ABSTRACT

A loss system in which the number of customers in the system can be described as a birth and death process with arbitrary birth and death intensities is considered. For this system we derive formulas for the variance of the number of lost calls as well as the variance of the time congestion, estimated by scanning with constant scanning intervals are calculated in [2]. In [3] there has been made a generalization in another direction. In this paper the assumption of Poissonian input has been replaced by the assumption of recurrent input. The present paper will follow the same lines as [2] and [3]. Concerning loss systems the variances of different estimates are calculated in a system that is a generalization of Erlang's loss system. For this latter system we will also derive variance formulas which are applicable when we use measuring methods which are different from those assumed in [3] - [5].

In [1] I have derived formulas for the calculation of the variances of observed mean waiting time, mean queue length and the proportion of calls that have to wait in an M/M/c waiting system. Formulas for calculation of the variances of the same quantities in a more general waiting system with a limited number of states were also derived. In this paper I will give some further formulas applicable to the M/M/c system.

1. INTRODUCTION

When there are measurements on simulated telecommunications traffic systems the conditions as a rule differ from those, valid when observations are made on real traffic. In the former case, contrary to the latter case, all system parameters are as a rule known, and the purpose of the simulation is to estimate one or several performance parameters such as losses or mean waiting times. In a simulation it is, at least theoretically, possible to make observations under stationary conditions during arbitrarily long time intervals. In addition to the measuring methods used in observing real traffic there are some methods which are specific to simulations, for example scanning with exponential distribution for the distances between successive scanning points or making observations in connection with the arrivals of the calls. When observing real traffic we often do this during a time interval of fixed length. If in a simulation we use the method of the inbedded markov chain we have no control of the continuous time and we therefore have to use an other stopping criteria such as the number of incoming calls or the number of measurements made.

An important question in the planning of a simulation run is: "how long should the run be to ensure that the estimates of the performance parameters will be accurate enough?" A simple stopping rule is: stop the simulation when the time congestion measured by continuous observation or by different scanning methods. For the Erlang loss system an exact formula for the variance of the number of lost calls is derived for the case when the observations comprise a fixed number of calls.

For observations on the Erlang waiting system comprising a fixed number of calls, formulas are derived for the variance of the mean waiting time, the proportion of delayed calls and the mean queue length observed when calls arrive.

2. A GENERAL MARKOVIAN LOSS SYSTEM

We consider a service system with c servers. The arrival intensities \( \lambda_1, \lambda_2, \ldots, \lambda_c \) are assumed \( > 0 \). The departure intensities \( \mu_1, \mu_2, \ldots, \mu_c \) are assumed \( > 0 \) and \( \mu_c = 0 \). Arrivals (calls) that occur when all servers are busy are lost.

With all \( \lambda_1 \), \( \lambda_2 \), \( \ldots \), \( \lambda_c \) and \( \mu_1 \), \( \mu_2 \), \( \ldots \), \( \mu_c \) we have the Erlang loss system and with \( \lambda_1 = (k - V) / \alpha \), \( k > c \), and \( \mu_1 = V \) we have the Engset loss system with N call sources with call intensity \( \alpha \) for each free source. A further model that is included in the system described, is studied in [5], where the call intensity is an increasing linear function of the states.

2.1 THE TRANSFORM OF THE JOINT DISTRIBUTION OF THE DISTANCE BETWEEN TWO SUCCESSIVE LOST CALLS AND THE NUMBER OF CALLS IN THIS INTERVAL

Define for a system with c servers:

- \( X_c \) = the distance between two successive lost calls
- \( K_c \) = the number of calls between two successive lost calls
- \( c \) = the number of lost calls including the last lost call

Let us also consider a system with c - 1 servers, in which \( \lambda_c \) and \( \mu_c = 0 \), \( \lambda_1 \), \( \lambda_2 \), \( \ldots \), \( \lambda_c \) are the same as in the first system.

Now assume for the moment that at time 0 a transition from c to c - 1 occupied servers has taken place in the first system and that at time 0 a call has been lost in the second system.

In connection with the first system we define a stochastic process \( \{ X_1(t), t \geq 0 \} \) as follows:

For \( V = 0, 1, \ldots, c - 1 \), \( X_c(t) = V \), if at time \( t \) \( V \) servers are occupied and if in the interval \( (0, t] \) all servers have not been occupied simultaneously. We define \( X_1(t) = c \) if at some time in the interval \( (0, t] \) all servers have been occupied simultaneously.

