Mean and Variance of the Overflow Traffic of a Group of Lines Connected to One or Two Link Systems

J. de Boer
Philips Telecommunications Industrie Bo Vo, Hilversum, Netherlands

ABSTRACT

Traffic overflowing from a group of lines connected to a link network may be characterized by its mean and variance. These moments can be expressed in the Poisson traffic offered, the number of lines and the characteristic blocking quantities \( P_i \) of the network. \( P_i \) is the probability that no one of \( i \) free lines is accessible from a given free inlet of the network under given loading conditions.

The paper consists of two distinct parts. After a short survey of basic formulas, the case of two link networks is considered in the first part. Here some of the lines are connected to one network with quantities \( P_i \) and the other lines are connected to a second network with quantities \( P_{i+1} \).

By a random choice it is decided whether the establishment of a connection should first be tried via the first network or via the second. For this situation the relations between the \( P_i \) of the combined network and the \( P_{i+1} \) are derived. It is shown that the combined network may be treated as the case of one network with quantities \( P_i \).

In the second part of the paper approximations for the mean and variance of the overflow traffic are derived which are suitable for pocket calculators.

1. INTRODUCTION

A call which is offered to a link network and destined for a certain direction, must occupy one of the outgoing lines of that direction, which are connected as outlets to the network. In the blocking of such calls, the network plays a role besides the group of lines. For this purpose the network may be characterized by the quantities \( P_i \) (\( i = 1, 2, \ldots \)) where \( P_i \) is the probability that - under the given loading conditions - no one of \( i \) free lines is accessible from the free inlet where the call enters. It is assumed that the loading of all inlets is equal and consequently the blocking probability for a call from a given inlet to the desired route is independent of the traffic state of the network. Then the blocking or overflow probability for a call from a given inlet to the desired route of \( n \) lines is (route included):

\[
\Phi^{(n)}(i) = \sum_{j=0}^{i} \frac{(1-P_j)\cdot P_{n-j}}{P_n} \quad (1)
\]

where \( \Phi^{(n)} \) is the probability that \( i \) of the \( n \) lines are occupied.

In most practical cases \( P_i \leq 1 \) for \( i = 2, 3, \ldots, n \), and consequently the distribution defined by (1) and (2) is often approximated by an Erlang distribution (e.g. in [1]). The approximation \( P_i = P_i \) is also used sometimes [2].

The overflow traffic has a mean \( \mu \) and a variance \( \nu \).

\[
\begin{align*}
\mu &= \lambda \nu = \sum_{i=0}^{\infty} \Phi^{(i)} \cdot P_i \\
\nu &= \sum_{i=0}^{\infty} \Phi^{(i)} \cdot P_i \cdot i \\
\end{align*}
\]

and

\[
\begin{align*}
\Phi^{(i)} &= \sum_{j=0}^{i} \frac{(1-P_j)\cdot P_{n-j}}{P_n} \\
\end{align*}
\]

where \( \lambda = \mu / \nu \) and \( g(i, k) \) is the probability that \( i \) lines of the route and \( k \) overflow lines are occupied when blocked calls are supposed to occupy lines of an overflow group with an infinite number of lines (\( i = 0, 1, \ldots, n \) and \( k = 0, 1, \ldots, \ldots \)).

The quantities \( M(n+1) \) may be determined from the recurrence relations

\[
egin{align*}
(i+1)M(n+1) &= M(n) - a(1-P_n) \cdot M(n-1) \\
\end{align*}
\]

for \( i = 0, 1, \ldots, n \), with the conventions \( P_1 \) and \( (n+1) \).

The probabilities \( \Phi^{(i)} \) may be found from (2) and the normalizing equation

\[
\sum_{i=0}^{\infty} \Phi^{(i)} = 1
\]

The result is

\[
\Phi^{(i)} = \Phi^{(0)} \cdot \frac{P_1 \cdot P_n}{P_i} \quad (3)
\]

and

\[
\Phi^{(0)} = \left[ \sum_{j=0}^{i} \frac{P_1 \cdot P_n}{P_j} \right]^{-1} \quad (4)
\]

2. BASIC FORMULAS

In a link network, the lines of a certain route usually form only a small part of the outlets of the network. This means that, as an approximation, we may assume that the number of occupied lines is independent of the traffic state of the network. Then the blocking or overflow probability for a call from a given inlet to the desired route of \( n \) lines is (route included):

\[
w = \mathcal{A}^{(1)} \cdot P_{n-i} \quad (1)
\]

where

\[
\Phi^{(i)}(i) = (1-P_i) \cdot P_{n-i} \quad (2)
\]

for \( i = 0, 1, \ldots, n \), with the conventions \( P_1 \) and \( (n+1) \).

