A Probabilistic Model for Optimization of Telephone Networks

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ABSTRACT

In this paper a probabilistic model for a telephone traffic network is developed. A system of state equations is presented for the basic building block of an alternate routing network and an explicit solution is given. From this model, the blocking probabilities at any network node may be readily obtained. As a consequence, it is possible to formulate an optimization problem for the minimization of the variance of the traffic arriving at the X-tandem (subject to cost and any other required constraints).

NOTATION

D_{ij} \quad j^{th} destination in the Y_j tandem area.
O_{kl} \quad k^{th} origin parented on the k^{th} X-tandem X_k.
\alpha_{ij} \quad Proportion of calls from O_{kl} that are destined for D_{ij}.
\omega_{lj} \quad Number of circuits busy on the direct route from the origin O_{kl} to D_{ij}.
\nu_1 \quad Vector containing all \omega_{lj} in the form \omega_{l1} \omega_{l2} \ldots \omega_{lj} \ldots , \omega_{lY}.
\nu_{Y} \quad Number of busy circuits on the route from O_{kl} to the Y_1 tandem.
\nu_{Y1} \quad Vector containing all \nu_{lj} .
\nu_{X1} \quad Number of busy circuits on the O_{kl}-X_k route.
\nu_{Y1} \quad Number of busy circuits on the X_k-Y_1 route.
\nu_{Y1} \quad Number of busy circuits on the Y_1-D_{ij} route.
\beta_{ij}(\nu_{lj}) \quad Probability of blocking on the direct route to D_{ij} given \omega_{lj} circuits are busy.
\beta_{Y1}(\nu_{Yj}) \quad Probability of blocking on the O_{kl}-Y_1 route given \omega_{lj} circuits are busy.
\beta_{X1}(\nu_{Xj}) \quad Probability of blocking on the O_{kl}-X_k route given \omega_{lj} circuits are busy.
\beta_{Y1}(\nu_{Yj}) \quad Probability of blocking on the X_k-Y_1 route given \omega_{lj} circuits are busy.
\beta_{Y1}(\nu_{Yj}) \quad Probability of blocking on the Y_1-D_{ij} route given \omega_{lj} circuits are busy.
\beta_{Y1}(\nu_{Yj}) \quad Probability that the network is in state \nu, \nu_{Y1}, \nu_{X1}.
\beta_{ij}(\nu_{lj}) \quad Probability P(\mu, \nu_{ij}, \nu_{X1}) with the ij component of \nu equal to \nu_{lj} - 1.
\beta_{ij}(\nu_{lj}) \quad Probability P(\mu, \nu_{ij}, \nu_{X1}) with the ij component of \nu equal to \nu_{lj} + 1.
\beta_{ij}(\nu_{lj}) \quad Probability P(\mu, \nu_{ij}, \nu_{X1}) with the Y_1 component of \nu equal to \nu_{Yj} - 1.
\beta_{ij}(\nu_{lj}) \quad Probability P(\mu, \nu_{ij}, \nu_{X1}) with the Y_1 component of \nu equal to \nu_{Yj} + 1.

INTRODUCTION

A prerequisite for the formulation of any optimization problem is a satisfactory mathematical model. The state equation representation of a traffic system is useful for this purpose if an explicit and unique solution to the state equations can be found. In this paper we present the state equations for the basic alternate routing model shown in fig. 1; an explicit solution to these equations is also given. The approach taken here is based upon the work of Whitaker [1] on multi-server, multi-queue systems.

In fig. 1 the broken lines are intended to indicate the presence of many X and Y tandems and many origins and destinations.

2. THE STATE EQUATIONS

In fig. 1, the node O_{kl} is one of r origins (the l^{th}) parented onto the k^{th} X-tandem X_k. The node D_{ij} is one of s_{ij} destinations (the j^{th}) in the Y_j tandem area. We assume that the network contains a number x of X-tandems and y of Y-tandems.

The traffic originating at O_{kl} destined for D_{ij} takes the direct route to D_{ij} if a line is available; if this route is blocked, it overflows onto the i^{th} Y-tandem route and if this route is also blocked it overflows onto its X-tandem route X_k. If this final route is blocked, the call is lost.

