Some Applications to Telephone Traffic Theory
Based on Functional Limit Laws for Cumulative Processes

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ABSTRACT

Cumulative processes appear frequently in traffic theory. If regeneration points in time can be chosen so, that the intervals between those points are i.i.d. r.v.'s. and the behaviour of the increments in the process over these intervals also are i.i.d., then we can call the process cumulative. Let e.g. the beginning of each congestion period be a regeneration point, then the total time congestion in \( (0,t] \) can be studied as a cumulative process, \( w(t) \).

Some results can be achieved from functional limit theorems for \( w(t) \).

To apply those theorems some constants have to be known. In simple cases they can be calculated, but in more complicated situations simulation is more useful. To illustrate the special technique and the advantages of having the results in a functional form, we have chosen some simple processes, where the norming constants are known. Functionals, stopping times and especially the random change of time method will be discussed. Applications to scanning processes are also given.

THEORETICAL BACKGROUND

The theoretical background of this article will be some functional limit theorems proved in \([5]\). These limit theorems concern cumulative processes in the sense of Smith, \([8]\) and \([9]\).

A process \( w(t), t \geq 0 \) is said to be cumulative, if

\[
S_0 = 0, S_1 = \sum_{j=1}^{i} \zeta_j \text{ are regeneration points and}
\]

\[ w(S_i) - w(S_{i-1}) \text{ are i.i.d. random vectors, } P(\zeta_j = 0) = 0 \]

\[ w(t) = \int |dw(t)| \text{ is finite w.p.1 for } 0 \leq t < \infty \]

\[ w(S_i) - w(S_{i-1}) \text{ are i.i.d. r.v.'s.} \]

Put \( w(0) = 0 \) for simplicity.

Define

\[
\begin{align*}
\kappa &= E(w(S_1) - w(S_{i-1})) = Ew(S_1) \\
\zeta_1 &= Ew(S_1), \zeta_2 = E(w^2(S_1)) \\
\mu &= \zeta_1, \nu_2 = \zeta_2^2
\end{align*}
\]

and

\[
\begin{align*}
\Delta &= E(w(S_i) - w(S_{i-1}) - \frac{\kappa}{\mu}(S_i - S_{i-1}))^2 \\
\Sigma &= E(w(\zeta_1) - \frac{\kappa}{\mu} \zeta_1)^2
\end{align*}
\]

The following theorems are proved in \([5]\).

Theorem 1 If \( \zeta < \infty, \kappa < \infty \) then

\[
\sup_{0 \leq t \leq 1} \frac{|w(nt)|}{n} = \frac{\kappa}{\mu} t \rightarrow 0 \text{ as } n \rightarrow \infty.
\]

Theorem 2 If \( \zeta = \infty, \zeta_2 < \infty \) and \( \Delta < \infty \) then

\[
X_n(t) = \frac{w(nt) - n\kappa t}{\sqrt{\Delta n/\mu}} \rightarrow W_0,
\]

where \( W_0 \) denotes weak convergence in the Skorokhod \( J_1 \)-topology in the space \( D[0,1] \), and \( W \) is the Wiener measure (see \([1]\)). Compare also \([7]\).

The special case \( t = 1 \) and \( n = u \) yields Smith's theorem in \([9]\),

\[
\frac{w(u) - u/\mu}{(\Delta u/\mu)^{1/2}} \rightarrow Z \text{ as } u \rightarrow \infty.
\]

Here \( \Rightarrow \) denotes weak convergence in the Skorokhod \( J_1 \)-topology in the space \( D[0,1] \), and \( \rightarrow \) denotes convergence in distribution.

However the functional form of the result, has advantages. Corresponding results for different kinds of stopping times can be achieved. For the ordinary first passage time,

\[
\alpha(c) = \inf(t > 0 : w(t) > c)
\]

we have the following result.

Theorem 3. Assume that \( \kappa > 0 \) and that the conditions of Theorem 2 are satisfied. If

\[
Y_n(t) = \frac{w(nt) - n\kappa nt}{(\Delta n/\mu)^{1/2}}, 0 \leq t \leq 1,
\]

then

\[
Y_n \Rightarrow W \text{ as } n \rightarrow \infty.
\]
The most important reason for proving the convergence in the form of Theorem 2 is that it implies convergence in distribution for continuous functionals of \( X_n \) to the corresponding functional of the limit element. An example is the result,

\[
\sup_{0 \leq t \leq 1} X_n(t) \overset{d}{\rightarrow} \sup_{0 \leq t \leq 1} W(t) \quad \text{as } n \to \infty, \tag{2}
\]

where \( X_n(t) \) is defined in Theorem 2.