The probabilities \( \Phi^{(i)} \) may be found from (2) and the normalizing equation

\[
\sum_{i=0}^{\infty} \Phi^{(i)} = 1
\]

The result is

\[
\Phi^{(i)} = \Phi^{(0)} \cdot \frac{P_1 \cdot P_n}{P_i} \quad (3)
\]

and

\[
\Phi^{(0)} = \left[ \sum_{j=0}^{i} \frac{P_1 \cdot P_n}{P_j} \right]^{-1} \quad (4)
\]

where \( i = 0, 1, \ldots, n \) and \( P_1 \).

In most practical cases \( P_i \leq 1 \) for \( i = 2, 3, \ldots, n \), and consequently the distribution defined by (3) and (4) is often approximated by an Erlang distribution (e.g. in [1]). The approximation \( P_i = P_i \) is also used sometimes [2]. Here, however, we use the general form of (3) and (4).

The overflow traffic has a mean \( \mu \) and a variance \( \nu \).

\[
\mu = \lambda \nu = \sum_{i=0}^{\infty} \Phi^{(i)} \cdot P_i \\
\nu = \sum_{i=0}^{\infty} \Phi^{(i)} \cdot P_i \cdot i \\
\]

and

\[
\begin{align*}
\Phi^{(i)} &= \sum_{j=0}^{i} \frac{(1-P_j)\cdot P_{n-j}}{P_n} \\
\end{align*}
\]

where \( \lambda = \mu / \nu \) and \( g(i, k) \) is the probability that \( i \) lines of the route and \( k \) overflow lines are occupied when blocked calls are supposed to occupy lines of an overflow group with an infinite number of lines (\( i = 0, 1, \ldots, n \) and \( k = 0, 1, \ldots, \ldots \)).

The quantities \( M(n+1) \) may be determined from the recurrence relations

\[
egin{align*}
(i+1)M(n+1) &= M(n) - a(1-P_n) \cdot M(n-1) \\
\end{align*}
\]

for \( i = 0, 1, \ldots, n \), with the conventions \( P_1 \) and \( (n+1) \), and the normalizing equation

\[
\sum_{i=0}^{\infty} M(i) = 1
\]

Formulas (6) and (7) have been given by Wallström [3].
Suppose that $n_0$ out of a group of $n$ lines of a route are connected to a network $N_1$, while the other $n_0$ out of $n$ lines of the route are connected to a network $N_2$. The quantities $P_{i,j}$ ($i=0,1,\ldots,n$) characterize $N_1$, and the same goes for the quantities $P_{i,j}$ ($i=0,1,\ldots,n$) with respect to $N_2$. A Poisson traffic of an erl. is offered to the route. For each call there is a random choice whether the establishment of a connection should first be tried via $N_1$ (probability $p=1-q$) or first via $N_2$ (probability $q=1-p$). This implies that a Poisson traffic of $a_1$ erl. is first offered to $N_1$, and a traffic of $a_2$ erl. is first offered to $N_2$. Traffic blocked on $N_1$ or the $n_0$ lines, is offered to the $n_1$ lines via $N_2$, and traffic blocked on $N_2$ or the $n_0$ lines is offered to the $n_0$ lines via $N_1$.

The networks $N_1$ and $N_2$ together form a network $N$ with quantities $P_i$ ($i=0,1,\ldots,n$). Regarding the distributions of numbers of occupied lines, the following notation is used:

\[
\phi(i,j) = \text{probability that } i \text{ of the } n_1 \text{ lines are occupied and } j \text{ of the } n_2 \text{ lines are occupied (} i=0,1,\ldots,n_1 \text{ and } j=0,1,\ldots,n_2 \text{).}
\]

With these definitions it is clear that

\[
\phi(i,0) = \sum_{k=0}^{i} f(k,i-k),
\]

and

\[
P(i) = \frac{1}{n!} \sum_{k=0}^{n} f(k,n-k).P_{i,n-k}P_{n-i,k},
\]

for $i=0,1,\ldots,n$.

