A TELETRAFFIC PROBLEM ARISING FROM THE DESIGN OF LARGE MESSAGE SWITCHES

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ABSTRACT

In loss systems (e.g. telephony) it is permissible to design for a traffic load less than the theoretical maximum and to ignore (or lose) traffic in excess of design. It is only necessary for the design value to be such that the probability of loss is acceptably small.

In non-loss systems such as message switching, offered traffic must be accepted. However the internal storage can buffer short input peaks if the expected duration can be found.

The author develops a formula for the duration of a peak in excess of a given design value. By similar methods he also develops a formula for the mean time below the design value and shows that these two formulae when combined reduce to the cumulative sum of binomial terms — as one would expect.

Knowledge of these parameters can lead to more economically designed large non-loss systems rather than the previous 'safe' design of being able to handle 1 Erlang on all lines subject only to the size of the consequent queues.

INTRODUCTION

A message switching centre receives digital messages, reads the message header, determines the required output channel(s) and when the channel(s) is free sends the message on its way. Unlike a circuit switching exchange, it is not necessary for the output to be free before the message can be input. For heavily used networks and for International networks where several centres in sequence this is a big advantage as the sender is not held up by meeting 'Busy' conditions. In fact, in most cases, this way of handling messages (known as 'Store-and-Forward') results in quicker message delivery than a circuit switching network. Another advantage is that messages requiring several destinations can easily be handled.

The Store-and-Forward technique is also used by Data switching and Packet Switching networks. Message switching networks emphasise one aspect of message handling which the new Data and Packet switching networks either do not provide, or on which they place less emphasis. This aspect of message switching is called Message Assurance. It is a requirement to record and trace the handling of each message throughout the network in such a way that no message can be completely lost in spite of equipment failures — even a total centre failure. The worst that is allowed is to know which messages must be rerun from some previous storage point in the network.

This requirement of Message Assurance arises because a Message Switching Network assumes responsibility for delivering the messages it receives. The Public Telegram service is one example of such a network.

A Data Switching network is usually passing data messages between terminals and a computer. There is less need for Message Assurance as checks can be, and are, put into the terminal procedures and the computer programs to see that the required information is forthcoming. For example, if a terminal makes a request on the computer then, if no reply is received in a given period, the terminal will enquire again - there is no need for the network to tell it to do so.

A Packet Switching network is an interesting combination of Store-and-Forward combined with not accepting input if the destination is unavailable. This is both to give the message immediate knowledge of the deliverability of the message (and hence avoid taking responsibility for it), and to avoid the possible need for large buffer stores to hold messages while waiting for free outputs. It is thus a combination of Store-and-Forward with some characteristics of circuit switching.

IMPLICATIONS ON DESIGN

In the interest of cost effectiveness most Telephone Exchanges are designed so that in peak traffic conditions a small proportion of calls are lost. While a similar approach could be taken with the design of a Message Switching centre this idea is not acceptable. Due to the Message Assurance requirement, it would result in the need to get reruns of the lost messages each time a peak traffic load occurred. This of course is just the time when there are a lot of other things to do. Also the rerun traffic would, unless delayed, add to the peak!

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Consequently most message switching centres are designed to handle all incoming and outgoing lines simultaneously busy - subject of course to any consequent queues not exceeding the internal store.

In the case of a small centre - say less than 50 lines - the chance of all lines being simultaneously busy is quite likely. Also the economic penalty of designing to cope with this situation is not significant. However, for a large centre of several hundred lines not only is the likelihood of all lines being simultaneously busy extremely remote, but also the cost of designing for this becomes severe.

A possible way of designing a large centre economically would be to ensure that it can adequately cope with the normal peak loads, but would deal with an abnormal peak by reducing its output activity. In this way it is possible, without providing extra equipment, to handle all input lines simultaneously busy. Of course this results in the received messages being merely stored and thus it is necessary to calculate, not only the probability of all input lines busy, but also the expected duration. It is important to know whether it is likely to last one element, one character, one message or one hour!

