AN ANALYTICAL MODEL TO DESCRIBE THE INFLUENCE OF THE REPEATED CALL ATTEMPTS

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1. INTRODUCTION

An increasing interest, from both theoretical and practical standpoints, can be observed in the implications that actual subscriber behaviour has on the traffic performance of networks and the methods used in traffic theory [1 to 5].

Current traffic methods for both network and switch calculations are based on lost-calls-cleared hypotheses. Subscriber behaviour may then be defined by the distribution functions for the:

- Interarrival period between fresh calls.
- Holding times of successful calls.
- Interarrival distribution function for successive attempts.
- Subscriber perseverance function.
- Holding time distribution for both successful and unsuccessful attempts.

In practice, in order to make use of the conventional teletraffic formulae (Erlang, Engset, etc.) an approximation which expresses the fresh and repeated attempts by a single interarrival distribution function is made.

This approximation may be reasonably good in particular cases such as in systems with low loss probability, large separations between reattempts, etc.; but becomes invalid in cases which are not so favorable.

The difficulty is due to the fact that the increment in the calling rate caused by reattempts at a certain instant strongly depends on the congestion situation (or any other cause of losing calls) in all previous instants. This increment in the calling rate as well as the holding time of unsuccessful calls may be different for each route of a network or for each stage in an exchange.

Let there be a network of switches (metropolitan or national for example). Let us consider a traffic relation between two terminal exchanges with a certain number \( r + 1 \) of intermediate tandems. The effects of the reattempts are different for the different sections of the network and depend on the number of links \( r + 1 \) to be used per call. For example, if a certain time \( h \) is required to switch the call through each exchange, the maximum holding times of the unsuccessful attempts would be \( h \) and \( (r+1)h \) for the links between the \( r \)th tandem and terminating exchanges and between the originating and last tandem exchange respectively. Analogously, the minimum interarrival time between two reattempts corresponding to the same call will...
be (r+l)\*h for those calls lost after the rth tandem and h for those which are lost between the originating exchange and the 1st tandem; the minimum interarrival times may be increaed by subscriber decisions about the initiation of reattempts.

In the previous paragraphs it has been foreseen that traffic increase in the links produced by subscriber reattempts stems from: short or long separation between reattempts and short or long holding times of unsuccessful attempts.

The objective of this article is to evaluate by means of an approximate analytical model the effect that reattempts have on traffic performance in a section of a network in the above mentioned cases.

2. DESCRIPTION OF THE MODEL.

Let us consider a section of the network described previously in the introduction. With the aim of studying the effect of repeated call attempts created by congestion in the network this section is considered as a full availability group with Poissonian fresh offered traffic and with negative-exponentially distributed holding times for successful attempts. In order to take into account the impact of reattempts on the interarrival distribution, an "equivalent birth and death" process is defined. In this new process the birth coefficients are no longer independent of the system's state. In fact, due to the time dependency between two consecutive attempts for the same call the system's state probabilities found by the reattempt will normally be different from the stationary state probabilities of the process. For that reason, the conditional probability that a reattempt will appear when the system is in a certain state during a time interval of duration \( dt \) is usually different, for each state as is the birth coefficient. In the article a method to calculate the values of these coefficients is given.

3. HYPOTHESES

In this section two sets of hypotheses are given. The first set is related to the system under study and the traffic sources characteristics; the simplifications accepted for the calculation method. It has to be pointed out that the model proposed is not necessarily restricted to the assumptions of the first set; however, only the case mentioned has been verified at the present time.

Hypotheses related to the system and the sources.

a) The system is composed of an infinite number of sources which have full access to \( n \) devices.

b) Sources originate fresh traffic according to a Poissonian distribution.

c) The holding time of those calls that do not find congestion in the system is negative-exponentially distributed with mean \( \Sigma \) and is independent of the number of attempts required to handle the call. This time \( \Sigma \) will be the time unit hereafter.