Thus

\[
P(\sup_{0 \leq t \leq 1} \frac{w(nt)-\xi nt}{(\Delta n/\mu)^{1/2}} \leq x) \rightarrow \int_0^x e^{-u^2/2} du \tag{3}
\]

as \( n \to \infty \).

The limit distribution can be achieved either as the distribution of

\[
\sup_{0 \leq t \leq 1} W(t), \quad \text{or as the limit distribution of}
\]

\[
\sup_{0 \leq t \leq 1} X_n(t), \quad \text{where in } X_n(t) \text{ a simple case of}
\]

\( w(t) \) is used, e.g. the sum of independent r.v.'s taking the values \( \pm 1 \) with probability \( 1/2 \) each. This so called invariance principle is discussed in [1], where also a number of other interesting functionals are treated.

Next we apply random change of time to a simple process appearing in traffic theory.

**A SIMPLE EXAMPLE FROM TELETRAFFIC THEORY**

Consider a lost call system with \( r \) available devices. Let the arrival process be a Poisson process with intensity \( \lambda \) and let the occupation times be exponentially distributed with mean one, for simplicity. Let \( w(t) \) denote the number of lost calls in the "time"-interval \( (0,t] \).

Sometimes it is practical to measure time in the number of arriving calls. Then by "time" \( t \) we mean the point where \( \lceil t \rceil \) calls have arrived and the proportion \( t-\lfloor t \rfloor \) of the interval in seconds between call \( \lceil t \rceil \) and \( \lceil t \rceil +1 \) has passed.

Now \( w(t) \) will satisfy the conditions of a cumulative process and the regeneration points will be at the beginning of each congestion period. With notations from above we get

\[
u = \lambda / r + 1/E_r - 1 = \lambda / r E_r^{1/2}
\]

and

\[
\kappa = \lambda / r
\]

so

\[
\kappa / \mu = E_r^{1/2}.
\]

Here \( \mu \) is the mean "time" (number of calls) between two adjacent regeneration points, and \( \kappa \) is the mean number of lost calls during that time. The corresponding \( \Delta \) can also be calculated for each \( r \). Results from [6], p. 84-85 can be used.

Now we are ready to apply Theorem 2.

Set

\[
X_n(t) = \frac{w(nt)}{(\Delta n/\mu)^{1/2}} \overset{d}{\rightarrow} 0 \leq t \leq 1,
\]

then

\[
X_n \Longrightarrow W. \tag{4}
\]

We can put \( w_0(t) = w(t)-E_r t \) and also write (4) in terms of \( w_0(t) \).

There is no essential difference to prove (4) in the space \( D[0,c] \), for any \( c>0 \) and thus (4) holds in the extended space \( D[0,\infty) \). For the rest of this section let \( \Rightarrow \) denote weak convergence in \( D[0,\infty) \). We want a result with time measured in ordinary time. Let \( N(t) \) be the number of calls arriving in \( (0,t] \) (t is measured in ordinary time)

Define

\[
\phi_n(t) = \frac{N(nt)}{n}, \quad t \geq 0
\]

and

\[
\phi(t) = At, \quad t \geq 0.
\]

Now

\[
\sup_{0 \leq t \leq c} \left( \frac{N(nt)}{n} - At \right) \overset{P}{\longrightarrow} 0 \quad \text{as } n \to \infty
\]

for any \( c>0 \), and thus

\[
P \phi \overset{\Rightarrow}{\longrightarrow} \phi \quad \text{in } D[0,\infty).
\]

If \( X_n \Rightarrow X \) and \( \phi \Rightarrow \phi \) it follows from [1], Theorem 4.4 that

\[
(X_n, \phi_n) \Rightarrow (X, \phi).
\]

From [10] Theorem 3.1 we get that composition is continuous in this case i.e.

\[
(X_n, \phi_n) \Rightarrow (X, \phi)
\]

implies that

\[
X_n \phi_n \Rightarrow X \phi.
\]

( \( X \phi(t) = X(\phi(t)) \) )

Therefore it follows that

\[
X_n \phi_n \Rightarrow W \phi,
\]

where

\[
W \phi(t) = W(At) = A^{1/2} W(t).
\]
Thus, with
\[ Z_n(t) = \frac{w(N(nt)) - E_n N(nt)}{(\Delta n/\mu)^{1/2}} = X_n \phi_n(t) \]
we have
\[ Z_n \Rightarrow A^{1/2} W, \text{in } D[0,\infty). \]
In particular (5) holds in the space \( D[0,1] \). From a result for \( w_0(t) \), the value of a process after \( [t] \) calls, we have got a result for \( w_0(N(t)) \), the value of the same process after \( t \) units of time. Furthermore note that the limit element changes by a factor \( A^{1/2} \).