As an example of this design problem: If a normal peak was judged to be less than 60% of all lines busy and would require 225 cells of core for input, then for all input lines busy we require 225 x 100 or 375 cells. If also the output lines with a 60% load need 150 cells then by temporarily ceasing output we have available for input 225 + 150 or 375 cells - just enough! This is of course, an over simplification of the actual problem.

PROBLEM 1

To find the probability of exceeding I (out of N) lines busy given an expected traffic load.

This is not so much a mathematical problem as a practical one as there are not many tables, graphs or charts which will give the answer when N = 300!

Mathematically we have:

$P_i = P_{i-1} + \frac{N!}{(N-I)!} P_{i-1} \cdot \frac{Q^N}{Q^i}$

and each term of the Binomial Expansion is the probability of that state. Thus to find the probability of I or more lines busy we must sum the first (I+1) terms.

i.e. we need

$S = \sum_{j=I}^{N-I} \frac{N!}{(N-j)!} P_{i-j} \cdot \frac{Q^N}{Q^i}$

The solution was to write a computer program to calculate this. A DEC PDP-6 was available and the program was written in FOCAL.

The program is given in the Appendix.

The interesting result is the extremely rapid fall off in probability above the expected number. For example with N = 500 and P = 0.33 we expect 100 lines busy. The probability of exceeding 140 is 0.6 x 10^-6 and of 150 is 0.9 x 10^-9!

PROBLEM 2

Given that it is extremely unlikely for more than I lines (out of N) to be simultaneously busy (say 1 in 10^-15), (a) how often does it happen and (b) for what duration. For example, 1 in 10^-15 might be 1 μs every 3 hours or it might be 1 hour every 10^5 years!

The problem proved amenable using a Transition State Diagram as follows:

Assume we wish to find the mean time that x or more lines are busy - and that both busy and free periods are negative exponential distribution. The transition probabilities are (a) Number of busy lines x death probability that the number will decrease and (b) Number of free lines x birth probability that the number will increase. We can now draw the Transition Diagram:

Note that the diagram has no transition from state (x-1) to x. Thus if one commences at x or above, one must eventually drop to (x-1) and there all transitions cease.

A Matrix equation can be written:

$|P_2(t)| = |a_{ij}| |P_1(t)|$

where $P_2(t)$ is the probability of being in state j and $a_{ij}$ is the transition rate from state j to state i

Integrating with respect to t we get:

$\int_0^t |P_2(t)| dt = \int_0^t |a_{ij}| |P_1(t)| dt$

Since $|a_{ij}|$ is constant

$\int_0^t |P_2(t)| dt = |a_{ij}| \int_0^t |P_1(t)| dt$

where $T_i = \int_0^t |P_1(t)| dt$

If we assume that at time 0 then x lines are busy and that at time (x-1) are busy we have:

$P_2(0) = 1$  and  $P_2'(0) = 1$

all other values being zero. $T_i$ is the time the system is in state j. Thus we want to find:

$\theta = T_0 + T_1 + \ldots + T_{N-1} + T_N$

Partially expanding Matrix equation I above, we get:

$$\theta = -\frac{N!}{(N-x)!} \cdot \frac{Q^N}{Q^i}$$

$$\theta = \frac{N!}{(N-x)!} \cdot \frac{Q^N}{Q^i}$$

$T_i = \int_0^t |P_1(t)| dt$

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$T_i = \int_0^t |P_1(t)| dt$
Taking the bottom row we have
\[ l = x b T_x \quad \text{or} \quad b T_x = \frac{1}{l} \]

Adding the bottom \((i-x+1)\) rows we get
\[ 0 = i b T_i - (N-i+1) T_{i-1} \]
\[ : T_i = \frac{(N-i+1) T_{i-1}}{i} \]

We also have from the top row
\[ T_N = \frac{1}{N} T_1 \]

Let \( \theta_x \) be the mean time that the system stays with \( x \) or more lines busy.