In connection with the second system we define a stochastic process \( \{ X_0(t), t \geq 0 \} \) as follows:
For \( V = 0, 1, \ldots, c - 1 \) \( X_2(t) = V \) if at time \( t \) \( V \) servers are occupied and if in the interval \((0, t]\) no call has been lost. We put \( X_2(t) = c \) if in the interval \((0, t]\) at least one call has been lost.

\( \{X_i(t), t > 0\} \) then forms for \( i = 1, 2 \) a birth and death process on the integers in \([0, c]\) with the same birth and death intensities and with an absorbing barrier in \( c \). As furthermore \( X_i(0) = c - 1 \) for \( i = 1, 2 \) the two processes have identical probability structures. This implies i.e. that the length of a non busy period and the number of calls in this period in the first system has the same joint distribution as the distance between two successive lost calls and the number of calls in this interval including the last lost call in the second system. For each of the two systems these two quantities correspond to the time until absorption and the number of positive steps before absorption in the birth and death process.

Let \( X'_c \) the length of a non busy period in a system with \( c \) servers \( K_c \) the number of calls in such a period

\[
\begin{align*}
X'_c & \sim \text{geometric}\{\lambda_c + \mu_c\}, \\
K_c & \sim \text{Poisson}\{\lambda_c + \mu_c\}.
\end{align*}
\]

Define furthermore

\[
X''_c \sim \text{exponential}\{\lambda_c + \mu_c\}, \quad K''_c \sim \text{Poisson}\{\lambda_c + \mu_c\}.
\]

We then have because of the markov property

\[
\begin{align*}
(X'_c, K'_c) & \sim (X''_c, K''_c), \\
(X'_c, K'_c) & \sim (X''_c, K''_c) - (X'_c, K'_c).
\end{align*}
\]

Define furthermore

\[
X''_c \sim \text{exponential}\{\lambda_c + \mu_c\}, \quad K''_c \sim \text{Poisson}\{\lambda_c + \mu_c\}.
\]

We then have because of the markov property

\[
\begin{align*}
(X''_c, K''_c) & \sim (X''_c, K''_c), \\
(X''_c, K''_c) & \sim (X''_c, K''_c) - (X''_c, K''_c).
\end{align*}
\]

Let us now consider a system with \( c \) servers at an instant when a call has been lost. By \( U \) we designate the time distance to the first event (arrival or departure). \( U \) is exponentially distributed with parameter \( \lambda_c + \mu_c \), so

\[
E\left[ e^{-sU} \right] = \frac{\lambda_c + \mu_c}{s + \lambda_c + \mu_c}.
\]

Independent of \( l \) the probability that the first event is an arrival is

\[
\frac{\lambda_c}{\lambda_c + \mu_c}
\]

and a departure

\[
\frac{\mu_c}{\lambda_c + \mu_c}
\]

If the first event is an arrival we have

\[
(X'_c, K'_c) = (U, 1)
\]

If the first event is a departure, the remaining part of the time to the next lost call may be decomposed into two parts, namely the first part being a non busy period and the second part the time from the beginning of a busy period until the time when the first call is lost. When the first event is a departure, occurring at \( U \), we have

\[
(X'_c, K'_c) = (U, 0) + (X''_c, K''_c) + (X''_c, K''_c)
\]

Because of the markov property, (1) and (2) we get

\[
\begin{align*}
E\left[ e^{-sX'_c} K'_c \right] & = \frac{\lambda_c}{\lambda_c + \mu_c} \cdot E\left[ e^{-sU} \right] + \frac{\mu_c}{\lambda_c + \mu_c} \cdot E\left[ e^{-sU} \right] \cdot E\left[ e^{-sX''_c} K''_c \right], \\
E\left[ e^{-sX''_c} K''_c \right] & = E\left[ e^{-sX''_c} \cdot K''_c \right].
\end{align*}
\]

