General Telecommunications Traffic Without Delay

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ABSTRACT
It is intended to demonstrate how the general formula relating to telephone traffic processes without waiting may be obtained. The lost call and repeat attempt models are considered, essentially for the case of an arbitrary service time distribution. The explicit formulae are then extremely complicated when the arrival process is general. They become simple only for a certain category of process which are called here "pseudo-poisonian". This simplicity takes the form of a simple distribution function for the "remaining service times" of the "indistinguishable" calls in progress, a distribution already well known for the Erlang model.

Two cases are then examined for which the preceding results may be utilized. The dependences of the non-independence of the arrival process in relation to the epochs of system congestion: firstly, the case of overflow traffic and then that of repeat attempts. The consequences of the dependences are not the same in the two cases.

1. INTRODUCTION

At the preceding International Teletraffic Congress [cf. Ref. 6] the author introduced a presentation of the application of stochastic integrals to the definition of the formalism relating to telephone traffic processes without waiting, under the most general hypothesis. This study has since been pursued for the lost calls cleared model [cf. Ref. 2 and 8]. The principal results obtained, relative to a single full availability group of circuits are resumed in Section 3. The case of exponentially distributed service times will not be considered here since this case is already well known and it presents the danger of obscuring the real difficulties which arise when one wishes to tackle the usual case of service time distributions "with memory".

The case considered is therefore, essentially, that of an arbitrary service time distribution. If the arrival process is also arbitrary, the formulae describing the process rapidly become extraordinarily complicated as the number of circuits increases, even in the case of stationarity. It will be shown in Section 4, however, that these formulae remain simple for a certain category of arrival processes which will be called "pseudo-poisonian processes". For such a process it is possible to obtain a result concerning the distribution of the "remaining service times" which is already known for the Erlang model. The possibilities offered by the use of this distribution have already been indicated in Ref. 6. The conditions which must be fulfilled by the arrival process were not, however, completely specified and this led to the incorrect application of the distribution to regenerative arrival processes which are not in fact "pseudo-poisonian".

The use of this distribution requires the independence of the arrival process in relation to the epochs of system congestion. It would be interesting to know what happens when this independence is not satisfied.

Firstly, in Section 5, the case of overflow traffic is examined and it is concluded that the global formulae (at stationarity) are independent of the service time distribution when the traffic offered to the first choice group is poissonian. This result is contrary to that which was indicated in Ref. 4. This derives from the fact that it was not noticed that the above mentioned simple distribution function for the "remaining service times" is only applicable to "indistinguishable communications".

In Section 6 the much more difficult case of the repeat attempt model is considered, for a single full availability group. The treatment is limited to the hypothesis of constant parameters of the calling subscriber. It will be indicated that for this model, at stationarity, the simple distribution of the "remaining service times" remains valid for the calls served at the first attempt. On the other hand, the formulae for the traffic corresponding to repeat attempts are the more modified, the more rapidly are the refused calls repeated. This property highlights the interest of theoretical studies in this domain, despite the simplified results which can be obtained for very large repetition intervals; cf. Ref. 3 and 4.

Section 2 presents a recapitulation of the hypotheses and notations already used in Ref. 3 and 4.

2. HYPOTHESES AND NOTATIONS

2.1 Only the case of traffic offered to a single full availability group is considered. It was shown in Ref. 6 how the results obtained for this, the simplest case, can be used to study networks (on a purely theoretical level).

2.2 For the sake of simplicity, the system is supposed empty at time zero.

2.3 The calls are assumed to be set-up instantaneously so that the holding time of a circuit corresponds to the service time, which, in the present case, is the duration of the communication.

2.4 CALL ARRIVAL PROCESS

The arrival process is assumed to be independent of the epochs of system congestion. The number of sources is therefore assumed to be unlimited. Multiple arrivals are supposed inextinct. This process is defined by the random variable N(t), the number of arrivals in the time interval (0, t]. It is analogous to a successive, non-simultaneous point process \( \{ N(t), t \geq 0 \} \) defined on the real line. This process is completely defined by the knowledge of the probability of arrival of an event in each of the elementary sub-intervals \( (t_i, t_{i+1}) \), \( i = 1, 2, \ldots, n \); \( n \) a positive integer. This probability may be represented by the symbol

\[
E \left[ e^{0 \cdot N(t_2) \ldots N(t_m)} \right] \quad (1)
\]

where \( E \) is the expectation operator. The random function \( 1_{(t_1)} \) takes the value 1 or 0 according to whether a call arrives or does not arrive in the interval \( (t_1, t_{1+1}) \).

The \( n \)th factorial moment of \( N(t) \) is given by

\[
M_n = \int_0^t s^n e^{-s} ds N(s) \quad 0 \leq s \leq t \quad (2)
\]

For any \( m \) and finite \( t \), this quantity is supposed finite corresponding to the fact that the possibility of explosive productions of calls has been eliminated.

It will also be useful to consider the process consisting of \( n \) distinct arrivals

\[
N(t) = (N(t) - 1)_+ N(t) - (N(t) - 1)_+ \quad (3)
\]

\[
\int_0^t e^{0 \cdot N(t_2) \ldots N(t_m)} ds
\]

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where \( dn(t_1), dn(t_2) = 0 \) if \( t_1 = t_2 \).

In Ref. 7, formula (2) the following stochastic relation was indicated:
\[
\frac{d}{dt} \sum_{n=1}^{\infty} \left( e^{2n} a \right)^n \int_0^t \! dn(a,\epsilon) \! \int_0^t \! dn(a_m) \\
(4)
\]
where \( a \) is any other variable such that \( \{ a = 1 \} \).