SOME FURTHER EXAMPLES
Once we have discussed reasons for having limit theorems in a functional form, we shall list some further applications of Theorem 2.

Define
\[
\begin{align*}
   w_1(t) &= \text{the amount of time at least one call is waiting in a } M|M|r \text{ queueing system,} \\
   w_2(t) &= \text{the total busy time in a } M|M|r \text{ queueing system,} \\
   w_3(t) &= \text{the total busy time in a } M|M|r \text{ lost call system,}
\end{align*}
\]
taken over the interval \((0,t]\).
The mean service time is one.

Set
\[
\begin{align*}
   X_n(t) &= \frac{w_1(nt) - D_r(A)nt}{\sigma_1 n^{1/2}}, \\
   X_n(t) &= \frac{w_2(nt) - D_r(A)nt}{\sigma_2 n^{1/2}}, \\
   X_n(t) &= \frac{w_3(nt) - E_r(A)nt}{\sigma_3 n^{1/2}} \quad 0 \leq t \leq 1.
\end{align*}
\]
In all these cases the regeneration points are chosen at the beginning of a new period of the kind that the process is measuring.

Note that \( \xi_2 < \xi_2 = \) in all cases.

Thus
\[
\begin{align*}
   X_n \Rightarrow W \text{ as } n \to \infty, \quad i=1,2,3.
\end{align*}
\]
where
\[
\begin{align*}
   \sigma_i^2 &= \frac{2AD_r^3}{(rE_r)^2} + \frac{A^2}{r^2} (r-A)D_r^3 E_r^2 (r-A)D_r^3 E_r^2 \quad (8)
\end{align*}
\]
\[
\begin{align*}
   \sigma_3^2 &= rE_r^3 E_r^2 \quad (9)
\end{align*}
\]
\[
\begin{align*}
   (D_r - D_r(A)) = E_r(A).
\end{align*}
\]
Here \( X_2 \) is the length of the non busy period, so
\[
\begin{align*}
   E_r^2 X_2^2 = 2(r-1)! \frac{r-1}{A(E_r)^2} \frac{A}{1} \frac{A}{1} \frac{1}{E_r^2} \quad (10)
\end{align*}
\]
The list can be much longer. In the following section some scanning processes will be discussed.

RANDOM SCANNING
Now, consider the traffic process discussed before, starting with a congestion period at \( t=0 \), with ordinary time scale. \((M|M|r, \text{ lost calls})\). Scanning takes place at points generated by a Poisson process, with intensity \( \gamma \), which is independent of the traffic process. Let \( U(t) \) be the number of scans registering congestion in the interval \((0,t]\). Then \( U(t) \) is a cumulative process with regeneration points at the beginning of each congestion period. We use the following notations.

\[
\begin{align*}
   Y &= \text{the number of scans during a congestion period} \\
   X_1 &= \text{the length of a congestion period} \\
   X_2 &= \text{the length of a non busy period.}
\end{align*}
\]

\( X = X_1 + X_2 \).

We note that \( U(t) \) is increasing and that the second moments of \( X \) and \( Y \) are finite. Now with notations from above we have
\[
\begin{align*}
   \kappa &= EY = \gamma/r \quad (11) \\
   \mu &= 1/rEY, \quad \kappa/\mu = \gamma EY^{-1} \quad (12)
\end{align*}
\]
Furthermore
\[
\begin{align*}
   \Delta &= E(Y - \frac{E}{\mu} X_1)^2 \\
   &= E(Y - \frac{E}{\mu} \left(X_1 + E X_2\right))^2 \frac{E}{\mu} \text{Var } X_2, \\
   &\text{since } X_2 \text{ is independent of } Y \text{ and } X_1.
\end{align*}
\]
Hence,
\[
\begin{align*}
   \Delta - \frac{E}{\mu} \text{Var } X_2 &= E(Y - \kappa/EY X_1 - 1/r)^2 \quad (10)
\end{align*}
\]
From \( E(X_1 Y|X_1) = X_1^2 Y \) it follows
\[
\begin{align*}
   E(X_1 Y) &= 2 \frac{EY}{r^2} \text{ i.e.} \\
   \text{Cov (Y, X_1)} &= \frac{EY}{r^2}. 
\end{align*}
\]
Since $Y$ is geometrically distributed it follows that

$$\text{Var} Y = \frac{Y}{Y} (1+Y/r),$$

and since

$$\text{Var}(X_1) = 1/r^2,$$

the right hand side of (10) becomes

$$\frac{Y}{Y} \frac{Y}{Y} \frac{Y}{Y} \frac{Y}{Y} - \frac{Y}{Y} - \frac{Y}{Y} \frac{Y}{Y} \frac{Y}{Y} .$$

Therefore

$$\Delta/\mu = E(r_{Y}+\frac{1}{r}(1-E_{r})^{2})y^{2}E_{r}(EX_{r}^{2}-(\frac{1-EX_{r}}{rE_{r}})^{2}) \text{ (11)}$$

Thus Theorem 2 can be applied to $U(t)$ with

$$\kappa/\mu = \gamma E_{r}$$

and $\Delta/\mu$ given by (11).