In formulating the mathematical model we assume the following:

(i) The traffic arriving at any origin O_{kl} is Poisson.
(ii) The holding times of calls are distributed exponentially.
(iii) The process is in statistical equilibrium.
(iv) In the infinitesimal time interval dt, no more than one call arrives or is terminated.

\[
\begin{align*}
D_{ij} & \quad j^{th} \text{ destination in the } Y_j \text{ tandem area.} \\
O_{kl} & \quad k^{th} \text{ origin parented on the } k^{th} \text{ X-tandem } X_k. \\
\alpha_{ij} & \quad \text{Proportion of calls from } O_{kl} \text{ that are destined for } D_{ij}. \\
\omega_{lj} & \quad \text{Number of circuits busy on the direct route from the origin } O_{kl} \text{ to } D_{ij}. \\
\nu_1 & \quad \text{Vector containing all } \omega_{lj} \text{ in the form } \omega_{l1} \omega_{l2} \ldots \omega_{lj} \ldots , \omega_{lY}. \\
\nu_Y & \quad \text{Number of busy circuits on the route from } O_{kl} \text{ to the } Y_1 \text{ tandem.} \\
\nu_{Y1} & \quad \text{Vector containing all } \nu_{lj}. \\
\nu_{X1} & \quad \text{Number of busy circuits on the } O_{kl}-X_k \text{ route.} \\
\nu_{Y1} & \quad \text{Number of busy circuits on the } X_k-Y_1 \text{ route.} \\
\nu_{Y1} & \quad \text{Number of busy circuits on the } Y_1-D_{ij} \text{ route.} \\
\beta_{ij}(\nu_{lj}) & \quad \text{Probability of blocking on the direct route to } D_{ij} \text{ given } \omega_{lj} \text{ circuits are busy.} \\
\beta_{Y1}(\nu_{Yj}) & \quad \text{Probability of blocking on the } O_{kl}-Y_1 \text{ route given } \omega_{lj} \text{ circuits are busy.} \\
\beta_{X1}(\nu_{Xj}) & \quad \text{Probability of blocking on the } O_{kl}-X_k \text{ route given } \omega_{lj} \text{ circuits are busy.} \\
\beta_{Y1}(\nu_{Yj}) & \quad \text{Probability of blocking on the } X_k-Y_1 \text{ route given } \omega_{lj} \text{ circuits are busy.} \\
\beta_{Y1}(\nu_{Yj}) & \quad \text{Probability of blocking on the } Y_1-D_{ij} \text{ route given } \omega_{lj} \text{ circuits are busy.} \\
\beta_{Y1}(\nu_{Yj}) & \quad \text{Probability that the network is in state } \nu, \nu_{Y1}, \nu_{X1}. \\
\beta_{ij}(\nu_{lj}) & \quad \text{Probability } P(\mu, \nu_{ij}, \nu_{X1}) \text{ with the } ij \text{ component of } \nu \text{ equal to } \nu_{lj} - 1. \\
\beta_{ij}(\nu_{lj}) & \quad \text{Probability } P(\mu, \nu_{ij}, \nu_{X1}) \text{ with the } ij \text{ component of } \nu \text{ equal to } \nu_{lj} + 1. \\
\beta_{ij}(\nu_{lj}) & \quad \text{Probability } P(\mu, \nu_{ij}, \nu_{X1}) \text{ with the } Y_1 \text{ component of } \nu \text{ equal to } \nu_{Yj} - 1. \\
\beta_{ij}(\nu_{lj}) & \quad \text{Probability } P(\mu, \nu_{ij}, \nu_{X1}) \text{ with the } Y_1 \text{ component of } \nu \text{ equal to } \nu_{Yj} + 1. 
\end{align*}
\]
The state equilibrium equation is written out below and followed by an explanation.

\[ y_{\mathbf{S}1}(1 - \theta_{\mathbf{S}1}(m_{\mathbf{S}1} - 1)) \times \left[ (1 - \theta_{\mathbf{S}1}(m_{\mathbf{S}1} - 1)) \times (1 - \theta_{\mathbf{S}1}(m_{\mathbf{S}1} - 1)) \right] \]

The left hand side of this equation gives the probability of transition out of the state \( \mathbf{S}1, \mathbf{S}2, \mathbf{Y}1 \) and the right hand side gives the probability of transition into this state. Since the terms involved in both sides are similar we give an explanation for one side only.