When the actual subscriber behaviour and the influence of the rest of the network over the full availability group is taken into account the following assumptions have been implicitly included in the last two hypotheses:

d) Those calls that occupy a free device have the same average duration and holding time distribution independently of whether they succeed or not in subsequent connections.

e) The probability of failure of one attempt after a device has been seized is a constant.

f) The reattempts of those calls that fail after a device has been seized are distributed according to the Poisson law, and are considered as fresh traffic.

g) and h) may be realistic if the size of the group as well as the number of subscribers accessed from it are sufficiently large. In fact, when that situation occurs, the proportion of attempts lost after seizing a device will be nearly uniform over time, because the primary reason for failure is the busy state of the called subscriber.

h) A fresh attempt that finds all devices busy is lost.

i) The probability of generating a new attempt after attempt \( k \) fails (perseverance function) only depends on \( k \) and the separation between two successive attempts of the same call is regulated by a certain function that will be defined later on. The perseverance function will be chosen so as to maintain the process stationary.

Hypotheses related to the analytical model.

j) The actual process may be approached by a birth and death process in which the probability of one attempt appearance during a time interval of duration \( dt \) when the system is in state \( j \) only depends on \( j \). This probability will be the sum of the mathematical expectations of the probabilities of one attempt appearance during \( dt \) corresponding to fresh attempts and one to each of the reattempts. The corresponding birth coefficients for the different system states will depend on the transition probabilities of the B&D process, and will be calculated in accordance with this dependency.

4. NOTATION

Let there be a time axis represented in fig. 1, for the instants \( t_i \), \( i = 0, 1, 2, \ldots, m \) (\( t_0 = T \)). For each pair of values \( i, i+1 \), the instant \( t_{i+1} \) comes before the instant \( t_i \). They are separated by the time interval \( a_i \).

\[ \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots \]
\[ t_0 = T \]
\[ t_1, t_2 \]
\[ t_3, t_4, t_5, \ldots \]

Fig. 1. Time Diagram.

\( n \) - number of devices in the group.

\( j \) - number of busy devices in the group.

\( \lambda dt \) - probability of a fresh attempt's appearance in the interval (\( T, T+dt \)), \( \lambda \) is the birth coefficient for the stationary birth and death process associated with fresh calls. It is assumed to be independent of the number of busy devices.

\( \lambda_j dt \) - probability of an attempt's appearance (either fresh or not) between \( T \) and \( T+dt \), having \( j \) busy devices in \( T \) (\( \lambda_j \) is the birth coefficient of the birth and death process associated with both fresh and repeated attempts).
(j_i, t_i) or (j, t) - System's state with j devices in the instant t.

m - Maximum number of reattempts for a single call.
k - Attempt index (fresh call) or 1, 2, ..., m (reattempts).
H_k - Probability of initiating the k-th reattempt after k failures. It is assumed to be constant for each k.
R_k(t_i, t+1) - Interarrival distribution function which governs the intervals between the k-th attempt (in the instant t_i) and the k+1 attempt (in the instant t_i+1) of a single call.

The above considerations show that the birth and death process of fresh attempts will correspond to both fresh and repeated attempts. According to hypothesis a6, making a fresh attempt with probability H_0 and carrying it out in the interval (T, T+dt) with the system in the state j. The mathematical expectation for this probability is
\[ \text{d}t \int_{-\infty}^{T} \lambda P_T[(n, t)] [\{j, T]\} \cdot H_0 \cdot dR_0(t, T) \]  

- Probability of a second reattempt appearance. In a manner similar to the previous case the mathematical expectation may be evaluated by the expression:
\[ \text{d}t \int_{-\infty}^{T} \lambda P_T[(n, t)] [\{j, T\}] \cdot H_1 \cdot dR_1(t, T) \]  

5. Expression of the Birth Coefficients as a function of the Fresh Traffic and the subscriber reattemping characteristics.