The stationary Poisson process corresponds to the case where the \( dn(t) \) are mutually independent and such that:
\[
E \left( \frac{dn(t)}{dt} \right)^2 = \lambda, \\
(5)
\]
being the arrival density.

2.5 DISTRIBUTION OF SERVICE TIMES

In the most general case, the random variable \( T_U \), representing the duration of a communication, depends on the epoch \( u \) at which the call was established. In other words, \( T_U \) depends on the past history of the system. Introduce the random function \( R(u,t) \) defined by:
\[
R(u,t) = \begin{cases} 
1 & \text{if } u < u + T_u \\
0 & \text{if } u < u + T_u \text{ or } u > u + T_u 
\end{cases} \\
(6)
\]
Thus, \( R(u,t) \) is equal to 1 if the communication, starting at the epoch \( u \), is still in progress at the epoch \( t \), and is equal to 0 otherwise.

Under the usually considered hypothesis of independence of the communication time \( T_U \) in relation to \( u \) and the other communications, a hypothesis which will be specified when it is used here, the distribution function of the holding time will be designated by \( F(\Theta) \). In this case:
\[
E \left( R(u,t) \right) = 1 - F(t-u) \\
(7)
\]
Under this latter hypothesis, the mean communication time is taken as the unit of time. Thus:
\[
\int_0^t dF(\Theta) = t = \int_0^t [1-F(\Theta)] d\Theta \\
(8)
\]

2.6 STOCHASTIC INTEGRALS

By way of an example, consider an infinite group of circuits. There is clearly no blocking. The random function \( Y(t) \), representing the number of communications in progress at the epoch \( t \), is the number of calls, \( k \), which arrive in the interval \( (0,t) \), are served and last still at the epoch \( t \). \( Y(t) \) is given by the stochastic relation:
\[
Y(t) = \int_0^t R(u,t) \! dn(u) \\
(9)
\]
This representation, using a stochastic integral, was first introduced by R. FORST in Ref.

2.7 NUMBER OF COMMUNICATIONS IN PROGRESS

For the usual case of a finite group, let \( L \) be the number of circuits. Let \( N_0(t) \) be the number of arrivals in the interval \( (0,t) \) which are served (or are being served). The stochastic relation (9) then becomes:
\[
Y(t) = \int_0^t R(u,t) \! dn(u) \\
(10)
\]
It is convenient to introduce the random function:
\[
\begin{align*}
Y(t) &= \int_0^t R(u,t) \! dn(u) \\
(11)
\end{align*}
\]
In the interval \( (0,t) \), \( R(u,t) \! dn(u) \) is a point process which also satisfies (4), from which may be deduced the stochastic relation:
\[
\frac{d}{dt} \sum_{n=1}^{\infty} \left( e^{2n} a \right)^n \int_0^t \! dn(a,\epsilon) \! \int_0^t \! dn(a_m) \\
(4)
\]
where \( a \) is any other variable such that \( \{ a = 1 \} \).

It is also useful to introduce the random function \( Z(k,t) \), which is equal to 1 or 0 if, at the epoch \( t \), the number of calls in progress is equal to \( k \) or not. Then, \( Z \) satisfies the stochastic relation:
\[
\frac{d}{dt} \sum_{n=1}^{\infty} \left( e^{2n} a \right)^n \int_0^t \! dn(a,\epsilon) \! \int_0^t \! dn(a_m) \\
(4)
\]

Comparison of (12) and (13) gives:
\[
\frac{d}{dt} \sum_{n=1}^{\infty} \left( e^{2n} a \right)^n \int_0^t \! dn(a,\epsilon) \! \int_0^t \! dn(a_m) \\
(4)
\]
In particular, for \( Y = L \):
\[
X(L,t) = Z(L,t) \\
(15)
\]
Let:
\[
E \left[ \frac{d}{dt} \sum_{k=1}^{\infty} \left( e^{2k} a \right)^n \int_0^t \! dn(a,\epsilon) \! \int_0^t \! dn(a_m) \\
(4)
\]

P \( k,t \) is therefore the probability that \( k \) communications are in progress at the instant \( t \) and the \( S(V,t) \) are the binomial moments of the distribution \( P(k,t) \).

The probability that a call arrives and is blocked at the instant \( t \), in the lost call model, is then:
\[
\frac{d}{dt} \sum_{k=1}^{\infty} \left( e^{2k} a \right)^n \int_0^t \! dn(a,\epsilon) \! \int_0^t \! dn(a_m) \\
(4)
\]

3. LOST CALL MODEL

3.1 In this section and in sections 4 and 5, attention is restricted to a full availability group of circuits, offering traffic directly and operating according to the lost calls cleared model.

It is useful at this stage to introduce the following random function which is more general than (11):
\[
X(o,\theta) = 1 \\
X(V,\theta; t_1, t_2, ..., t_V) = \prod_{\epsilon=1}^{V} \int_0^{t_\epsilon} R(a_{\epsilon}, t_\epsilon) \! dn(a) \\
(19)
\]
for \( V = 1, 2, ..., L \). The usefulness stems from the fact that the quantity
\[
\frac{d}{dt}. E \left[ X(V,\theta; t_1, t_2, ..., t_V) \right] \\
(20)
\]
represents the probability that \( V \) specific circuits are busy at the epoch \( \theta \) and that the \( 1 \) th circuit is still busy at the epoch \( t_i \) (\( i = 1, 2, ..., V \)). Obviously, it is assumed that the order of selection is random so that there is a symmetry among the states of occupation.

For an infinite group, the expression (19) is denoted by:
\[
X_{\infty}(V,\theta; t_1, t_2, ..., t_V) \\
(21)
\]
In the following, some results which were derived in Ref. 8 will just be stated.

