Behaviour of Overflow Traffic and the Probabilities of Blocking in Simple Gradings

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ABSTRACT

This paper essentially is a condensed and modernised version of some of author's early work which is nearly inaccessible due to war circumstances. A system is studied in which several Poisson groups of sources each dispose of a number of individuals and having overflow to an infinity of commons (better called "secondaries" now, as there are no competing groups of sources). Two formulae are obtained for the distribution function of the number of occupied secondaries in those subsystems. For the original problem a (formal) exact solution is obtained as well as bounds to this solution.

1. INTRODUCTION

The main subject of the present paper is the so-called simple grading, which has been depicted in figure 1. There are a number of sources, each generating demands for service according to Poisson processes with arrival rates $\rho_i$. There are a number of servers, able to help those demands. A successful demand causes a server occupation during the so-called holding time. Non-occupied servers are free. A demand stemming from group i of sources first of all tries to find a free one out of a group of $c_i$ servers (further called individuals). If none is found, the demand tries to find a free one out of $y$ servers, common to all sources (the commons). When no free common is found either, the demand is lost (is said to meet with blocking). It is asked to determine the probabilities $B_i$ of blocking (or loss factors) for the various groups under the assumption of stationarity. The holding times are supposed to possess a negative exponential duration, with average $\mu$. The problem at hand and some generalisations have received considerable attention in the past in the area of telecommunication engineering [18,2,3,6,9,19]. Identical models are to be found recently in the field of file organisation [12,15,16,17]. To the exception of the last section, the present paper is a condensed version of author's 1942 doctoral thesis [10].

2. GENERAL SOLUTION; BOUNDS

Let $r_1, r_2, \ldots, r_m$ denote the steady-state probability of the state with $r_i = r_j (i = 1, \ldots, m)$ occupied individuals of the $m$ groups and $s$ occupied commons. The probabilities satisfy a set of generalised birth-and-death equations, to be obtained in the regular way (cf. e.g. [3,11,19]):

\[
\sum_{i=1}^{m} \left[ c_i r_{r_1, \ldots, r_i-1, \ldots, r_m} + (r_i+1) r_{r_1, \ldots, r_i, \ldots, r_m} \right] = \sum_{i=1}^{m} \frac{\rho_i}{\mu} \left[ \left( r_i \prod_{j=1}^{i-1} r_j \right) + s \right] r_{r_1, \ldots, r_i, s+1} \\
= \left\{ \prod_{i=1}^{m} \left( \frac{\rho_i}{\mu} \right) + s \right\} r_{r_1, \ldots, r_m} \\
= d_i \cdot r_{r_1, \ldots, r_m} \quad (r_i = 0, \ldots, c_i; \sum_{i=1}^{m} r_i = s; \mu > 0, y) \quad (1)
\]

with two sets of "boundary-conditions", expressed as modifications of (1):

- Whenever $r_i = c_i$ for some $i$, the last term between $\mu$ is replaced by $\rho_i r_{r_1, \ldots, c_i, \ldots, r_m, s-1}$.

- Whenever $s = y$ the following modifications are made (apart from those on account of (2)),

\[3.1. \text{The term of the second line of (1) is deleted.} \quad (3)\]

\[3.2. \text{From the last member of (1) coefficients} \quad \rho_i \text{are deleted for all indices} \quad i \text{for which} \quad r_i = c_i. \quad (4)\]

The number of this set of homogeneous equations equals the number of unknowns, viz. $(c_1+1) \ldots (c_m+1)(y+1)$. As the set is simply linearly dependent - total addition results in an identity - they may be solved in conjunction with the fact that the total of all unknowns is unity.

After solution of the equations the probability of blocking $B_i$ for group $i$ and the overall probability $B$ are determined from:

\[B_i = \sum_{r_1=0}^{c_i} \ldots \sum_{r_m=0}^{c_m} \sum_{s=0}^{y} \text{with} \quad r_i = c_i (i = 1, \ldots, m), \quad r_1 + \ldots + r_m + s = y. \quad (4)\]

\[B = \sum_{i=1}^{m} B_i \quad (5)\]

where fat dots denote summation over all possible values, $c_i, y$. The size of the set of equations mostly is $r_i = 1$ prohibitive. As, however, $B_i$-values normally are small, it is a fair guess that the behaviour of the system with commons (to be denoted by upper index (=)) will be indicative of the actual system's behaviour. Indeed

\[B^+ = \sum_{r_1 = 0}^{c_1} \ldots \sum_{r_m = 0}^{c_m} \sum_{s=0}^{y} \text{where} \quad r_i = 1 \quad B^+ \quad (6)\]

is a conservative (upper) bound to $B$. Let $y$ be the number of occupied common commons and $\rho_i = \lambda_i$ that of occupied additional commons, respectively. Then in this system:

\[B_1 = \text{Pr}(r_i = c_i, \lambda \mu = y) \quad (7)\]

\[B_1 = \text{Pr}(r_i = c_i, \lambda (\mu y + y) > y) > B_i \quad (8)\]

which concludes the proof.