The first term on the right hand side represents a transition into the state \( \mathbf{S}1, \mathbf{S}2, \mathbf{Y}1 \) due to an arrival and no blockage on the direct routes. The second term represents a transition into \( \mathbf{S}1, \mathbf{S}2, \mathbf{Y}1 \) due to an arrival and no blockage on the \( \mathbf{Y}1 \) routes. The third term represents an arrival and no blockage on the \( X_k \) route.

The remaining three terms represent a transition into state \( \mathbf{S}1, \mathbf{S}2, \mathbf{Y}1 \) due to the termination of a call on the direct routes, \( \mathbf{Y}1 \) routes and \( X_k \) route respectively.

3. SOLUTION OF THE STATE EQUATIONS

Equation (1) describes a Birth and Death process where the number of states is finite. The solution of the equation is therefore unique and describes a genuine probability distribution [2]. It is given by

\[ \left[ \begin{array}{c} y_{\mathbf{S}1} \\ y_{\mathbf{S}2} \\ \vdots \\ y_{\mathbf{Y}1} \\ y_{\mathbf{X}k1} \end{array} \right] = \left[ \begin{array}{c} \frac{1}{m_{\mathbf{S}1}!} \left( m_{\mathbf{S}1} \right)^{\mathbf{S}1} \left( 1 - \theta_{\mathbf{S}1}(m_{\mathbf{S}1} - 1) \right) \left( 1 - \theta_{\mathbf{S}1}(m_{\mathbf{S}1} - 1) \right) \\ \frac{1}{m_{\mathbf{S}2}!} \left( m_{\mathbf{S}2} \right)^{\mathbf{S}2} \left( 1 - \theta_{\mathbf{S}2}(m_{\mathbf{S}2} - 1) \right) \\ \frac{1}{m_{\mathbf{Y}1}!} \left( m_{\mathbf{Y}1} \right)^{\mathbf{Y}1} \left( 1 - \theta_{\mathbf{Y}1}(m_{\mathbf{Y}1} - 1) \right) \\ \frac{1}{m_{\mathbf{X}k1}!} \left( m_{\mathbf{X}k1} \right)^{\mathbf{X}k1} \left( 1 - \theta_{\mathbf{X}k1}(m_{\mathbf{X}k1} - 1) \right) \\ \frac{1}{m_{\mathbf{X}k1}!} \left( m_{\mathbf{X}k1} \right)^{\mathbf{X}k1} \left( 1 - \theta_{\mathbf{X}k1}(m_{\mathbf{X}k1} - 1) \right) \end{array} \right] \times \left[ \begin{array}{c} \frac{1}{m_{\mathbf{S}1}!} \left( m_{\mathbf{S}1} \right)^{\mathbf{S}1} \left( 1 - \theta_{\mathbf{S}1}(m_{\mathbf{S}1} - 1) \right) \left( 1 - \theta_{\mathbf{S}1}(m_{\mathbf{S}1} - 1) \right) \\ \frac{1}{m_{\mathbf{S}2}!} \left( m_{\mathbf{S}2} \right)^{\mathbf{S}2} \left( 1 - \theta_{\mathbf{S}2}(m_{\mathbf{S}2} - 1) \right) \\ \frac{1}{m_{\mathbf{Y}1}!} \left( m_{\mathbf{Y}1} \right)^{\mathbf{Y}1} \left( 1 - \theta_{\mathbf{Y}1}(m_{\mathbf{Y}1} - 1) \right) \\ \frac{1}{m_{\mathbf{X}k1}!} \left( m_{\mathbf{X}k1} \right)^{\mathbf{X}k1} \left( 1 - \theta_{\mathbf{X}k1}(m_{\mathbf{X}k1} - 1) \right) \\ \frac{1}{m_{\mathbf{X}k1}!} \left( m_{\mathbf{X}k1} \right)^{\mathbf{X}k1} \left( 1 - \theta_{\mathbf{X}k1}(m_{\mathbf{X}k1} - 1) \right) \end{array} \right] \]

By taking the appropriate summations we obtain the desired equilibrium equations.