3.2 GENERAL EQUATIONS

Consider (19) with \( V = L \) and \( t_1 = \theta \) for \( i > \lambda \).

In this case, (19) will be written:
\[
X(\lambda, \theta; t_\lambda, ..., t_\lambda | L-\lambda, \theta, \theta, \theta) \\
(23)
\]
Let:
\[
\frac{d}{dt}. E \left[ X(\lambda, \theta; t_\lambda, ..., t_\lambda | L-\lambda, \theta, \theta, \theta) \right] \\
(24)
\]
The basic equation for \( Y = 1, \ldots, L \), may be written:

\[
\left\{ \begin{array}{l}
\mathbf{X}(\nu, \theta) \cdot \mathbf{t}_s \ldots \mathbf{t}_v = \mathbf{X}(\nu, \theta) \cdot \mathbf{t}_s \ldots \mathbf{t}_v \\
- \left( \frac{1}{\nu} \right) \mathbf{X}(\nu, \theta) \cdot \mathbf{t}_s \ldots \mathbf{t}_v \end{array} \right.
\]

where \( \mathbf{t}_s \ldots \mathbf{t}_v \) represents the set \( \{ t_1, \ldots, t_v \} \) with the extraction of \( t_s \). It may be deduced by induction that for \( Y = 1, \ldots, L \):

\[
\mathbf{X}(\nu, \theta) \cdot \mathbf{t}_s \ldots \mathbf{t}_v = \mathbf{X}_\infty(\nu, \theta) \cdot \mathbf{t}_s \ldots \mathbf{t}_v
\]

The summation \( \sum_{\nu} \) is defined as follows: a given subset \( \nu \) of \( \nu \) of the terms \( \{ t_1, \ldots, t_v \} \) is denoted by \( \{ t_1, \ldots, t_v \} \) and the remaining terms by \( \{ t_{v+1}, \ldots, t_n \} \); the summation is taken over all possible subsets. In the multiple stochastic integral,

\[
\nu = \min \{ \nu_1, \ldots, \nu_{\lambda} \} \quad \text{(25)}
\]

The expression (24) defines the stochastic behaviour of the system if (21) is known. This is given by the solution of the following stochastic integral equation which corresponds to (24) for \( Y = 1 \):

\[
\mathbf{X}(\lambda, \theta) \cdot \mathbf{t}_s \ldots \mathbf{t}_v = \mathbf{X}_\infty(\lambda, \theta) \cdot \mathbf{t}_s \ldots \mathbf{t}_v \\
- \left( \frac{1}{\lambda} \right) \mathbf{X}(\lambda, \theta) \cdot \mathbf{t}_s \ldots \mathbf{t}_v
\]

The summation \( \sum_{\lambda} \) is defined as follows: a given subset \( \lambda \) of \( \lambda \) of the terms \( \{ t_1, \ldots, t_v \} \) is denoted by \( \{ t_1, \ldots, t_v \} \) and the remaining terms by \( \{ t_{v+1}, \ldots, t_n \} \); the summation is taken over all possible subsets. In the multiple stochastic integral,

\[
\mathbf{X}_0(\lambda, \theta) \cdot \mathbf{t}_s \ldots \mathbf{t}_v = \mathbf{X}_\infty(\lambda, \theta) \cdot \mathbf{t}_s \ldots \mathbf{t}_v \\
- \left( \frac{1}{\lambda} \right) \mathbf{X}(\lambda, \theta) \cdot \mathbf{t}_s \ldots \mathbf{t}_v
\]

The summation \( \sum_{\lambda} \) is defined as follows: a given subset \( \lambda \) of \( \lambda \) of the terms \( \{ t_1, \ldots, t_v \} \) is denoted by \( \{ t_1, \ldots, t_v \} \) and the remaining terms by \( \{ t_{v+1}, \ldots, t_n \} \); the summation is taken over all possible subsets. In the multiple stochastic integral,

\[
\mathbf{X}(\lambda, \theta) \cdot \mathbf{t}_s \ldots \mathbf{t}_v = \mathbf{X}_\infty(\lambda, \theta) \cdot \mathbf{t}_s \ldots \mathbf{t}_v \\
- \left( \frac{1}{\lambda} \right) \mathbf{X}(\lambda, \theta) \cdot \mathbf{t}_s \ldots \mathbf{t}_v
\]

The summation \( \sum_{\lambda} \) is defined as follows: a given subset \( \lambda \) of \( \lambda \) of the terms \( \{ t_1, \ldots, t_v \} \) is denoted by \( \{ t_1, \ldots, t_v \} \) and the remaining terms by \( \{ t_{v+1}, \ldots, t_n \} \); the summation is taken over all possible subsets. In the multiple stochastic integral,

\[
\mathbf{X}(\lambda, \theta) \cdot \mathbf{t}_s \ldots \mathbf{t}_v = \mathbf{X}_\infty(\lambda, \theta) \cdot \mathbf{t}_s \ldots \mathbf{t}_v \\
- \left( \frac{1}{\lambda} \right) \mathbf{X}(\lambda, \theta) \cdot \mathbf{t}_s \ldots \mathbf{t}_v
\]

The summation \( \sum_{\lambda} \) is defined as follows: a given subset \( \lambda \) of \( \lambda \) of the terms \( \{ t_1, \ldots, t_v \} \) is denoted by \( \{ t_1, \ldots, t_v \} \) and the remaining terms by \( \{ t_{v+1}, \ldots, t_n \} \); the summation is taken over all possible subsets. In the multiple stochastic integral,

\[
\mathbf{X}(\lambda, \theta) \cdot \mathbf{t}_s \ldots \mathbf{t}_v = \mathbf{X}_\infty(\lambda, \theta) \cdot \mathbf{t}_s \ldots \mathbf{t}_v \\
- \left( \frac{1}{\lambda} \right) \mathbf{X}(\lambda, \theta) \cdot \mathbf{t}_s \ldots \mathbf{t}_v
\]