It is further to be conjectured that

\[B_1 = \sum_{r_1 = 0}^{c_1} \ldots \sum_{r_m = 0}^{c_m} \ldots \sum_{s=0}^{y} \text{with} \quad r_i = c_i (i = 1, \ldots, m), \quad r_1 + \ldots + r_m + s = y. \quad (9)\]

\[B^+ = \sum_{r_1 = 0}^{c_1} \ldots \sum_{r_m = 0}^{c_m} \sum_{s=0}^{y} \text{is a lower bound, in any way practically so.} \quad (10)\]

In the (=) case the numbers $c_1, \ldots, c_m$ of commons occupied by the various groups are uncorrelated, both unconditional and conditional on $r_1, \ldots, r_m$. The solution for the complete (=) system now is a convolution of solutions $f_{r_1}^{(i)}$ (called "partial solutions") $f_{r_1}^{(i)} \ldots f_{r_m}^{(i)}$ for one-group (=) systems (called sub-systems):

\[f_{r_1, \ldots, r_m} = \prod_{i=1}^{m} f_{r_i}^{(i)} \quad (i = 1, \ldots, m) \quad \text{for m one-group (=) systems called sub-systems.} \quad (11)\]

\[f_{r_1, \ldots, r_m} = \prod_{i=1}^{m} f_{r_i}^{(i)} \quad (i = 1, \ldots, m) \quad \text{for m one-group (=) systems called sub-systems.} \quad (11)\]
These partial solutions satisfy the following set of equations, from which suffixes i to $\phi$, and $c_1$, as well as upper indexes to $f_i^n$ have been omitted "for reasons of simplicity:"

\[
\begin{align*}
df_r^n + r f_r^{n-1} + (a+n) f_r^n &= p \times (r+1) f_r^n \quad (r=0, \ldots, c-1; \ n=0, 1, \ldots, m) \\
df_r^c + p f_c^{c-1} + (a+n+1) f_c^n &= (p+c+1) f_c^n \\
& \quad (s=0, 1, \ldots, m)
\end{align*}
\]

The function $F_r(Y)$ and $f_{rs}$ now are (cf. A3; A4) formally transformed into:

\[
\begin{align*}
F_r(Y) &= \sum_{n=0}^{m} f_r^n \ e^{-nY} \\
f_{rs} &= \sum_{s=0}^{m} f_{rs} \ e^{-sY}
\end{align*}
\]

In the (s) systems (or subsystems) the relations (3) no longer are relevant. They should be replaced by the requirement of "good behaviour" for $s=0$, i.e. finite moments. It can, however, be verified that any convolution (8) of formal solutions of $n$ systems (9,10) formally satisfies the system (1,2). In the subsystems the "commons" will also be called secondaries.

3. PARTIAL SOLUTIONS; BEHAVIOUR OF OVERFLOW TRAFFIC

We now come to solving the systems (9,10). Let the generating function (GF) of $f_{rs}$ with respect to $s$ be denoted by $F_r(Y)$ ($r=1, \ldots, c$). Then (9) and (10) transform into:

\[
\begin{align*}
dF_r^{n+1} + r F_r^n + \frac{dF_r^n}{dy} &= (p+r) F_r^n + \frac{dF_r^n}{dy} \\
& \quad (r=0, 1, \ldots, c-1) \\
dF_r^{c} + p F_r^{c-1} + \frac{dF_r^{c}}{dy} &= (p+c) F_r^{c} + \frac{dF_r^{c}}{dy}
\end{align*}
\]

Assume (11) to be valid also for $r \geq c$ (the added equations determining added unknown functions $F_c, F_{cs}(y)$, etc.). Now let $F_k(x,y)$ be the GF of $F_r(Y)$ with respect to $r$. The translation of (11) is:

\[
\begin{align*}
\frac{dF}{dx} + \frac{dF}{dy} &= \frac{dF}{dy} + p F + \frac{dF}{dy}
\end{align*}
\]

Its general solution is:

\[
\begin{align*}
F(x,y) &= \left(1-x^{-1}\right) e^{-y(1-x)}
\end{align*}
\]

where $K$ is some arbitrary function.