Specifically $P_n = 1$. The case $n=\infty$ is discussed later. The quantities $f(i,j)$ where either $i$ or $j$ or both are negative, are defined to be zero in (8) and (9) and following formulas.

The equations of state for the $f(i,j)$ are:

\[
f(i+1,j) = (a_1+a_2)P_{i+1,j-1} + (a_1+a_2)P_{i-1,j+1} + (a_1+a_2)P_{i-1,j-1}
\]

for $i=0,1,\ldots,n$ and $j=0,1,\ldots,n$.

Starting with $i=1$, $j=1$, it is possible to express $f(1,0)$ in $f(0,0)$. Analogously $i=0$, $j=0$ gives $f(0,1)$ expressed in $f(0,0)$. Then $P_{1,0}$ may be calculated from (8) and (9). Using (10) consecutively with values $(2,0)$, $(1,1)$ and $(0,2)$ for $(i,j)$, the probabilities $f(2,0)$, $f(1,1)$ and $f(0,2)$ can be expressed in $f(1,0)$ and $f(0,1)$, and hence in $f(0,0)$. Then (8) and (9) yield $P_{2,0}$, etc.

By performing this process numerically we get the quantities $P_i$, which depend, of course, on $P_0$, $P_1$, and $P_2$, but also on $a_1$ and $a_2$. Doing the same process algebraically we are able to demonstrate by rather cumbersome calculations that equations (3) hold for the network $N$. This may be shown for the first step where $i=1$, $j=0$ in (10) leads to

\[
f(1,0) = (a_1+a_2)P_{2,0} = f(0,0).
\]

Furthermore, $i=0$, $j=1$ yields

\[
f(0,1) = (a_1+a_2)P_{2,1} = f(0,0).
\]

Summation gives

\[
\phi(0) = \phi(0)(a_1+a_2)P_{2,0} + \phi(0)(a_1+a_2)P_{2,1} - P_2.
\]

Because $P_2 = P_{1,n_0}P_{2,n_0}$, we have

\[
\phi(0) = \phi(0)(a_1+a_2).
\]

Continuing in the same way, but with much more algebra, it may be shown that

\[
\phi(2) = \phi(0)^2(1-P_2)^2 + \phi(0)(1-P_2)^2
\]

and so on. For $i=0,1,\ldots,n_0$:

\[
P_i = \sum_{j=0}^{\text{Max}(0,i-n)} \phi(n-j,i-j) \phi(n-i,j) P_{n-j,i-j} P_{n-i,j-n+j} P_{2,0}.
\]

Thus the combined network $N$ may be treated as one network with quantities $P_i$ which are calculated from (8), (9) and (10).

4. APPROXIMATION OF THE MEAN OF THE OVERFLOW TRAFFIC

Computer methods for numerical calculation of the moments of the overflow traffic are given in [4] and also [5]. Here we are concerned with practical approximations for the mean $\bar{M}$ and the variance $V$, which are suitable for pocket calculators.

Approximating $M$ is equivalent to approximating $w$ because $M = \bar{w}$.

Now, combining (1) and (2):

\[
w = \phi(n)P_0 + \phi(n-1)P_1 + \cdots + \phi(n)P_n
\]

where

\[
\phi(n) = \left[ \frac{1}{n!} \sum_{k=0}^{n} f(n-k,n-k) P_{n-k,n-k} \right]
\]

When $P_i$ is negligible compared to 1, for $i$ greater than a certain value $m$, $P_i$ may be replaced by 1, yielding

\[
\phi(n) \approx \sum_{i=0}^{n} \phi(n) \left[ \frac{1}{i!} \sum_{n-i}^{n} f(n-i,n-i) P_{n-i,n-i} \right]
\]

The value of $n$ will practically always be at most 5 as stated above. For larger $n$ also may be determined as follows:

Rearranging the terms in (12) we find

\[
\phi(n) = \sum_{i=0}^{n} \phi(n) \left[ \frac{1}{i!} \sum_{n-i}^{n} f(n-i,n-i) P_{n-i,n-i} \right]
\]