The summation \( \sum_{\lambda} \) is defined as follows: a given subset \( \lambda \) of \( \lambda \) of the terms \( \{ t_1, \ldots, t_v \} \) is denoted by \( \{ t_1, \ldots, t_v \} \) and the remaining terms by \( \{ t_{v+1}, \ldots, t_n \} \); the summation is taken over all possible subsets. In the multiple stochastic integral,

\[
\mathbf{X}(\lambda, \theta) \cdot \mathbf{t}_s \ldots \mathbf{t}_v = \mathbf{X}_\infty(\lambda, \theta) \cdot \mathbf{t}_s \ldots \mathbf{t}_v \\
- \left( \frac{1}{\lambda} \right) \mathbf{X}(\lambda, \theta) \cdot \mathbf{t}_s \ldots \mathbf{t}_v
\]

The summation \( \sum_{\lambda} \) is defined as follows: a given subset \( \lambda \) of \( \lambda \) of the terms \( \{ t_1, \ldots, t_v \} \) is denoted by \( \{ t_1, \ldots, t_v \} \) and the remaining terms by \( \{ t_{v+1}, \ldots, t_n \} \); the summation is taken over all possible subsets. In the multiple stochastic integral,

\[
\mathbf{X}(\lambda, \theta) \cdot \mathbf{t}_s \ldots \mathbf{t}_v = \mathbf{X}_\infty(\lambda, \theta) \cdot \mathbf{t}_s \ldots \mathbf{t}_v \\
- \left( \frac{1}{\lambda} \right) \mathbf{X}(\lambda, \theta) \cdot \mathbf{t}_s \ldots \mathbf{t}_v
\]

The summation \( \sum_{\lambda} \) is defined as follows: a given subset \( \lambda \) of \( \lambda \) of the terms \( \{ t_1, \ldots, t_v \} \) is denoted by \( \{ t_1, \ldots, t_v \} \) and the remaining terms by \( \{ t_{v+1}, \ldots, t_n \} \); the summation is taken over all possible subsets. In the multiple stochastic integral,
The expressions describing the process for just two circuits are already exceptionally complicated and it is evident that for general $l$ they become yet more so. For a general treatment of this case the reader is directed to Ref. 8.

Even when the arrival process is a renewal process and only the stationary limit is considered, the expressions remain extremely complicated. The equations for this latter case were established by F. Pollaczek in Ref. 8, but it is not difficult to understand why they have not yet been solved for arbitrary distributions $F(t)$ and $F_2(t)$.

The difficulty arises, in the main part, from the fact that, for a given sequence of served arrivals $u_i$, the probabilities in (19) depend in general on the permutation of the epochs $t_i$ considered. In otherwords, if each $u_i$ must be associated to a particular $t_i$ (i.e., if the communications are required to be distinguishable) the distribution of "remaining service times" $t_i - 	heta_i$ depends on the permutation $t_i$ considered. This difficulty does not arise if the holding time distribution is exponential because of the no-memory property. Unfortunately, the use of this distribution cannot reasonably be justified when the call setting-up process is long and complex as it can be in large modern networks or when the conversation time is influenced by the charging intervals.

In Ref. 6 certain convergence theorems were used to obtain a simple result which considerably simplified the calculations. This result was, however, wrongly applied to renewal arrival processes which, as was demonstrated above, do not in fact lead to a significant simplification of the general formulae. The class of arrival processes for which the convergence theorems are applicable is determined in the following section.

4. PSEUDO-POISSONIAN ARRIVAL PROCESSES

Theorems 2 and 3 of Ref. 6 require the existence of the limit of expression (19):

$$\lim_{\theta \to \infty} E\{v(\theta; \{t_1, \ldots, t_p\})\},$$

the "remaining service times" $t_i - \theta_i$, being fixed. In fact these two theorems were derived from theorems 7 and 8 of the appendix on supposing, more exactly, the existence of the following limit:

$$\lim_{\theta \to \infty} \frac{1}{\theta} \int_0^\infty \left\{ \sum_{i=1}^p R(u_i, t_i) dN_0(u_i) \right\},$$

the "remaining service times" $t_i - \theta_i$ being fixed. In general this function and consequently its limit depend on the quantities $(\theta_i - \theta_{i-1})$ which are not necessarily bounded.

It is necessary to determine the conditions under which the limit is independent of $(\theta_i - \theta_{i-1})$.

This problem is considered in Ref. 8. It proves necessary to assume that the call holding times are independent of their starting epochs and of the arrival process. The holding times are also assumed to be mutually independent with the same distribution function $F(t)$. In these circumstances the random function $R(u,t)$ may be written as $R(t-u)$. It is shown in Ref. 8 that, providing the limits "almost certainly" exist (i.e., they exist with probability one), then for a very large $n$:

$$\int_0^t R(t-u) dN_0(u) \bigg| \int_0^t R(t-u) dN_0(u)$$

It is then possible to state the following theorem.

THEOREM 4.1 (key theorem)

The holding times are superposed independent of their starting epochs and of the arrival process. They are also assumed to be mutually independent and have the same arbitrary (and therefore not exponential) distribution function. If the epochs $\theta_i$ (i=1, ..., p) increase unboundedly and independently while the "remaining service times" $(t_i - \theta_i)$ remain fixed, then in order that the following limit almost certainly exists

$$\lim_{\theta \to \infty} \frac{1}{\theta} \int_0^\infty \left\{ \sum_{i=1}^p R(u_i, t_i) dN_0(u_i) \right\},$$

it is necessary and sufficient that the following limit almost certainly exists

$$\lim_{\theta \to \infty} \frac{1}{\theta} \int_0^\infty \left\{ \sum_{i=1}^p R(t_i - \theta_i) dN_0(u_i) \right\},$$

and is a random variable independent of the fixed positive quantities $\theta_i$ (i=1, ..., p).