Let

\[
\varphi_c(s, \omega) = E\left[ e^{-sX'_c} \cdot K'_c \right]
\]

and use (3) in (6). We then get

\[
(s + \lambda_c + \mu_c - \mu_c) \cdot \varphi_{c-1}(s, \omega) \cdot \varphi_c(s, \omega) = \lambda_c \cdot \varphi_c(s, \omega)
\]

The arguments used in the derivation of (7) may be used for all \( c \). If we replace \( c \) by \( r \) we get the recurrence relations

\[
(s + \lambda_r + \mu_r - \mu_r) \cdot \varphi_{r-1}(s, \omega) \cdot \varphi_r(s, \omega) = \lambda_r \cdot \varphi_r(s, \omega)
\]

For \( r = 0 \) the first event is always an arrival and the distance between two successive (lost) calls is exponentially distributed with parameter \( \lambda_0 \), so

\[
\varphi_0(s, \omega) = E\left[ e^{-sU} \cdot K'_0 \right] = \frac{\lambda_0 \cdot \omega}{s + \lambda_0}
\]

2.2 THE TRANSFORM OF THE MARGINAL DISTRIBUTION OF \( X'_c \)

Define

\[
\Pi_r(s) = E\left[ e^{-sX'_r} \right]
\]

By putting \( \omega = 1 \) in (8) and (9) we get the recurrence relations

\[
\Pi_r(s) = \frac{\lambda_r}{s + \lambda_r + \mu_r - \mu_r} \cdot \Pi_{r-1}(s)
\]

with

\[
\Pi_0(s) = \frac{\lambda_0}{s + \lambda_0}
\]

We write \( \Pi_r(s) \) in the form

\[
\Pi_r(s) = \frac{T_r(s)}{N_r(s)}
\]

Insertion of this in (10) gives

\[
T_r(s) = \lambda_r \cdot N_{r-1}(s)
\]

\[
N_r(s) = (s + \lambda_r + \mu_r) \cdot N_{r-1}(s) - \mu_r \cdot T_{r-1}(s)
\]

If we define

\[
N_{-1}(s) = 1 \quad \text{and} \quad N_0(s) = s + \lambda_0
\]

we get

\[
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\]
\[ N_r(s) = (s + \lambda_r + \mu_r) \cdot N_{r-1}(s) - \mu_r \cdot \lambda_{r-1} \cdot N_{r-2}(s) \] (13)

and

\[ \Pi_r(s) = \frac{\lambda_r \cdot N_{r-1}(s)}{N_r(s)} \] (14)

From (12) and (13) it is seen that \( N_r(s) \) is a polynomial of degree \( r+1 \) and it is easily shown by induction that the \( r+1 \) zeros of \( N_r(s) \) are distinct, negative and separated by the \( r \) zeros of \( N_{r-1}(s) \). Therefore \( \Pi_r(s) \) may be written

\[ \Pi_r(s) = \sum_{j=0}^{c} \frac{B_j}{s - \alpha_{jc}} \] (15)

where \( \alpha_{jc}, j = 0, 1, \ldots, c \), are the zeros of \( N_c(s) \) and \( B_j \) are calculated from

\[ B_j = \frac{\lambda_{j+1} \cdot (\alpha_{jc} - \alpha_{j+1,c})}{\prod_{r=0}^{j} (\alpha_{jc} - \alpha_{rc})} \] (16)

From the inequalities

\[ \alpha_{0c} > \alpha_0, c-1 > \alpha_{c-1} > \ldots > \alpha_{c-1,c-1} > \alpha_c, c \]

it follows that the number of negative factors in the numerator and denominator of (16) are equal for all \( j \) and therefore

\[ B_j > 0 \quad j = 0, 1, \ldots, c \] (17)

2.3 THE TRANSITION PROBABILITY \( p_{cc}(t) \)

We now define the stochastic process \( \{X(t), t \geq 0\} : X(t) = \nu \) if \( \nu \) servers are occupied at \( t \), \( \nu = 0, 1, \ldots, c \).