Let us assume the possibility of a series-expansion:

\[
\begin{align*}
F(x,y) &= \sum_{n=0}^{m} \beta_n \left(1-x^{-1}\right)e^{-y(1-x)} \\
F_r(Y) &= \sum_{n=0}^{m} \beta_n \Phi_e^{a+n}(1-y)^{a+n} \\
f_{rs} &= \sum_{s=0}^{m} \beta_{rs} \Phi_{cs}^{a+s}(1-s)^{a+s}
\end{align*}
\]

Comparison of (12) and (11) for $r = c$ yields:

\[
\begin{align*}
\sum_{n=0}^{m} \beta_n (1-x^{-1})e^{-y(1-x)} &= 0 \\
\text{Substitution of (16) into (18) and equating the coefficients of } (1-y)^{a+n} \text{ to zero results in:}
\text{or, using (A7):}
\end{align*}
\]

\[
\begin{align*}
\sum_{n=0}^{m} \beta_n \Phi_e^{a+n-1} &= 0 \\
\sum_{n=0}^{m} \beta_{rs} \Phi_{cs}^{a+s-1} &= 0
\end{align*}
\]

As $\beta_s \neq 0$, $\beta_{rs} \neq 0$, then the denominator should vanish for $n = 0$, which yields the indicial equation:

\[
\omega_c^{a+1} - 1 = 0
\]

The roots of this equation are: (i) $a_1 = 0$, and (ii) the values $a = a_j = n_j-1$, (j=1, ..., c), where the $n_j$ are the $r$ roots of $\phi_c = 0$, which are known to be negative real and different. Each of the $n+1$ roots of the indicial equation engenders a different solution (17), where the coefficients $\beta_r$ follow from (19), apart from a multiplicative constant.

In the case $a = 0$, $F_r(y)$ is analytical in $y = 1$. It is the actual solution for the $(r,c)$ case, contrary to the $c$ aforementioned solutions, which are only formal solutions. From (A8) and (19) it follows (quotient criterium) that for $a = 0$, the series (17) converges absolutely. It is normed by requiring that:

\[
\sum_{n=0}^{m} \beta_n \Phi_e^{a+n} = 0 \\
or: $\beta_0 = 1/\phi_c^n$
\]

The final result is:

\[
\begin{align*}
F_r(Y) &= \sum_{n=0}^{m} \beta_n \Phi_e^{a+n} \\
\text{and:}
\end{align*}
\]

\[
\begin{align*}
f_{rs} &= \sum_{s=0}^{m} \beta_{rs} \Phi_{cs}^{a+s}
\end{align*}
\]

This result may be explained as follows. The quantities to the left and to the right are the absolute rates of transitions $\pm s$ - 1 and $s - 1$ respectively. Under conditions of stationarity these rates should be equal.

4. SIMPLIFICATION OF THE GENERAL SOLUTION

According to section 3 there exist $c+1$ different partial solutions for the subsystems (9,10), $i = 1, \ldots, m$. Hence, a linear aggregate of the $(c+1)$ systems $(9,10)$ (say $N$) possible convolutions (8) will formally satisfy (1,2). This solution still has to satisfy the $N$ relations (1,2) which are modified sub (3). Those $N$ relations are homogeneous. The modifications sub (3) specify for every boundary state $F_1, \ldots, F_m$ the fact that influx from and efflux to the group of states ...; $F_{m+1}$ cancel. This of course entails the global result (addition):

\[
\sum_{n=0}^{m} \beta_n \Phi_e^{a+n} = 0 \\
\text{On the other hand this result already follows by addition from (1,2). Hence, the } N \text{ relations modified sub (3) are linearly dependent. They may be solved for the } N \text{ coefficients of the aggregate but for a multiplicative constant, which results from normalizing. The original set of } N \text{ equations with a sparse matrix has been replaced by a set of } N \text{ equations with a dense matrix. Whether the "simplification" pays depends on circumstances.}
\]

It should be mentioned that Bech [1] has derived another simplification based on matrix methods. It is reported to be capable of handling the problems by the use of computers.
5. VARIANT OF THE SOLUTION OF SECTION 3

As \( q_n \) has \( c \) different zeros \( n=n_1, \ldots, n_c \) the following relation exists:

\[
\frac{1}{\psi_c} = n \sum_{i=1}^{c} A_i \frac{(-p)^k}{k!(n-n_i+k+1)}
\]

where \( A_i \) is the residue in \( n=n_i \). Insertion in (23) yields (the \( n_i \) are negative):

\[
f_{cs} = \psi_c \sum_{i=1}^{c} A_i \int_{0}^{1} t^{n_i} \psi(t) dt = \psi_c \sum_{i=1}^{c} (\Delta)^{n_i} \left\{ \frac{1}{(s-n_i+1,s)} - \frac{1}{(s-n_i+1,a)} \right\}
\]

with:

\[
\psi(t) = e^{-pt} \sum_{i=1}^{c} A_i t^{-n_i}
\]

(39)

The "closed expression" for \( f_{cs} \) with incomplete f-functions is of little value as the available tables [14] necessitate higher order interpolation in coarse double-entry tables!