This theorem, applied to random functions, is a generalisation of the theorem of D. Blackwell applied to the first factorial moment in the case of renewal processes.

Setting $h_i = -\Delta \theta_i$ leads to the following corollary.

COROLLARY 4.2 (pseudo-poissonian processes)

The condition (39) is equivalent to the condition

$$\lim_{\theta \to \infty} \frac{1}{\theta} \int_0^\infty \left\{ \sum_{i=1}^p R(u_i, t_i) dN_0(u_i) \right\} = K(\theta),$$

where the process $dN(u)$ is poissonian and the $dN(u_i)$ are independent of the random variable $K(\theta)$.

DEFINITION

Under the assumptions of theorem 4.1 the arrival processes, satisfying (40) for any positive integer $n$, will be called pseudo-poissonian. For such processes it is no longer necessary to order the arrival instants.

Under the conditions of the above theorem, consider (19) and let

$$\lim_{\theta \to \infty} \frac{1}{\theta} \int_0^\infty \left\{ \sum_{i=1}^p R(u_i, t_i) dN_0(u_i) \right\} = X(\theta),$$

On replacing $R(u_i, t_i)$ by the notation $R(\theta_i - u_i)$ the following corollary may be deduced.

COROLLARY 4.3 (Independence of the "remaining service times")

Under the conditions of theorem 4.1, and therefore only for pseudo-poissonian carried traffic, the limit (38) may be expressed:

$$X(\theta) = \lim_{\theta \to \infty} \left\{ \sum_{i=1}^p R(u_i, t_i) dN_0(u_i) \right\},$$

on condition that the mean holding time is taken as the unit of time and the communications are considered to be indistinguishable.

The hypothesis of indistinguishability is fundamental. If, on the contrary, the sequence $t_i \theta_i$ is associated with a given permutation $u_i$, expression (19) becomes:

$$\lim_{\theta \to \infty} \frac{1}{\theta} \int_0^\infty \left\{ \sum_{i=1}^p R(u_i, t_i) dN_0(u_i) \right\},$$

The limit corresponding to (38) no longer has the same value and, in particular, the preceding corollary no longer applies.
Corollary (4.3) has previously been demonstrated (for expectations) in the case of an Erlang carried traffic [cf. Ref. 14] although the necessity for indistinguishability was not noted.

Consider then a pseudo-poissonian traffic offered for which the communications are indistinguishable. Let \( Z(k) \) be the "almost certain" limit of the random function \( Z(k,t) \) introduced in Section (2.7). In other words, \( Z(k) \) is a random variable which, in the stationary limit, takes the values 1 or 0 if, at any given epoch, the number of calls in progress is equal to \( k \) or not. Let \( X_{\infty} (v) \) be the limit corresponding to (41) for an infinite group of circuits. From equations (24) and (26) the following theorem is derived in Ref. 8.

**THEOREM 4.4 (pseudo-poissonian traffic)**

a) A traffic is called pseudo-poissonian if it satisfies the assumptions and conditions of theorem 4.1 and the communications are indistinguishable.

b) If the traffic offered to a group of \( L \) circuits is pseudo-poissonian then

\[
\begin{align*}
Z(k) &= X_{\infty} (k) - \sum_{\ell = 1}^{L} X_{\infty} (k+\ell) - \sum_{\mu \neq 1}^{L} \left( \sum_{\ell = 1}^{L} X_{\infty} (k+\ell+\mu) - \sum_{\ell = 1}^{L} X_{\infty} (k+\ell) \right) \\
&= \cdots
\end{align*}
\]

(4.4)

For \( k > 1 \).

c) The traffic carried is then also pseudo-poissonian and expression (44) is independent of the service time distribution.

Adapting the calculation rule

\[
X_{\infty} (k_{1}) \cdot X_{\infty} (k_{2}) \rightarrow \frac{L_{1} + L_{2}}{M_{1} + M_{2}} X_{\infty} \left( \frac{L_{1} + L_{2}}{M_{1} + M_{2}} \right)
\]

(4.5)

the stochastic expression (44) may be considered as the development in series of the symbolic expression

\[
Z(k) = X_{\infty} (k) + \sum_{\mu \neq 1}^{L} X_{\infty} (\mu)
\]

(4.6)

In the same conditions, the following corollary may be deduced.

**COROLLARY 4.5**

Under the conditions of theorem 4.4, the pseudo-poissonian traffic carried is defined by the random variable \( K(v) \) of expression (49). In symbolic notation, this is given by:

\[
K(v) = \sum_{\lambda = 0}^{L} X_{\infty} (\lambda) \left( \nu_{0} = 1, \ldots, L \right)
\]

(4.7)

and, in addition (in symbolic notation):

\[
X (v) = K (v), \quad X_{\infty} (v) \quad (v = 1, \ldots, L)
\]

(48)

All these symbolic expressions are generalisations of the usual Erlang model.

The above results may easily be generalised to include symmetric networks by the use of theorem 5 of Ref 6 (or theorem 5.1 of Ref 7).

Thus, in a symmetric network offered diverse poissonian traffics, the currents of carried traffic are pseudo-poissonian (at stationarity). This fact demonstrates the usefulness of the development of the above concepts which yield results which have a much greater potential for practical application than those given in Section 3 for arbitrary processes.