In this section the effects of the reattempts due to system congestion will be treated. Let us consider a group of n devices as defined in hypotheses a1 to a5 of section 3. Assume the system state (j, t) at instant t and assume also that the last congestion state occurred at an instant sufficiently removed to allow us to consider that most of the reattempts have already been handled. The calling rate will for the most part include only fresh attempts. Consider another state (j', t+a) at instant t+a such that a congestion state (n, t) occurred at an instant t' (t<t+a) not very removed from t+a thus allowing most reattempts to remain pending. It may easily be concluded that the birth coefficient B at the instant t will be nearly \( \lambda \) while at the instant t+a it will be normally higher due to the reattempts.

Moreover, if another state (j', t+a') exists at instant t+a' where j' is very small with respect to the average carried traffic, a' will normally be large enough to allow most of the reattempts, originated by the congestion, to be carried.

The above considerations show that the birth coefficient depends on the system's state as well as on the instant corresponding to the occurrence of this state.

According to hypothesis a6, the state probabilities of the system will be calculated assuming a birth and death process in which the birth coefficients for each system state will be the mathematical expectations of the actual birth coefficients for these states over time. They will correspond to both fresh and repeated attempts. According to the following expressions for the probability of one attempt appearing in the interval (T, T+dt) when the system's state at the instant T is \( \{j, T\} \):

- Probability of one fresh attempt appearance:
\[ \lambda \cdot \text{d}t \]

\[ B = \int_{-\infty}^{T} \lambda P_T[(n, t)] [\{j, T\}] \cdot H_0 \cdot dR_0(t, T) \]  

\[ C = \int_{-\infty}^{T} \lambda P_T[(n, t)] [\{j, T\}] \cdot H_1 \cdot dR_1(t, T) \]

- Probability of a second reattempt appearance.
\[ \int_{-\infty}^{T} \lambda P_T[(n, t)] [\{j, T\}] \cdot H_2 \cdot dR_2(t, T) \]

- Probability of one first reattempt appearance due to a fresh call that appeared at any previous instant t_1, finding congestion state n, deciding to make a first reattempt with probability H_2 and carrying it out in the interval (T, T+dt) with the system in the state j. The mathematical expectation for this probability is
\[ \text{d}t \int_{-\infty}^{T} \lambda P_T[(n, t)] [\{j, T\}] \cdot H_2 \cdot dR_2(t, T) \]  

- Probability of a second reattempt appearance. In a manner similar to the previous case the mathematical expectation may be evaluated by the expression:
\[ \text{d}t \int_{-\infty}^{T} \lambda P_T[(n, t)] [\{j, T\}] \cdot H_1 \cdot dR_1(t, T) \]  

The above formula may then be expressed as follows:
\[ \lambda \cdot \text{d}t \]

\[ B = \int_{-\infty}^{T} \lambda P_T[(n, t)] [\{j, T\}] \cdot H_0 \cdot dR_0(t, T) \]  

\[ C = \int_{-\infty}^{T} \lambda P_T[(n, t)] [\{j, T\}] \cdot H_1 \cdot dR_1(t, T) \]

If the intervals between successive attempts for a single call were governed by the same distribution function R (R_k=R for all k's), the above formula could be expressed as follows:
\[ F_1 \cdot H_0 \cdot L_1 [1 + \Sigma_{n=1}^{m} \Pi_{H_n}] \cdot \frac{k}{k+1} \cdot \text{d}t \]  

\[ F_1 \cdot H_0 \cdot L_1 [1 + \Sigma_{n=1}^{m} \Pi_{H_n}] \cdot \frac{k}{k+1} \cdot \text{d}t \]  

\( F_1 \) is the ratio between the birth coefficients associated with the repeated and fresh attempts for system state j.
with \( L_j = I(j,k) \) for all \( k \)'s. \( \text{(5.11)} \)

It can be seen that the method, which allows one to consider any perseverance function that verifies the hypothesis \( a_5 \), shows \( H_2 \) to be the most influential factor in the expression \( \text{(5.10)} \).

6. ALGORITHM FOR THE DETERMINATION OF THE BIRTH COEFFICIENTS.

As the transition probabilities are functions of the birth coefficients \( \lambda_j \), and the latter are expressed as functions of the former ones, the following iterative algorithm will be necessary to solve these recursive expressions and obtain the \( \lambda_j \)'s.