Denote by \( p_{cc}(t) \) the transition probability

\[ p_{cc}(t) = P \left\{ X(t) = c \mid X(0) = c \right\} \] (18)

and its Laplace transform

\[ \mathcal{L}\{p_{cc}(t)\} = \frac{s}{s - \lambda_c} \cdot p_{cc}(t) \cdot dt = \psi_c(s) \] (19)

Now the event \( \{X(t) = c \mid X(0) = c\} \) may occur in two different ways

1. in the interval \((0, t)\) there is no departure,
2. a. at \( x \to c \) there is a first departure and
   b. at \( x < u < t \) there is a first return to the state \( c \) and
c. there is a transition from \( c \to c \) in the interval \((u, t)\)

Thus

\[ p_{cc}(t) = e^{-\lambda_c t} + \int_{0}^{t} \mu_c \cdot e^{-\mu_c x} \cdot f_{c-1}((u - x)) \cdot \] (20)

where \( f_{c-1}(x) \) is the density function of the length of a non busy period.

Taking Laplace transforms in (20) gives

\[ \psi_c(s) = \frac{1}{s + \mu_c} + \frac{\mu_c}{s + \mu_c} \cdot \psi_{c-1}(s) \] (21)

or

\[ \psi_c(s) = \frac{1}{s + \mu_c - \lambda_c \cdot \psi_{c-1}(s)} \] (22)

Using (13) and (14) we can write this

\[ \psi_c(s) = \frac{N_{c-1}(s)}{N_c(s) - \lambda_c \cdot N_{c-1}(s)} \] (23)

or finally

\[ \psi_c(s) = \frac{\pi_c(s)}{\lambda_c \cdot (1 - \pi_c(s))} \] (24)

To find the inverse of \( \psi_c(s) \) we have to calculate the zeros of \( 1 - \pi_c(s) \). According to the preceding section the denominator of \( \pi_c(s) \), \( N_c(s) \), has \( c+1 \) zeros, which are distinct and negative. Let us take a closer look at the function \( \pi_c(s) \).

By differentiating in (15) we get

\[ \pi_r'(s) = - \sum_{j=0}^{c} \frac{B_j}{(s - \alpha_{jc})^2} \] (25)

As all \( B_j \) are > 0 this means that

\[ \pi_c'(s) < 0 \] (26)

whenever it exists.

From (15) and (24) follows

1. When \( s \) varies from \(+\infty\) to \( \alpha_{0c} \), \( \pi_c(s) \) increases monotonically from \( 0 \) to \( +\infty \) and takes the value 1 for \( s = 0 \).
2. When \( s \) varies from \( \alpha_{j,c} \) to \( \alpha_{j+1,c} \) \( j = 1, 2, \ldots, c-1 \), \( \pi_c(s) \) increases monotonically from \( -\infty \) to \( +\infty \) and thus takes the value 1 once and only once in this interval.
3. When \( s \) varies from \( \alpha_{c,c} \) to \( -\infty \) \( \pi_c(s) \) increases monotonically from \( -\infty \) to \( 0 \).

This means that the zeros of \( 1 - \pi_c(s) \) are distinct, real and, if we denote them \( \delta_j, j = 0, 1, \ldots, c \), that

\[ \delta_0 > \alpha_{0c} > \delta_1 > \alpha_{1c} > \ldots > \delta_c > \alpha_{c,c} \]

We can therefore write

\[ \frac{\pi_c(s)}{1 - \pi_c(s)} = \frac{A_0}{s} + \sum_{j=1}^{c} \frac{A_j}{s - \delta_j} \]

\( A_0 \) is calculated from

\[ A_0 = \lim_{s \to 0} \frac{s \cdot \pi_c(s)}{1 - \pi_c(s)} = \frac{1}{\pi_c(0)} = \frac{1}{E(X_1)} \] (27)

For \( j = 1, 2, \ldots, c \) we get

\[ A_j = \frac{\pi_c(\delta_j)}{-\pi'_c(\delta)} = \frac{1}{\pi'_c(\delta_j)} \] (28)
From (24) follows that \( A_j > 0, \ j = 0,1, \ldots \ c \)

Inversion now gives

\[
\lambda^{-1} \left[ \frac{\pi_c(s)}{1 - \pi_c(s)} \right] = \frac{1}{E[X_0^c]} + \sum_{j=1}^{c} A_j \cdot e^{\delta_j t}
\]  

(27)

and

\[
\rho_{ec}(t) = \frac{1}{\lambda_c E[X_0^c]} + \frac{1}{\lambda_c} \sum_{j=1}^{c} A_j \cdot e^{\delta_j t}
\]  

(28)

From (10) we can get the recurrence formulas

\[
E[X_0^c] = 1 + \mu_\pi E[X_{r+1}^c] \lambda_r
\]  

for calculation of \( E \left[ X_c \right] \) with the start value

\[
E[X_0^c] = \frac{1}{\lambda_0}
\]  

(29)

(30)

By letting \( t \to \infty \) in (28) we get

\[
\rho_c = \lim_{t \to \infty} \rho_{ec}(t) = \frac{1}{\lambda_c E[X_0^c]}
\]  

(31)

where \( p_c \) denotes the equilibrium probability of finding all \( c \) servers occupied.