The properties of traffic flow in networks may be further enlightened by the use of another very useful theorem. Note that, for two pseudo-poissonian traffics which are not necessarily independent, the product of their respective expressions (40) is also of the same type in which the resultant random variable \( K(v) \) is, however, the product of two random variables which are not necessarily independent. Recall also that, in (38) and (39), it is assumed that

\[
d_{0} (u_{i}) \cdot d_{0} (u_{j}) = 0 \quad {\text{if}} \quad u_{i} = u_{j}
\]

The following theorem may now be deduced.

**THEOREM 4.6 (Sum of pseudo-poissonian traffics)**

The sum or linear combination of two, not necessarily independent, pseudo-poissonian traffics is also pseudo-poissonian.

It is important to note that in Sections 3 and 4 the arrival process is assumed to be independent of the system state and its congestion epochs. It is now proposed to study a first case where this condition is not satisfied, the case of overflow traffic.

5. OVERFLOW TRAFFIC (Lost call model)

The traffic offered to the first choice route is assumed to be poissonian.

It may be noted that the overflow traffic is the difference of two pseudo-poissonian traffics: that carried by the combined group of first choice and overflow circuits and that carried by the first choice group alone. By theorem (4.6) it follows that the overflow traffic is also pseudo-poissonian. While the traffic is not independent of that on the first choice group, it follows from corollary (4.1), applied simultaneously to these two pseudo-poissonian traffics (in the stationary regime) that the joint occupancy distribution is independent of the service time distribution. By repeated application of theorem (4.6) it may be deduced that, in a hierarchical network where diverse high usage routes overflow successively onto diverse overflow routes, the various carried traffics are all pseudo-poissonian and the occupancy distributions are independent of the service time distribution. The above is summarised in the following theorem.

**THEOREM 5.1 (Hierarchical networks)**

a) As for "symmetric" networks, in the stationary regime, if a "hierarchical" network is offered pseudo-poissonian traffics, the "direct" and "overflow" carried traffics are also pseudo-poissonian.

b) The occupancy distributions are then independent of the service time distribution.

This leads immediately to the following corollary.

**COROLLARY 5.2**

a) For both symmetric and hierarchical networks offered poissonian traffics, the occupancy distributions in the stationary regime are independent of the service time distribution.

b) These distributions can be determined from a markovian model in which the service time distribution is taken to be exponential (but the process of successive overflow instants is not the same).

These results might be found surprising, but it should be remembered that they rely on the assumption that the
communications are indistinguishable at least in terms of their starting epochs.

If on the contrary, these starting epochs are distinguished, the properties are much more complex. For example, if the service times are exponentially distributed, it is known that the process of overflow instants is a renewal process. It is then more convenient to use a Poissonian [but the property (42) is still satisfied]. For an arbitrary service time distribution, F. Pollaczek has shown in Ref.10 that the process of successive overflow instants is extremely complex and difficult to study. In fact, by using the above theorem it has been possible to obtain simple results on condition that the overflow instants are not ordered.

For a constant service time, P.J. Burke found in Ref.2 that the occupation distribution in the stationary regime differs from that obtained for an exponential service time distribution. This divergence from the results proved here may be explained by the fact that, in using Takacs' theorem on the distribution of the "remaining service times", Burke uses properties of combinatorial analysis which suppose the communications to be distinguishable.

In the next section, the much more difficult case of traffic with repeat attempts is tackled.

6. REPEAT ATTEMPT MODEL

6.1 HYPOTHESES AND NOTATIONS

a) For the purpose of understanding the properties of telephone traffic with repeated attempts, it is sufficient to consider the simple theoretical case of traffic offered to a full availability group of \( L \) circuits. The model is that already described in Ref.4. An intended call or "call intent" is characterised by its first attempt which will be called here, a "fresh call". As was noted in Ref.3, the type of distribution function considered for the repetition intervals relative to the same call intent has little influence on the results. For the sake of simplicity, therefore, the repetition intervals will be taken to be exponentially distributed with mean \( \mu \).

b) The influence of repetitions of calls falling due to "no-answer" by the called party is in general, much less than that of repetitions following blocking. The influence often only concerns the first two attempts and can be considered as corresponding to a well defined traffic integrated to the normal traffic offered. The treatment of the present simplified model will therefore be limited to the case of repetitions due uniquely to blocking in the considered circuit group. The perseverence rate \( H \) will be supposed constant and therefore independent of the number of previous failures of a call intent. Clearly, \( 0 < H \leq 1 \).

c) As the chosen distribution for the repetition intervals is Poissonian, it is sufficient to characterise the system as in Section 3 for an arbitrarily fresh arrival process \( dN(t) \) with just the additional specification of the random number of calls.\( \llbracket \llbracket \), in a state of repetition (not yet served) at the epoch \( t \). It is not necessary to specify the previous history of these calls (with the epochs of their successive failures) and this leads to a considerable simplification. Let \( X(\{m; \mu, \nu, \theta; t_1, t_2\}) \) be the random function which takes the value (19) if the number of calls in repetition is \( m \) (at the epoch \( t \)) and equals 0 otherwise. It will in fact be more convenient to use the random function

\[
X_{\mu}(\nu, \theta; t_1, t_2, \ldots, t_p) = \sum_{m, r_0} \binom{m}{r_0} X(m; \mu, \nu, \theta; t_1, t_2) \quad (49)
\]

