blocking probability \( P_f \) be distributed as in Equation 3, this yields

\[
x(n) < n^\left( \frac{\log \frac{P}{k} - \log \frac{2}{P}}{\log \left( \frac{2}{P} \right)} \right) \left( \frac{\log k - \log 2}{\log \left( \frac{2}{P} \right)} \right)
\]

5.10

which can then be minimized as a function of \( k \) and \( c \). As in the previous section, the important point is the order of growth, in this case

\[ x(n) = O(n \log n) + O(n \log \frac{1}{P_f}) \]

5.11

which proves Theorem 5.1.

6. CONCLUSIONS

The key results of this paper are shown in theorems 4.1 and 5.1. The latter shows that networks can be built with a complexity proportional to the Shannon bound with the blocking only affecting a second order term.

Perhaps equally important is the methodology which is used. It is hoped that the concepts introduced will lead to further results in Teletraffic Theory.

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STAGE NUMBER

$\ell - 1 \quad \ell - 2 \quad \ell - 3$

$C = \text{FANOUT}$

n INPUTS

$\cdots$

$\cdots$

$\cdots$

n OUTPUTS

Kn NODES IN THESE STAGES

FIGURE 1. HOMOGENEOUSLY STRUCTURED CONNECTING NETWORK

FIGURE 2. PROBABILITY OF CONNECTION FROM STAGE J AS A FUNCTION OF PROBABILITY FROM STAGE J-1

FIGURE 3. CONSTRUCTION OF VALUES FOR $p_j$ GIVEN $p_1$
Figure 4. Linear bound for $P_j$

Figure 5. A non-homogeneously structured connecting network