2.4 THE VARIANCE OF THE NUMBER OF LOST CALLS IN \( (0,t) \)

We assume that the traffic process is in statistical equilibrium at time \( 0 \) and consider the stochastic sequence that is formed by the distance to the first lost call and after that the distances between successive lost calls. This sequence forms a so called equilibrium renewal process. We can then use formulas from the theory of these processes.

Designate by \( R(t) \) the number of lost calls in \( (0,t) \). Formula (3) p. 46 of [6] then gives

\[
E[R(t)] = \frac{t}{E[X_0^c]}
\]  

(32)

From (32) and formula (5) p. 56 of [6] follows

\[
\left[ \text{Var} \left[ R(t) \right] \right] =

\frac{2 \pi_c(s)}{E[X_0^c]^2 \lambda^2} \left[ s^2 \left( 1 - \pi_c(s) \right) \right] + \frac{1}{E[X_0^c]^2 \lambda^2} - \frac{2}{E[X_0^c]^2 s \lambda^2}
\]  

(33)

This expression may be broken up into partial fractions of the form

\[
\left[ \text{Var} \left[ R(t) \right] \right] = \frac{C_0}{s^3} + \frac{C_1}{s^2} + \frac{C_2}{s} + \sum_{j=1}^{c} D_j \cdot e^{-\delta_j t}
\]  

(34)

where \( \delta_j \) are the non-zero roots of \( \pi_c(s) - 1 = 0 \).

To determine \( C_0 \), we multiply both sides of (33) by \( s^3 \) and let \( s \to 0 \). We then get

\[
C_0 = \lim_{s \to 0} \left( \frac{\pi_c(s)}{1 - \pi_c(s)} \right) \frac{2}{E[X_0^c]^2}
\]  

As \( \pi_c(0) = 1 \) and

\[
\lim_{s \to 0} \frac{s}{1 - \pi_c(s)} = \frac{1}{\pi_c(0)} = \frac{1}{E[X_0^c]}
\]

we have \( C_0 = 0 \).

This means that \( \text{Var} \left[ R(t) \right] \) is of the form

\[
\text{Var} \left[ R(t) \right] = C_1 t + C_2 + \sum_{j=1}^{c} D_j \cdot e^{-\delta_j t}
\]  

(35)

From (26) and (33) we get

\[
D_j = \frac{2 A_j}{E[X_0^c] \delta_j}
\]  

(36)

or using (31)

\[
D_j = \frac{2 \lambda_c \rho_c A_j}{\delta_j}
\]  

(37)

To calculate \( C_2 \) we put \( t = 0 \) in (35) and get

\[
C_2 = - \sum_{j=1}^{c} D_j = - \frac{2 \lambda_c \rho_c}{\delta_j} \sum_{j=1}^{c} A_j
\]  

(38)

We differentiate in (35) and put \( t = 0 \):

\[
\frac{d}{dt} \text{Var} \left[ R(0) \right] = C_1 + \sum_{j=1}^{c} D_j \cdot e^{-\delta_j}
\]

We easily find

\[
\frac{d}{dt} \text{Var} \left[ R(0) \right] = \rho_c \cdot \lambda_c
\]

and then using (37)

\[
C_1 = \lambda_c \rho_c \left( 1 + 2 \sum_{j=1}^{c} \frac{A_j}{-\delta_j} \right)
\]

and finally

\[
\text{Var} \left[ R(t) \right] =

= \lambda_c \rho_c \left( 1 + 2 \sum_{j=1}^{c} \frac{A_j}{-\delta_j} \right) t - 2 \lambda_c \rho_c \sum_{j=1}^{c} \frac{A_j}{-\delta_j} \left( 1 - e^{-\delta_j t} \right)
\]  