The latter expression is of the same kind as (14). For example, if the number \( m \) could not be greater than a fixed bound \( N \), the quantity

\[
\frac{\frac{d}{dN} \left[ \binom{m}{r_0} X(m; \mu, \nu, \theta; t_1, t_2) \right]}{\left[ \binom{m}{r_0} X(m; \mu, \nu, \theta; t_1, t_2) \right]} \quad E \left[ X_{\mu}(\nu, \theta; t_1, t_2, \ldots, t_p) \right]
\]

would represent the probability of there being \( \mu \) specific calls in a state of repetition and \( V \) specific circuits busy at the epoch \( \theta \), the \( i \)th circuit still being busy at the epoch \( t_i \) (\( i = 1, \ldots, V \)). The \( \mu \)th binomial moment of the random number of calls in a state of repetition at the epoch \( \theta \) has the value \( X_{\mu}(0, \theta) \).

d) In the following, the symbol \( n(t; c) \) will designate the Poisson arrival process of density \( C \). For example, the event corresponding to the random function (49) and to the attempt (at the epoch \( \theta \)) of one of the considered \( \mu \) specific calls in the state of repetition at the epoch \( \theta \), has the random function

\[
dN(\theta; \lambda) \cdot \mu X_{\mu}(\nu, \theta; t_1, \ldots, t_p) \quad (50)
\]

since, because of the exponential distribution of repetition intervals, the production of repeat attempts at a given instant is a Poisson process.

e) Consider a random duration \( T \) having the exponential distribution with mean (\( \frac{1}{\lambda} \)). Let \( P(t; \phi) \) be the random variable equal to 1 if \( T \geq t \) and 0 if \( T < t \).

f) Similarly, introduce the random variable \( T \) which is equal to 1 if a caller decides to repeat following an unsuccessful attempt and equal to 0 otherwise.

g) In analogy with (22), let:

\[
\sum_{r_0} U_r(\nu, \theta; t_1, \ldots, t_p) = X_{\mu}(\nu, \theta; t_1, \ldots, t_p)
\]

This random function corresponds to the event of expression (49) where, however, the probability of blocking at the epoch \( \theta \) is excluded.

6.2 GENERAL EQUATIONS

Let \( \Psi \) be the most recent epoch (earlier than \( \theta \)) which contributed to the value of the random function (49). This epoch corresponds to either

- the arrival of a call (repeated or not) which was served,
- or the arrival of a call (repeated or not) which was refused but will be repeated with probability \( H \).

In fact, if only the calls still lasting at the epochs \( t_j \) are retained, the contributions to (49) can come from the following four eventualities:

a) Served "fresh" call finding the system in state \( [k] \):

\[
\frac{\lambda}{L} \sum_{t_j} \left[ U_r(\nu, \theta; t_1, \ldots, t_p) \right] \cdot \left[ R(\nu, \theta; t_1, \ldots, t_p) \right] \cdot dN(\nu(t_j)) \quad (52)
\]

where \( (t_1, \ldots, t_p) \) is the set \( (t_1, \ldots, t_p) \) with the exclusion of \( t_j \). It is worth recalling that the arrival process of fresh calls \( dN(\nu) \) is arbitrary and therefore not necessarily Poissonian.

b) Served repeat attempt among the \( (\mu + 1) \) specific calls considered at the epoch \( \theta' \):

\[
\frac{\lambda}{L} \sum_{t_j} \left[ U_r(\nu, \theta; t_1, \ldots, t_p) \right] \cdot \left[ R(\nu, \{t_1, \ldots, t_p\}) \right] \cdot dN(\nu(t_j)) \quad (53)
\]

c) non-served "fresh" call which will be repeated, the number of specific call intents in state of repetition considered being \( (\mu + 1) \):

\[
\frac{\lambda}{L} \sum_{t_j} \left[ U_r(\nu, \theta; t_1, \ldots, t_p) \right] \cdot \left[ R(\nu, \{t_1, \ldots, t_p\}) \right] \cdot dN(\nu(t_j)) \quad (54)
\]

d) non-served repeat attempt which will be repeated, the number of calls in state of repetition therefore remaining unchanged at the epoch \( \theta' \):

\[
\frac{\lambda}{L} \sum_{t_j} \left[ U_r(\nu, \theta; t_1, \ldots, t_p) \right] \cdot \left[ R(\nu, \{t_1, \ldots, t_p\}) \right] \cdot dN(\nu(t_j)) \quad (55)
\]

In order that these four eventualities be taken into consideration in (49), it is necessary that the \( \mu \) specific call intents which are then in the state of repetition at \( \theta' \) still are in this state at the epoch \( \theta \). It is therefore necessary to multiply the preceding random functions by the random variable \( P((\theta') - \theta; \mu) \).

Finally, the general equation of the system for \( \nu > 0,1, \ldots, L \) and \( \mu \geq 0 \) is:
The following comments may now be made. Firstly, there is no difficulty in extending the properties deduced here to the "symmetric" networks considered in theorem 5 of Ref. 6 or theorem 5.1 of Ref. 7. It is sufficient to introduce the vector
\[ \mathbf{z}(\theta) = \left[ -\cdots - \mathbf{X}_{\mathbf{e}_{m}} \mathbf{r}_{m} \mathbf{e}_{m} \mathbf{r}_{m} \mathbf{e}_{m} \mathbf{r}_{m} \mathbf{e}_{m} \cdots \right] \]
made up of the random functions of the diverse currents of traffic. Generalisation of the same form as (57) is then obtained. Further, the call setting-up time may now be assumed to be non-zero. The resulting "ineffective" traffic, which loads the network, may be considered as supplementary traffic currents that satisfy the assumptions of symmetry and do not perturb the general properties of the other currents of traffic since the considered processes \( \mathrm{dN}(\theta) \) are general. Thus, the following theorem applies.