Then \( \theta_x = T_x + T_{x-1} + \ldots + T_1 + T_0 \)

\[ = T_x + T_{x-1} + \ldots + T_1 \left(1 + \frac{1}{N} (\frac{p}{q})\right) \]
\[ = T_x + \ldots + T_{x-N-1} \left(1 + \frac{1}{N} \left(\frac{p}{q}\right)^N \right) \]

Thus \( b \Theta_x = \frac{1}{l} \left[1 + \frac{1}{N} \left(\frac{p}{q}\right)^N \right] \)

Formula \( A \) can be also written as
\[ b \Theta_x = \sum_{i=0}^{N-x} \frac{(N-x)!}{(N-x-i)! (x+i)!} \left(\frac{p}{q}\right)^i \]

It will be noted that formula \( A \) presents a simple calculation method working from the inner bracket outward.

In fact, for small \( \frac{1}{N} \left(\frac{p}{q}\right)^N \) the value is close to 1 and in many cases the value of each successive bracket is not significantly different until the last few terms are evaluated. This fact can with care reduce significantly the amount of calculation.

It will be found that in many cases the expected value of \( \Theta_x \) (the mean time that \( x \) or more lines are busy) is a fraction of a mean message time.

Note also that \( \frac{1}{b} \) is the mean time the line is busy, NOT the mean message length \( \lambda \).

We have \( A = \frac{1}{b} \cdot \frac{1}{\lambda} \)

Also \( \frac{1}{b} = \frac{1}{\lambda} \) as with n.e.d. the mean time to the next birth is independent of the start point.

\[ : (1-A) \frac{1}{\mu} = A \frac{1}{\lambda} = \frac{\mu}{\lambda} \]
\[ : \frac{1}{b} = \frac{1}{\lambda} \cdot \frac{1}{1-A} \]

PROBLEM 3

How can one get confidence in formulae \( B \)? - at the extreme levels of probability practical measurement is out of the question and even speeded up simulation would be quite difficult.

The author realised that one test would be to use the same theory to develop a formula for the mean time that the system will be less than \( x \) lines busy and then combine these two to give the probability of exceeding \((x-1)\) busy.

This formula should be the sum of Binomial terms given in problem 1.

The Transition Diagram for the mean time below \( x \) lines busy is:

\[ \begin{array}{cccc}
T_{x-1} & T_x & T_{x-2} & \ldots & T_0 \\
\frac{1}{N-1} & \frac{b}{(N-1)p} & \frac{1}{N} & \frac{b}{Np} & \ldots & \frac{1}{N-x} & \frac{b}{N-xp} & \frac{1}{N-x+1} & \frac{b}{(N-x+1)p} \\
\end{array} \]

Thus we get

\[ \begin{array}{cccc}
0 & N-1 & N-2 & \ldots & 0 \\
0 & b & 2b & \ldots & N-x \cdot b \\
0 & 0 & \ldots & 0 \\
\end{array} \]

Giving us

\[ T_{x-1} = \frac{1}{l} \cdot \frac{1}{N-x} \]
\[ T_x = \frac{(i+1)}{(N-1) \cdot p} T_{i+1} \quad \left(\frac{b}{f} = \frac{\lambda}{f} \right) \]
\[ T_0 = \frac{1}{N} \cdot \frac{b}{f} T_i \]

Thus \( \phi_{x-1} \) is the mean time that the system has less than \( x \) lines busy.

\[ f \phi_{x-1} = \int T_{x-1} + T_{x-2} + \ldots + T_0 \]
\[ = \int T_{x-1} + \ldots + T_1 \left(1 + \frac{1}{N} \frac{b}{f} \right) \]
\[ = \int T_{x-1} \left[1 + \frac{2}{N} \frac{b}{f} \right] \left[1 + \frac{1}{N} \left(\frac{p}{q}\right)^N \right] \]
\[ = b \frac{1}{N-x} \left[1 + \frac{2}{N} \frac{b}{f} \right] \left[1 + \frac{1}{N} \left(\frac{p}{q}\right)^N \right] \]
\[ \text{or} \quad \phi_{x-1} = \sum_{j=0}^{x-1} \frac{(x-1)!}{(x-1-j)! (N-x-j)!} \left(\frac{p}{q}\right)^j \]