(39)

2.5 AN APPROXIMATE FORMULA FOR THE VARIANCE OF THE NUMBER OF LOST CALLS IN \( (0,t) \) IN THE ERLANG CASE

When all \( \lambda_v \) are equal (\( = \lambda \)) and \( \mu_\pi = \nu \) (the Erlang case), the variance of the number of lost calls is derived in [1]. Like in the generalized case treated here, the expression for the variance consists of a sum of a linear part

\[
C_1 t + C_2
\]

and a sum of \( c \) exponential terms with negative exponents, so that for large \( t \)

\[
\text{Var} \left[ R(t) \right] \approx C_1 t + C_2
\]

Formulas for the numerical calculation of \( C_1 \) and \( C_2 \) are
also given in [1].

These formulas are, as the authors state, rather tedious. By using formulas from renewal theory it is possible to get a simpler algorithm for the calculation of $C_1$ and $C_2$. These quantities can be expressed as simple functions of the three first moments of $X_c$. The formula required is (18) on page 58 of [6]. $C_1$ and $C_2$ may be given in the form

$$C_1 = \frac{E[X_c^2] - E[X_c]}{E[X_c]}$$  

and

$$C_2 = \frac{-2E[X_c]E[X_c^2] - 3E[X_c^3]}{6E[X_c]}$$

The calculation of the first three moments of $X_c$ may be done recursively as follows.

By putting $A_r = A$ and $l$ in (10) and (11) and expanding $N,(s)$, $N_2(s)$, and $N_3(s)$ in powers of $s$, we get the following recurrence relations

$$E[X_c] = \frac{1}{\lambda}$$

$$E[X_c^2] = \frac{1}{\lambda}E[X_c] + 2E[X_c]$$

$$E[X_c^3] = \frac{1}{\lambda}E[X_c^2] + 6E[X_c^2]E[X_c] - E[X_c]$$

with the start values

$$E[X_0] = \frac{1}{\lambda}$$

$$E[X_0^2] = \frac{2}{\lambda}$$

$$E[X_0^3] = \frac{6}{\lambda}$$

We have thus arrived at formulas, which give us the exact value of the linear part of the variance. To get an expression that is approximately valid for all values of $t$ we should try to approximate the exponential part. This part is more important the less $t$ is, while its value approaches 0 when $t \to \infty$. We should therefore try to find an approximation that is good in the first place for small values of $t$ and we have chosen the following. Put

$$A(t) = C_1t + C_2 - C_3\left(k \cdot e^{-at} + (1-k) \cdot e^{-bt}\right)$$

where $C_1$ and $C_2$ are calculated from formulas (41) - (47) and determine the parameters $k$, $A_1$, and $A_2$ so that the first three derivatives of $A(t)$ at $t = 0$ coincide with the corresponding quantities for the exact solution. This means that we have approximated the sum of $c$ exponential terms by the sum of two such terms.

We remind of the two theorems from the theory of Laplace transforms.

1. $\mathcal{L}\left[r^V(t)\right] = s \cdot \mathcal{L}\left[r^{V-1}(t)\right] - r^V(0^+)$  

2. $F^V(t)_{t=0} = \lim_{t \to -\infty} s \cdot \mathcal{L}\left[r^V(t)\right]$
The corresponding problem in the Erlang case is solved in [2]. When the service times are exponentially distributed and the inter-arrival times form a renewal process the solution is given in [3].

Denote by \( t_n \), \( n = 0, 1, 2 \ldots \) the time point when the \( n \)th call is lost and by \( z_n = t_n - t_{n-1} \) the distance between the \( n \)th and the \((n-1)\)st lost calls. By \( I_n \) we denote the number of incoming calls in \((t_{n-1}, t_n]\) and by \( X_n \) the vector

\[
x_n = (I_n, Z_n)
\]

Because of the markov property \( Z_n \) as well as \( X_n \), \( n = 1, 2, 3, \ldots \) are independent and equally distributed random variables, so \((X_n, Z_n), n = 1, 2, \ldots \) forms a multidimensional renewal process. Define

\[
\eta(t) = \sum x_n
\]

where the sum is taken for all \( n \) such that \( 0 < t_n < t \).