**Theorem 6.1 (Nature of traffic with repeat attempts)**

a) For a "symmetric" network in which: currents of "fresh" calls arrive according to arbitrary processes; failing calls are repeated with a certain persistence rate at exponentially distributed intervals; and the call holding time distribution is arbitrary; the diverse currents of traffic may be considered as the sums of the traffic due to the "fresh" calls with the lost call hypothesis, \( T_1 \), and the more complex traffic due to repeat attempts whether carried or not, \( T_2 \).

b) If the arrival processes of "fresh" calls are pseudo-Poisson, the traffic \( T_1 \) satisfies corollary (4.3) and expression (42) on the independence of the "remaining service times". This is not so for the traffic \( T_2 \), especially if it is required to evaluate the number of calls waiting in the state of repetition simultaneously.

To derive other simple properties, further treatment will be restricted to consideration of the stationary limit.

### 6.3 Stationary Limit

In this case it is possible to uncover new simple properties on the repetition traffic if the first call attempts are separated from the repeat attempts in another way. The traffic \( T_2 \) now becomes the traffic \( T_4 \) for which the right hand side of (57) is
\[ \left( \frac{\theta}{\theta - \mathbf{e}_{m}} \right) \mathbf{x}_{\mathbf{e}_{m}} \left( \mathbf{1}_{\mathbf{e}_{m}} \mathbf{x}_{\mathbf{e}_{m}} \right) - \left( \frac{\theta}{\theta - \mathbf{e}_{m}} \right) \mathbf{X}_{\mathbf{e}_{m}} \mathbf{r}_{m} \mathbf{e}_{m} \mathbf{r}_{m} \mathbf{e}_{m} \mathbf{r}_{m} \cdots \mathbf{dN}(\theta) \]
\[ \mathbf{dN}(\theta) \]

If the mean repetition interval was taken as the unit of time, its influence (through \( \lambda \)) would disappear from the equations (57) relative to \( T_4 \). To obtain the solution, it is sufficient to consider the case \( \lambda \) very small, the callers taking a very long time to repeat. From the study undertaken in Ref. 9, it is known that the arrival epochs of second and greater order call attempts form a Poisson process \( \mathbf{dN}(\theta) \), independent of the congestion epochs and of density (for a particular traffic current):
\[ A = \alpha \frac{\mathbf{H}_P}{\mathbf{1} - \mathbf{H}_P} \]
where \( \alpha \) is the arrival density of "fresh" calls which arrive according to the new stationary but otherwise arbitrary, process \( \mathbf{dN}(\theta) \). \( \mathbf{P} \) is the probability of congestion at any given instant.

Note that the random variable \( m_1 \), giving the number of calls in the state of repetition can be considered as a Poissonian traffic where the arrival distribution is exponential with mean \( \frac{1}{\lambda} \). Thus \( m_1 \) is the Poisson distribution with mean \( \frac{1}{\lambda} \) (for arbitrary \( \lambda \)).

\[ \bar{m}_1 = \frac{A}{\lambda} = \frac{A}{\lambda} \left( \frac{\mathbf{H}_P}{\mathbf{1} - \mathbf{H}_P} \right) \]

Without additional complication, the treatment may be extended to the nodal of the network in which the persistence term \( \mathbf{H}_P \) is replaced by the BHN function \( \mathbf{H}(\mathbf{\lambda}) \) giving the persistence rate at the \( m_1 \) attempt. Assume further that the mean repetition interval between attempts \( x \) and \( \left( k+1 \right) \)
In Ref's 3 and 4 it was not possible to uncover the supplementary variable $l$, which does not depend on $H(x)$ for $x > 1$. It is, however, the only one to subsist when calls are repeated very quickly.

6.4 MEAN NUMBER OF ATTEMPTS PER "CALL INTENT" AND THE PROBABILITY OF ABANDONING

Consider, for the moment, the process in which the calls which abandon are held waiting in the state of repetition until the following attempt. They are assumed to be abandoned just before this new attempt. Denote by $m(x)$ the mean number of calls in the state of repetition (at any given instant and which have made $x$ attempts) for this new process. Then, just before the $(x+1)$th attempt:

- Arrival density of repeat attempts: $\lambda_{\infty} H(x)$. $m(x)$
- Density of abandoned attempts: $\lambda_{\infty} [H(x)-m(x)]$.

The old process would have given the same value:
- Arrival density of repeat attempts: $\lambda_{\infty} H(x)$
- Density of abandoned attempts: $\lambda_{\infty} [H(x)-m(x)]$.

Theorem 6.3 (mean number of attempts per "call intent" and probability of abandoning)

In the conditions of theorem 6.2 and for a given current of traffic in the considered "symmetric" network, the mean number of attempts per call intent is:

$$\beta \equiv \frac{1}{\alpha} \lambda_{\infty} + \sum_{x=1}^{\infty} \frac{H(x)}{H(x)} \cdot P \left( -x \right)$$

(60)

In the same conditions the probability of abandoning is:

$$\tau = \sum_{x=1}^{\infty} \frac{H(x)}{H(x)} \cdot P \left( -x \right)$$

(74)
or alternatively, the probability of ultimate success is:

$$d - \tau = \left( d - P \right) \left( \beta - \frac{1}{\alpha} \lambda_{\infty} \right)$$

(72)

6.5 THE EFFICIENCY RATE

In Ref's 4 and 5 was introduced the notion of the efficiency rate $r$, a parameter easily observed:

$$r = \frac{\text{number of effective attempts}}{\beta}$$

An attempt is said to be effective if the answer signal (in automatic working) is received by the originating exchange. We deduce

COROLLARY 6.4 (Efficiency rate)

In the conditions of Theorems 6.2 and 6.3, the efficiency rate for the considered current of traffic is:

$$\eta = \left( d - P \right) \left( d - \frac{\lambda_{\infty}}{\beta} \right)$$

(74)

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