From the marginal probabilities we may also determine the average traffic on each route analysed

\[ E(m_{\mathbf{S}2}) = \sum m_{\mathbf{S}2} P(m_{\mathbf{S}2}) \]
\[ E(m_{\mathbf{Y}1}) = \sum m_{\mathbf{Y}1} P(m_{\mathbf{Y}1}) \]
\[ E(m_{\mathbf{X}k1}) = \sum m_{\mathbf{X}k1} P(m_{\mathbf{X}k1}) \]

4. DETERMINATION OF SIGNIFICANT TRAFFIC CHARACTERISTICS

Now to complete our model we need the probability expressions for the \( X_k-Y_1 \) route and the \( Y_1-D_1 \) route.

Consider first the \( X_k-Y_1 \) route. The proportion of the traffic from \( O_k \) to the \( Y_1 \) tandem area that overflows onto the \( O_k-Y_1 \) route is given by

\[ m_{\mathbf{X}k1} P(m_{\mathbf{X}k1}) \]

We will now show that the steady state solution described by the above relation satisfies the equilibrium equation (2). The following recurrence relations are readily obtained from equation (2):

\[ y_{\mathbf{S}1} = \left[ \begin{array}{c} y_{\mathbf{S}1} \\ y_{\mathbf{S}2} \\ \vdots \\ y_{\mathbf{Y}1} \\ y_{\mathbf{X}k1} \end{array} \right] = \left[ \begin{array}{c} \frac{1}{m_{\mathbf{S}1}!} \left( m_{\mathbf{S}1} \right)^{\mathbf{S}1} \left( 1 - \theta_{\mathbf{S}1}(m_{\mathbf{S}1} - 1) \right) \left( 1 - \theta_{\mathbf{S}1}(m_{\mathbf{S}1} - 1) \right) \\ \frac{1}{m_{\mathbf{S}2}!} \left( m_{\mathbf{S}2} \right)^{\mathbf{S}2} \left( 1 - \theta_{\mathbf{S}2}(m_{\mathbf{S}2} - 1) \right) \\ \frac{1}{m_{\mathbf{Y}1}!} \left( m_{\mathbf{Y}1} \right)^{\mathbf{Y}1} \left( 1 - \theta_{\mathbf{Y}1}(m_{\mathbf{Y}1} - 1) \right) \\ \frac{1}{m_{\mathbf{X}k1}!} \left( m_{\mathbf{X}k1} \right)^{\mathbf{X}k1} \left( 1 - \theta_{\mathbf{X}k1}(m_{\mathbf{X}k1} - 1) \right) \\ \frac{1}{m_{\mathbf{X}k1}!} \left( m_{\mathbf{X}k1} \right)^{\mathbf{X}k1} \left( 1 - \theta_{\mathbf{X}k1}(m_{\mathbf{X}k1} - 1) \right) \end{array} \right] \times \left[ \begin{array}{c} \frac{1}{m_{\mathbf{S}1}!} \left( m_{\mathbf{S}1} \right)^{\mathbf{S}1} \left( 1 - \theta_{\mathbf{S}1}(m_{\mathbf{S}1} - 1) \right) \left( 1 - \theta_{\mathbf{S}1}(m_{\mathbf{S}1} - 1) \right) \\ \frac{1}{m_{\mathbf{S}2}!} \left( m_{\mathbf{S}2} \right)^{\mathbf{S}2} \left( 1 - \theta_{\mathbf{S}2}(m_{\mathbf{S}2} - 1) \right) \\ \frac{1}{m_{\mathbf{Y}1}!} \left( m_{\mathbf{Y}1} \right)^{\mathbf{Y}1} \left( 1 - \theta_{\mathbf{Y}1}(m_{\mathbf{Y}1} - 1) \right) \\ \frac{1}{m_{\mathbf{X}k1}!} \left( m_{\mathbf{X}k1} \right)^{\mathbf{X}k1} \left( 1 - \theta_{\mathbf{X}k1}(m_{\mathbf{X}k1} - 1) \right) \\ \frac{1}{m_{\mathbf{X}k1}!} \left( m_{\mathbf{X}k1} \right)^{\mathbf{X}k1} \left( 1 - \theta_{\mathbf{X}k1}(m_{\mathbf{X}k1} - 1) \right) \end{array} \right] \]
Thus the average arrival at the Xk tandem of the traffic destined to the Y1 tandem is equal to

\[ A_{XkY1} = [a(k1)]^2 \cdot a(k1) + \cdots + a(k1)^r \]

where r is the number of origins.