Now probability of \( x \) or more lines busy is:

\[ P_x = \frac{\Theta_x}{\Theta_x + \phi_{x-1}} \]
\[ = \frac{b \Theta_x}{b \Theta_x + (x) \cdot f \phi_{x-1}} \]

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IN CONCLUSION

It is believed that a formula has been developed which gives the expected duration of a peak traffic load which exceeds \( x \) lines busy (out of \( N \)) given the average load. This shows that in many practical cases not only is the probability of such a peak extremely small (e.g. 1 in \( 10^9 \)) but also would last about 200 milliseconds (ms). Thus the mean time between occurrence is of the order of 5 years of average load. As the busy period is only about 3 hours per day, this results in an expected rate of 1 in 40 years!

Further, the storage required and the small message delays resulting from holding the output for 200 ms are quite acceptable.

Contrast this result with what would have been the result had the mean duration turned out to be 1 minute. Now the peak occurs every 10 seconds or approximately every \( \frac{1}{4} \) hour! Thus would have been quite unacceptable.

Thus it is believed that a practical economic solution has been found to the peak overload problem in large Message Switching Systems.

APPENDIX

FOCAL PROGRAM: BINOMIAL TERMS CALCULATOR

The program will calculate:

\[
S = \sum_{i=0}^{R} \binom{N}{i} \left( \frac{p}{q} \right)^{N-i} \frac{q}{q}^{i} \]

given \( N, J, R, P, Q \)

Consider:

\[(P\cdot Q)^N = P^N \cdot N \cdot P^J \cdot Q + \ldots + \binom{N}{N-J} \frac{P^{N-J} \cdot Q^J}{(N-J)!} \]

a) If \( R-J = 0 \) then \( S = P^N \)

b) If \( R-J = I \) and \( P = Q = 1 \) then \( S = \frac{N!}{I!(N-I)!} \)

c) If \( R-J = I \) then \( S = \frac{N!}{I!(N-I)!} \)

As the program can take a long time to calculate some results (especially for large \( N \)), the value of \( S \) is output for each ten terms above \( J \) before \( R \).

The program will produce erroneous results if any term is smaller than \( 10^{-6} \) or larger than \( 10^6 \). Thus \( P = .001 \) \( N = 300 \) \( R = J = 0 \) will produce a large answer! Knowing that the early terms should be negligible while middle terms are significant, such a problem may be solved by setting \( J \) equal to the first value of \( R-J \) which gives a small result.

The accuracy of the results may be tested by summing all terms of the expansion: if \( P+Q = 1 \) and \( R = N \), the result should be 1 within the accuracy of the calculation.

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01.20 TYPE "?N?", "?J?", "?R?", "?P?", "?Q?"; Q = 1
01.30 SET S:=SET X=-2
02.10 IF (R=J=0) THEN GOTO 1.2
02.20 SET C=1
03.11 IF (X=0) THEN GOTO 3.8
03.12 IF (X=1) THEN GOTO 3.8
03.21 IF (N=1) THEN GOTO 3.8
03.22 IF (N=1) THEN GOTO 3.8
03.30 SET Y=0
03.31 SET S=SET X=0
03.40 SET S=SET Y=+1
03.41 IF (C+C=N-Y)=R+Q/G THEN GOTO 3.8
03.50 SET S=SET C+C=N-Y=1+R+Q/G
03.60 IF (Y=1) THEN GOTO 3.8
03.70 IF (Y=1) THEN GOTO 3.8
03.80 SET S=SET Y=+1
03.90 SET S=SET C+C=N-Y=1+R+Q/G
04.10 TYPE "ERROR"; GOTO 1.2
05.10 SET X=0
05.20 TYPE %6,1-1," \nS IS "ITYPE %5,1!"