Let \( \lambda^p_j \) \((p = 0, 1, 2, \ldots; j = 0, 1, \ldots, n)\) be the birth coefficients obtained in the \( p \)-th step. Beginning with \( \lambda^0_j = \lambda \), the transition probabilities \( PT(j_{i+1} \mid j_i) \) are obtained from the \( \lambda^p_j \)'s in the same way.

The birth coefficients \( \lambda_j \) will be the set of limits

\[
\lambda_j = \lim_{p \to \infty} \lambda^p_j \quad j = 0, 1, \ldots, n
\]

The convergence of the sequences \( \lambda_j (p) \) (for each \( j \)), guaranteed by the stationary state of the process is rapid in all the cases treated.

The value of \( \lambda^p_j \) depends on \( \lambda \), \( H_i (i < s) \), \( I(j,k) \) and \( I(n,k) \). The values of \( \lambda \) and \( H_i \) are fixed for all steps \( p \) of the iteration; \( I(j,k) \) and \( I(n,k) \) are functions of the transition probabilities, the last ones being dependent on the birth coefficients obtained in step \( p-1 \).

In appendix 1 the transition probabilities \( PT(j_{i+1} \mid j_i) \) are expressed in terms of the elements of three matrices \( Q \), \( Q \) and \( BM \), according to the following formula:

\[
PT(j_{i+1} \mid j_i) = \sum_{s=0}^{n} qns qsj e^{-bsq} \quad \text{(6.1)}
\]

According to \( \text{(5.7)} \) the values \( I(j,k) \) is then

\[
I(j,k) = \int_0^\infty \left( \sum_{s=0}^{n} qns qsj e^{-bsq} \right) dR_k(x) \quad \text{(6.2)}
\]

In the article the interarrival distribution function \( R_k(x) \) will be assumed to be independent of \( k \). Three types of distribution functions \( R \) have been studied although the method allows the treatment of any distribution:

a) Constant separation \( a \). This distribution is not realistic but shows the situation in an extreme and unfavorable case.

b) Negative exponential with mean \( \bar{a} \). It is not realistic because it does not include the minimum time required by the subscriber to repeat his attempt.

c) Minimum constant separation \( a \) and additional negative exponential with mean \( \bar{a} \). This distribution seems to be realistic.

According to \( \text{(5.11)} \) and \( \text{(6.2)} \) the values of \( L_j \) may be expressed as follows:

a) In this case \( R(x) \) is defined as:

\[
R(x) = \begin{cases} 0 & x < a \\ 1 & x \geq a 
\end{cases}
\]

and \( L_j = \sum_{s=0}^{n} qns qsj e^{-bsq} \quad \text{(6.3)} \)

b)

\[
R(x) = 1 - e^{-\frac{x}{a}} \quad \text{(6.4)}
\]

c)

\[
L_j = \sum_{s=0}^{n} qns qsj e^{-a - bsq} \quad \text{(6.5)}
\]

7. PRESENT STATUS OF THE CALCULATION METHOD.

A program to calculate the state probabilities and birth coefficients according to \((5.9), (5.10), (6.3), (6.4) \) and \( (6.5) \) has been written. The calculation of the matrix \( BM \) by means of the "QR" algorithm and the calculation of the matrices \( Q \) and \( B \) with the algorithm described in appendix 1 presents, at present, problems of a numerical analysis type. These problems become more serious when the number \( n \) of devices increases.

To verify the validity of the hypotheses of the model the system defined by hypothesis \( a_5 \) of section 3 has been simulated in a time-true model. The comparison between calculation and simulation has not been possible due to the fact that to determine the degree of approximation obtained by application of the hypothesis \( a_6 \) with respect to the actual process, the error introduced in the analytical method by the calculation program must be significantly smaller than the one caused by the above mentioned hypothesis.

8. CONCLUSIONS.

General conclusions may be derived from the analytical expressions deduced in sections 5 and 6. Even though the lack of calculation results does not allow one to give concrete figures, the conclusions have been confirmed by simulation.