Then for large \( t \),

\[
\eta(t) = (R(t), N(t))
\]

We can now use formulas (25) and (26) of [8], which express the moments of \( R(t) \) and \( N(t) \) in \( t \) and the moments of \( Z_n \) and \( I_n \). Obviously we have for an arbitrary \( n \)

\[
(L_n, Z_n) \overset{d}{=} (K_n, X_n)
\]

Further the moments of \( K_n \) and \( X_n \) may be calculated recursively from the recurrence relations (8) and (9).

Formula (25) of [8] now gives

\[
E(R(t)) = \frac{t}{E(X_n)}
\]

and

\[
E(N(t)) = t \cdot E(K_n)
\]

Formula (26) gives

\[
\text{Var}(R(t)) = t \cdot \text{Var}(X_n) + \text{Var}(K_n)
\]

\[
+ t \cdot E(K_n) \left[ E\left(\frac{X_n}{X_n^2}\right)^2 - 2 \cdot \frac{E(K_n)}{E(X_n)} \cdot \text{Cov}(K_n, X_n) \right]
\]

and finally

\[
\text{Cov}(R(t), N(t)) = \frac{t}{E(X_n)} \left[ E(X_n^2) \cdot E(K_n) - E(K_n) \cdot E(X_n) \right]
\]

By differentiating in (8) and (9) once or twice with respect to \( \omega \) and/or \( s \) and then putting \( \omega = 1, s = 0 \) we get the following recurrence relations from which the moments of \( K_n \) and \( X_n \) may be calculated:

\[
E(X_n) = \frac{1}{\lambda_0} + \mu_r \cdot E(X_{n-1})
\]

\[
E(K_n) = \frac{\mu_r}{\lambda_r} \cdot E(X_{n-1}) + 2 \cdot E^2(X_r)
\]

\[
E(X_0^2) = \frac{2 \lambda_r}{\lambda_0}
\]

\[
E(K_0) = 1 + \frac{\mu_r}{\lambda_r} \cdot E(K_{r-1})
\]

\[
E(K_{r-1}) = 1 + \frac{\mu_r}{\lambda_r} \cdot E(K_{r-2})
\]

\[
E(K_{r-1}) = 1 + \frac{\mu_r}{\lambda_r} \cdot E(K_{r-2})
\]

\[
E(K_{r-1}) = 1 + \frac{\mu_r}{\lambda_r} \cdot E(K_{r-2})
\]

\[
E(x_0^2, K_0) = \frac{1}{\lambda_0}
\]

The system of formulas given makes it possible for us to get an expression for \( t \cdot \text{Var}(R(t), N(t)) \) that is independent of \( t \). The usefulness of the formulas has in the Erlang and Engset cases been confirmed by simulations.

2.7 THE VARIANCE OF THE NUMBER OF LOST CALLS AMONG A FIXED NUMBER OF CALLS IN THE ERLANG CASE

Let us again look at the Erlang model and assume that the process is in statistical equilibrium and that we observe a fixed number of calls, say \( M \). Define

\[
Z_i = 1 \quad \text{if call no. } i \text{ is lost}
\]

\[
Z_i = 0 \quad \text{" } i \text{ is successful}
\]

We want to calculate the variance of the estimate

\[
R_M = \frac{1}{M} \sum_{i=1}^{M} Z_i
\]

We have

\[
E(R_M) = E_1 c
\]

with \( E_1 c \) the Erlang loss formula and

\[
\text{Var}(R_M) = \frac{1}{M^2} \sum_{i=1}^{M} \sum_{j=1}^{M} \text{Cov}(Z_i, Z_j)
\]
Let
\[ u_k = P \left( Z_{kt+1} = 1 \mid Z_t = 1 \right) \]  
(66)

Then
\[ \text{Cov} \left( Z_{kt+1}, Z_t \right) = E_{1c} u_k - E_{1c}^2 \]

and
\[ \text{Var} \left( R_m \right) = E_{1c} + \frac{2 E_{1c}}{M^2} \sum_{k=1}^{M-1} (M-k) u_k - E_{1c}^2 \]  
(67)

We now have to calculate \( U_k \).