Repeated partial integration of the last member of (28) yields:

\[
f_{cs} = \psi_c \sum_{i=1}^{c} A_i \int_{0}^{1} t^{n_i} \psi(t) dt = \psi_c \sum_{i=1}^{c} (\Delta)^{n_i} \left\{ \frac{1}{(s-n_i+1,s)} - \frac{1}{(s-n_i+1,a)} \right\}
\]

with:

\[
\psi(t) = e^{-pt} \sum_{i=1}^{c} A_i t^{-n_i}
\]

(39)

The quantity \( \psi(t) \) equals \( z^{c-j-2} \) times a function, analytic in \( z=0 \) (cf. A4). Hence, \( z=0 \) for \( j=0, \ldots, c-2 \); moreover, \( R_{c-1}=0 \) and \( R_c=0 \). Now, (32) may be rewritten as

\[
f_{cs} = \sum_{k=0}^{c} d_k \psi_{c+s+k}
\]

with:

\[
d_k = \left\{ \begin{array}{ll}
1 & (k = 0) \\
(c+1) \cdots (c+k+1) p^k R_{c+k+1} & (k > 0)
\end{array} \right.
\]

The computation of the residues is simple though very tedious. Recursion formulae by the author [10] are useful until the computation of about \( d_9 \). More powerful formulæ by J. Giltay [5] provided an extension upto \( d_9 \). The results (Giltay) may be summarized as follows:

\[
d_k = \sum_{j=1}^{k} e_{kj} (z)^{-j-1}
\]

(40)

(38)

Whereas the series (23) becomes rather intractable for large values of \( s \), the series (38) is better for large \( s \) values.

6. MOMENTS OF \( \psi \) BEHAVIOUR OF OVERFLOW TRAFFIC

The unconditional moments of \( \psi(n) \) (secondaries), which to some extent characterise the overflowing traffic, are obtained in the following way. By summing (21) for \( r=0, \ldots, c \):

\[
F_s(y) = \sum_{n=0}^{c} \psi_c \frac{(s+1-n)!}{s!} \psi_n
\]

(41)

Hence, the integral in the remainder term is absolutely less than \( M_j/n(n+j) \) and the remainder vanishes for \( j>M \). Thus:

\[
\psi_{kij} = \int_{0}^{1} t^{n_i} \psi(t) dt
\]

(39)

When (29) is written as:

\[
\psi(1-x) = \sum_{i=1}^{c} A_i \frac{n_i}{(1-x)^{n_i}}
\]

(33)

we have an account of (A3):

\[
(\Delta)^{n_i} \frac{n_i}{(1-x)^{n_i}} = \sum_{i=1}^{c} A_i \frac{n_i}{(1-x)^{n_i}}
\]

(34)

Now, \( \phi_n \) is an integral function of \( n \) and hence:

\[
(\Delta)^{n_i} \frac{n_i}{(1-x)^{n_i}} = \sum_{i=1}^{c} A_i \frac{n_i}{(1-x)^{n_i}}
\]

(35)

(36)

By expressing the sum as a contour-integral (cf. Knopp [7]) and using the transformation \( z=1/n \) we obtain:

\[
(\Delta)^{n_i} \frac{n_i}{(1-x)^{n_i}} = J/n_j
\]

(37)

The quantity \( \psi(1-x) \) is the well-known Erlang B-formula for the probability of blocking of the group of individuals \( [3,11,19] \). Hence, it equals the density of the point-process of overflowing demands, as was to be expected. The
Finally it results that:
\[ f_0 = \frac{1}{x} \]

and from:
\[ f_1 = \frac{x}{x^2 - 1} \]

we obtain:
\[ \psi_{n+1} = \psi_n - \psi_{n+1} \]

and from:
\[ \frac{d}{dx} \frac{\psi_n}{x^n} = \frac{\psi_n}{x^n} + \frac{\psi_n}{(x^n)^+} \]

it results that:
\[ \psi_{n+1} = \psi_n^+ + \psi_n \]

The result (A6) is useful in constructing tables of \( \psi_r \) for natural values of \( n \).

Finally it follows from (A4) for natural \( n \):
\[ \lim_{n \to \infty} \frac{\psi_{n+1}}{\psi_n} = 1 \]

The equation in \( n \):
\[ \psi_n = 0 \]

is a polynomial equation (cf. A4), yielding \( r \) roots \( n=1, \ldots, n_r \). It can be shown that those roots are negative real. For the lemma's proof it is useful to include in the induction on \( r \) the statements:
(i) the roots \( n_r+1, \ldots, n_r \) are spaced apart by more than unity, and
(ii) the zero of \( \psi_r \) interlace those of \( \psi_r \).

REFERENCES

5. Giltay, J., "Opmerkingen bij een methode voor het berekenen de een door overloopverkeer belaste onbeperkte bundel" (in Dutch), De Ingenieur, 78 (1956) O 95-97.
Figure 1. Simple grading