With respect to the effect reattempts have on the interarrival law the following aspects can be witnessed assuming Poissonian arrival for fresh calls:

- As was mentioned in hypothesis \( a_6 \), obtained in formula \((5.9)\), the birth coefficients are no longer equal for all states. Figures 2, 3 and 4 show how they depart from a constant birth coefficient assumed in the Erlang formula according to the simulation results obtained for different mean separations between reattempts.
- In spite of this situation the interarrival law of the process with reattempts can be fitted very closely by a negative-exponential distribution with the same mean. The approximation obtained by a negative-exponential distribution is normally better than the accuracy that may be expected from traffic measurements. Fig. 5 shows the actual interarrival law corresponding to the same case as Fig. 3.

- The variation of the birth coefficients for different average separations between reattempts of the same call is observed in Fig. 2, 3 and 4. This discrepancy with respect to a constant calling rate (independent of the system's state) is still significant in the case of a separation between reattempts of \( \alpha = 0.5 \). For longer separations the birth coefficients become independent of the system's state, according to the analytical model. This approximation may be invalid for constant separations between reattempts. With respect to the effect of reattempts on the grade of service definition:

- The loss probabilities for fresh and repeated attempts are different as may be expected from the model and as is confirmed by the simulation. The first one coincides for Poissonian fresh traffic with the probability of all devices being busy. The second one is, in the model, independent of how many reattempts were previously made and coincides with the value \( L_1 \) of (5.11). The grade of service defined by a single value, as for example the loss probability, may not be very significant.

More specific conclusions will be possible when analytical results are obtained.

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Appendix 1. ALGORITHM FOR THE CALCULATION OF THE TRANSITION PROBABILITIES.

In order to determine the transition probabilities $PT(n, a | j)$ the following matrical method has been employed.

A B&D process, with B&D coefficients $\lambda_j$ and $\mu_j$ respectively, may be defined by the matrix $PM((m_{ij}))$, with

\[
\begin{align*}
    m_{00} &= \lambda_0 \\
    m_{jj} &= -((\lambda_j + \mu_j)) \\
    m_{NN} &= \mu_N \\
    m_{jj+1} &= \lambda_j \\
\end{align*}
\]

remaining elements being zero.

The eigenvalues of the matrix $PM$ are simple and real and can be calculated by any matrical method. One method that seems to be advantageous is the "QR" algorithm, which is based on the fact that the PM matrix is of the Hessenberg type.

The Jordan decomposition of matrix $A$ is:

\[ A = Q \cdot BM \cdot \bar{Q} \]

where $BM$ is the diagonal matrix whose elements are the eigenvalues $b_s (s=0,1, \ldots, N)$ of the PM matrix.

$Q=((q_{js}))$, $\bar{Q}=((\bar{q}_{js}))$ are the matrices of eigenvectors of the matrix.

The matrices $Q$ and $\bar{Q}$, in the particular case of the PM matrix, can be calculated with the following algorithms:

\[
\begin{align*}
    q_{s0} &= 1 \\
    q_{is} &= \frac{b_s - m_{0i}}{m_{0i}} q_{is-1} + \frac{(b_s - m_{ii})}{m_{ii}} q_{is} + \frac{(b_s - m_{is})}{m_{is}} q_{is-1} \quad i = 1, \ldots, (N-1)
\end{align*}
\]

and

\[
\begin{align*}
    \bar{q}_{s0} &= 1 \\
    \bar{q}_{si} &= \frac{b_s - m_{0i}}{m_{0i}} \bar{q}_{si-1} \quad i = 1, \ldots, (N-1)
\end{align*}
\]

for each value of $s=0,1, \ldots, m$, together with the normalizing condition $Q \cdot \bar{Q} = I$.

The transition probability from $i$ to busy devices in $n$ time units is then:

\[ PT(n, a | j) = \sum_{s=0}^{N} q_{ns} \cdot \bar{q}_{sj} \cdot e^{-bs \cdot a} \]

More detailed information about the subjects of this appendix is given in [10].