The right-hand member of this equation is of the same form as the part within braces in (12), with $n$ replaced by $n-1$, and $P_i$ replaced by $P_{i-1}$. Replacing $\phi(n)$ for the case of $n$ lines and quantities $P_0$, $P_1$, $\ldots$, $P_n$ by $\phi(n^i)$, we may write (14) as

\[
\phi(n) = \sum_{i=0}^{n} \phi(n) \left[ \frac{1}{i!} \sum_{n-i}^{n} f(n-i,n-i) P_{n-i,n-i} \right]
\]

This recurrence relation is a generalization of the recurrence relation for the Erlang loss probability $E_n(a)$.

Putting

\[
\phi(n) = \phi(n) \left[ \frac{1}{i!} \sum_{n-i}^{n} f(n-i,n-i) P_{n-i,n-i} \right]
\]

it may be used to calculate $\phi(n)$ in $n$ steps.

This procedure is essentially the same as the calculation by (13).

The calculation of $\bar{M}$ can be summarized as follows. Compute $\phi(n)$ by (13) or by repeated application of (15).

Determine the form within braces in (11) by taking at most 6 terms, and multiply it by $a(n)$.
5. APPROXIMATION OF THE VARIANCE OF THE OVERFLOW TRAFFIC

In order to approximate $V$ given by (6) we need an approximation of the quantities $M(i)$. They can be calculated exactly by (7) and (8), but that is not our intention here. Using (2) in (7) we get the equations

$$ g(i) = M(i) - M(i-1) - \Psi(i) \quad (i=0,1,\ldots, n) \quad (18) $$

Approximating the right-hand member by

$$ g(i) = \sum_{i=0}^{n} a_i (M(i) - \Psi(i)) \quad (i=0,1,\ldots, n) \quad (19) $$

where $a_i > 0$ is a quantity to be determined below, and putting

$$ g(i) = M(i) - M(i-1) - \Psi(i) \quad (i=0,1,\ldots, n) \quad (18) $$

we get

$$ g(i+1) = \sum_{i=0}^{n} a_i g(i) \quad (i=0,1,\ldots, n) \quad (20) $$

and hence

$$ g(i+1) = \sum_{i=0}^{n} a_i g(i) \quad (i=0,1,\ldots, n) \quad (21) $$

Substituted in (21) this yields

$$ g(0) = a(1-P_0) \quad (22) $$

where

$$ D_k = \sum_{i=0}^{n} a_i \Psi(i) \quad (23) $$

and from (20) follows

$$ M(k) = (a-n-1) D_k + \sum_{i=0}^{n} \Psi(i) \quad (24) $$

Now we may define a probability distribution $\hat{\Psi}(i)$

$$ \hat{\Psi}(i) = \frac{\Psi(i)}{D_k} \quad (i=0,1,\ldots, n) \quad (25) $$

Thus (22) becomes

$$ M(k) = (a-n-1) \frac{\hat{\Psi}(i)}{D_k} + \sum_{i=0}^{n} \hat{\Psi}(i) \quad (26) $$

We now place the requirement on the distribution $\hat{\Psi}(i)$ that in the extreme case of a full availability group (i.e. $P_0 = 1$ and $P_1 = 0$ for $i>0$) we get the right values for $M = M(i)$.

For a full availability group this sum $S$ is equal to $M(n)$ and according to (25)

$$ M(n) = \frac{1}{n+1-a} \sum_{i=0}^{n} \hat{\Psi}(i) \quad (27) $$

This agrees with the exact value if and only if $\hat{\Psi}(i) = \Psi(i)$ for all $i$, but for simplicity we define it.

We thus have the approximation

$$ M(k) = \frac{M(n)}{n+1-a} \sum_{i=0}^{n} \hat{\Psi}(i) \quad (28) $$

and hence $S$ may be approximated by

$$ S = \frac{M(n)}{n+1-a} \sum_{i=0}^{n} \hat{\Psi}(i) \quad (29) $$

This calculation is simple because $\hat{\Psi}(i)$, $\Psi(i)$, etc. have been determined by the calculation of $w$ by means of (6). Furthermore, as the terms of the series decrease rapidly, not all of them have to be calculated.