Note. This result of random scanning can also be used in another situation. Let $U(t)$ be the number of lost calls in $(0,t)$ and set $\gamma = A$. If we use a scanning intensity $\gamma = A$ and only count scans during congestion periods, it is possible to use the call-process as scanning process, since the length of the congestion period is independent of the calls during that period. Thus elements converge to $W$ under

$$\text{U}(t)$$

converge to $W$ under $J_1$, as $n \to \infty$,

where

$$U(t) = \text{number of lost calls in } (0,t]$$

and

$$\sigma^2 = \text{AE}(r_{Y}+A^2E_{r}^{2}EX_{r}^{2}). \text{ (12)}$$

We shall now consider another process,

$$W(t) = (1-E_{r})U(t)-E_{r}V(t) = U(t)-E_{r}(U(t)+V(t)),$$

where $V(t)$ is the number of scans, registering noncongestion period in $(0,t]$. Obviously this process satisfies the conditions of Theorem 2 and the constants involved will be calculated as follows,

$$\kappa = \gamma/r-E_{r}Y/rE_{r} = 0,$$

and

$$\mu = 1/rE_{r}.$$
So far it is proved that the elements \( \{Z_n\} \), converge to \( W \) as \( n \to \infty \) under \( J_1 \), where
\[
Z_n(t) = \frac{U(nt) - E_r(U(nt) + V(nt))}{\sqrt{n}}
\]
With the random change of time-method of [1] ch. 17, applied above, it can be proved that the elements
\[
Y_n(t) = \frac{U'(nt) - E_nU'(nt)}{\sqrt{\sigma^2 n/\gamma}}
\]
also converge to \( W \) under \( J_1 \) as \( n \to \infty \), where \( U'(t) \) denotes the number of scans showing congestion up to the \( [t] \)-th scan.

Thus, as \( n \to \infty \)
\[
U'(N) - E_N U'(N) \frac{d}{\sqrt{N \sigma^2 / \gamma}} \rightarrow Z
\]
where \( N \) is the number of scans,
\[
\sigma^2/\gamma = E_r(1-E_r) + \gamma E_r^3\kappa X^2.
\]
and \( Z \sim N(0,1) \).

CONSTANT SCANNING INTERVALS

Other types of scanning can also be treated in this way. In the case of constant scanning intervals, one can not choose the regeneration points at the beginning of each congestion period. A better choice will be at the first scan in the congestion period, but since there will be congestion periods without any scan, the calculation of \( \Delta \) will be more complicated in this case. However the result \([4], (36)\) can be used to achieve the norming constant in this case.

Thus, with scanning interval \( h \) and \( U(t) \) denoting the number of scans indicating congestion, weak convergence to \( W \) can be proved for elements
\[
U(nt) - E_r n \frac{V}{\sigma \sqrt{h}}
\]
where the norming constant is
\[
\sigma = E_r \frac{1}{h} \sum_{j=1}^{n} \prod_{i>k} (1 - \omega_j^{-1}) \coth \left( \frac{h \omega_j}{2} \right) / h
\]
The numbers \( \omega_j \), \( j=1,2,...,c \), are defined in \([4]\).

SIMULATION OF THE CONSTANTS

For many processes the conditions Cl-3 and the conditions of Theorem 2 are satisfied, but the constants \( \kappa/\mu \) and \( \Delta/\mu \) may be impossible to calculate. In such cases simulation is useful. We refer to a method by Crane and Iglehart, \([3]\), giving an approximate confidence interval for the constant \( \kappa/\mu = v \), when \( n \) regeneration times are simulated.

Let \( U_i = (Y_i, X_i) \), where \( Y_i = w(S_i) - w(S_{i-1}) \).
Let
\[
U(n) = \left( \frac{Y(n)}{X(n)} \right)
\]
and
\[
S(n) = \left( \begin{array}{c} s_{11}(n) \\ s_{12}(n) \\ s_{22}(n) \end{array} \right) = \frac{1}{n} \sum_{i=1}^{n} U_i \Xi(U_i - \bar{U})
\]
The i.i.d. random vectors \( U_i \) are simulated and the confidence interval is given in terms of \( Y(n), X(n), s_{11}(n), s_{12}(n) \) and \( s_{22}(n) \) in \([3], (4.2)\).

Also \( \Delta/\mu \) can be estimated by simulation.

REFERENCES