Once again assuming statistical equilibrium, no time correlation, and that probability of more than one arrival in time dt may be ignored, we may write the probability \( P(m_{XkY1}) \) of the number of calls on the Xk-Y1 route as follows

\[
P(m_{XkY1}) = \frac{m_{XkY1}^{m_{XkY1}-1}}{m_{XkY1}!} \left( 1 - \beta_{XkY1}(m_{XkY1}) \right) P(0)
\]

where \( P(0) \) is determined by the normalizing condition

\[
\sum_{m_{XkY1}=0} P(m_{XkY1}) = 1
\]

 Analysing the Y1-Dij route we obtain the following.

The proportion of traffic from Oij-Dij that reaches Y1 from Xk is given by

\[ a_{ij}B_{ij}B_{Y1}(1 - B_{Y1})(1 - B_{Yk1}) \]

The proportion of the traffic reaching Y1 from the Oki-Y1 route is given by

\[ a_{ij}B_{ij}(1 - B_{ij}) \]

... the traffic arriving at Y1 from Oki and destined for Dij is given by

\[ a_{ij}B_{ij}B_{Y1}(1 - B_{ij})(1 - B_{Yk1}) + (1 - B_{ij}) \]

Thus we have an expression for traffic reaching Y1 from all X tandems and the total arriving at Y1 and destined for Dij is given by

\[ a_{ij}B_{ij} \]

The probability \( P(m_{1j}) \) of the system being in state \( m_{1j} \) is then

\[
P(m_{1j}) = \begin{cases} \frac{m_{1j}^{m_{1j}-1}}{m_{1j}!} \left( 1 - \beta(t) \right) P(0) & \text{if } m_{1j} \geq 0 \\ 0 & \text{otherwise} \end{cases}
\]

subject to the normalising condition

\[
\sum_{m_{1j}} P(m_{1j}) = 1
\]

The blocking probability and average traffic for these routes may be found in exactly the same fashion as those in equations (4) and (5).

For example, the average blocking in the route between Y1 and Dij is given by

\[
\sum_{m_{1j}} B_{ij}(m_{1j})P(m_{1j})
\]

5. SYSTEM OPTIMIZATION

Now that we have all the significant probabilistic information for the system, we are in a position to formulate an optimization problem. For high efficiency, one would require maximum utilization of the final route. Consequently a reasonable approach would appear to involve minimisation of the variance of traffic arriving at each X-tandem. This approach is simplified through the fact that a unique set of origins is parented to each X-tandem. This allows us to choose the sum of the variances of the traffic arriving at all the X-tandems as the objective function in the optimization problem.

The primary constraint on the problem is cost. A second constraint that we impose is on the grade of service at each destination; we require the blocking probability of calls to a destination Dij in the network to be some predetermined value.

The programming of this approach is now under consideration and it is hoped that some preliminary results will be available at the congress.

CONCLUDING REMARKS

In this paper, we have presented a general-purpose model for a metropolitan telephone network. This model should prove useful as a design aid and with appropriate extensions should be valuable in the allocation of direct routes, switching availabilities and service protection routes.

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REFERENCES


MODIFICATIONS TO THE MODEL DISCUSSED IN PAPER 516

NOTATIONAL MODIFICATION:

\( A_{kl, ij} \) Traffic offered to origin \( O_k \) and destined for destination \( D_{ij} \)

Similar meaning extends to \( B, P, m \).

MODIFICATION TO THE MODEL:

Instead of considering three routes and then deriving the probabilities for the last two links based on the assumption of no correlation we may write the state equations for each link.

Thus for the Direct Route \( O_k \rightarrow D_{ij} \)

\[
A_{kl, ij} \left\{ 1 - B_{kl, ij} \left( m_{kl, ij} \right) \right\} + m_{kl, ij} P(m_{kl, ij})
\]

\[
= A_{kl, ij} \left\{ 1 - B_{kl, ij} \left( m_{kl, ij} - 1 \right) \right\} P(m_{kl, ij} - 1)
\]

\[
+ (m_{kl, ij} + 1) P(m_{kl, ij} + 1)
\]

Similarly we may write the state equations for each link. The method of solution and proof follows the one given in the paper.

CORRECTION: Equation (2) page 516-2 should be multiplied by \( P(\bar{0}) \)