6. SOME PROPERTIES OF THE APPROXIMATION OF THE OVERFLOW TRAFFIC

We have already seen in chapter 5 that the approximation of $S$ (and hence the approximation of $V$) is equal to the exact value for a full-availability group. The same is valid for the other extreme, i.e. a non-available group with $P_0 = 1$ for all $i$. In that case we have $S = M(n)$, and this has already been included in the approximation by means of (21).

In general $V$ is never underestimated. This may be demonstrated by considering the following identity:

$$ S = \frac{1}{n+1-a} \sum_{i=0}^{n} \hat{\Psi}(i) \quad (30) $$

In deriving the approximation we wrote

$$ H(i+1)-M(i)-\Psi(i) = a \frac{M(i)}{M(i)+1} \Psi(i) \quad (31) $$

and later, because $\hat{\Psi}(i)$ is the highest possible values of $M(i)-H(i)$, $\Psi(i)$, this equation gives

$$ a \frac{M(i)}{M(i)+1} \Psi(i) \leq \hat{\Psi}(i) \quad (32) $$

This means that for all $i$, the highest possible values of $M(i)-H(i)$ have been chosen, which are compatible with the boundary condition

$$ H(M(n)) = M(n)+1+2(M(n-1)+M(n-2))+\ldots+(n+1)(M(n-n)+M(n-n-2))+(n-n+1)(M(n-1)-M(n)) \quad (33) $$

or

$$ \sum_{i=0}^{n} \hat{\Psi}(i) \geq \sum_{i=0}^{n} \hat{\Psi}(i) = \frac{M(n)}{n+1-a} M(n) \quad (34) $$

By (28) we see that $S$ is not underestimated, and hence the same is valid for $V$.

Finally, it is easy to see that by using (27) $V/M$ tends to 1 for $a\to 0$ and for $a\to \infty$, as it should.

For $a\to 0$, namely, (27) gives for $\hat{\Psi}$

$$ \frac{M(n)}{w(n+1-a)} \sum_{i=0}^{n} \hat{\Psi}(i) = \frac{a}{n+1-a} \sum_{i=0}^{n} \hat{\Psi}(i) $$

ITC8

423-3
This form tends to 0 for \( a \to 0 \). Because \( M \to 0 \) too, we see from (6) that \( V/M \to 1 \).

For \( a \to 0 \), (27) gives 
\[
\frac{M}{n+1-a-M} \approx \frac{M}{n+1},
\]
and because the carried load \( a-M \) tends to \( n \) (we exclude the case \( P=1 \), i.e. the non-available group treated previously), this form tends to 
\[
\frac{M}{n+1} = M,
\]
which means in (6) that \( V/M \to 1 \).

7. CONCLUSIONS

This paper deals with link networks and groups of lines connected to them as outlets. A network is characterized by its blocking quantities \( P_i \), the probability that no one of \( i \) free lines of a group is accessible from a given free inlet of the network under given loading conditions. Two possibilities are considered: first, all lines of a group are connected to one network, and second, part of the lines is connected to one network and the other lines to another network. In the second case the networks have quantities \( P_{1i} \) and \( P_{2i} \) respectively and by a random choice for each call it is decided whether the establishment of a connection of an inlet with a free line of the group should be tried first through network no.1 or first through network no.2. It is shown that in this case the two networks together may be seen as one network with quantities \( P_i \), which can be derived from the \( P_{1i} \) and \( P_{2i} \). Considering further only the case that all lines of a group are connected to one network with given quantities \( P_i \), approximations for the mean and the variance of the traffic overflowing from the group of lines are derived. It is assumed throughout that the traffic offered is Poisson traffic.

The blocking probability of the traffic offered is calculated by (11) together with (13), or by (11) together with (15) and (16). By truncation of series a very good numerical approximation of the blocking probability, and hence of the mean of the overflow traffic, may be obtained. The variance of the overflow traffic is given by (6), where the sum is approximated by (27). The approximation thus obtained has the following properties:

a. It is equal to the exact value for the extremes of a full-availability group and a non-available group.

b. The ratio of variance to mean approaches 1 when the traffic offered approaches 0 or \( \infty \).

c. The variance is never underestimated.

REFERENCES