Define
\[ y = \text{the number of successful calls between two successive lost calls, and} \]
\[ h_i = P \{ Y = i \} \quad i = 0, 1, \ldots \]

Further
\[ H_c(x) = \sum_{i=0}^{\infty} h_i x^i \]

The function \( H_c(x) \) is thoroughly investigated in [9]. There it is proved that \( H_c(x) \) may be written
\[ H_c(x) = \lambda \cdot \frac{N_c(x)}{N_{c+1}(x)} \]

where \( N_c(x) \) is a polynomial, that may be formed according to the recurrence relations
\[ N_{m+1}(x) = (m+\lambda) N_m(x) - \lambda \cdot x \cdot m \cdot N_{m-1}(x) \]

with
\[ N_0(x) = 1 \quad \text{and} \quad N_1(x) = \lambda \]

In [9] the following properties of the polynomials are proved
(a) \( N_m(x) > 0 \) for \( x \leq 0 \), \( m = 0, 1, 2, \ldots \)
(b) The degree \( \nu = \nu(m) \) of \( N_m(\cdot) \) is the integral part of \( m/2 \)
(c) The coefficients of \( x^0, x^1, \ldots, x^{\nu} \) in \( N_m(x) \) are alternately positive and negative
(d) When \( x \to \infty \), \( N_m(x) \) tends to \( \infty \) if \( \nu(m) \) is even and to \( -\infty \) if \( \nu(m) \) is odd
(e) \( N_m(\cdot) \) has \( \nu(m) \) distinct zeros, which are positive and > 1
(f) \( H_c(x) \) is of the form
\[ H_c(x) = \sum_{i=1}^{\nu(c+1)} \frac{Y_i}{x-x_i} \]

where \( x_i \) are the zeros of \( N_{c+1}(x) \).
\[ Y_0 = 0 \quad \text{if} \ c \text{ is odd and} \quad Y_0 > 0 \quad \text{if} \ c \text{ is even.} \]
\[ Y_i, \quad i = 1, \ldots, \quad \nu(c+1) \text{ are > 0.} \]

Let us now define
\[ X = \text{the number of incoming calls between two successive lost calls, including the last lost call.} \]

Further
\[ g_i = P \{ X = i \} \quad i = 1, 2, \ldots \]

and
\[ G_c(x) = \sum_{i=1}^{\infty} g_i x^i \]

Obviously
\[ X = Y + 1 \]

so
\[ g_c(x) = x \cdot H_c(x) \]  
(68)

If we define
\[ U(x) = \sum_{k=0}^{\infty} U_k x^k \]

it follows from [10] that
\[ U(x) = \frac{1}{1-G_c(x)} = \frac{1}{1-x \cdot H_c(x)} \]  
(69)

Using the properties (a) - (f) above and (69) it is possible to prove that for all \( c \) the function \( 1-G_c(x) \) has a number of distinct zeros that is equal to \( \left\lfloor \frac{c}{2} \right\rfloor + 1 \). One zero is \( = 1 \) and the others > 1. We can then write
\[ U_c(x) = C_0 + \frac{C_1}{x-1} + \sum_{i=2}^{c+1} \frac{C_i}{x-Z_i} \]  
(70)

where \( Z_i \) are the zeros of \( 1-G_c(x) \). In this expression \( C_0 \) may \( = 0 \). We have
\[ C_1 = -\frac{1}{G'(1)} = -E_{1c} \]  
(71)

and
\[ C_i = -\frac{1}{G'(Z_i)} < 0 \quad i = 2, \ldots, \left\lfloor \frac{c}{2} \right\rfloor + 1 \]  
(72)

We write (70)
\[ U(x) = C_0 + \frac{E_{1c}}{1-x} + \sum_{i=2}^{c+1} \frac{-C_i}{Z_i} \cdot \frac{1}{1-x/Z_i} \]  
(73)

Developing the terms of (73) in series gives
\[ U(x) = C_0 + E_{1c} \sum_{k=0}^{\infty} x^k + \sum_{i=2}^{c+1} \sum_{k=0}^{\infty} \frac{-C_i}{Z_i^{k+1}} x^k \]  
(74)

Now \( U_k \) is the coefficient of \( x^k \) in this series. Thus
\[ u_k = E_{1c} + \sum_{i=2}^{c+1} \frac{-C_i}{Z_i^{k+1}} \quad k=1,2,\ldots \]  
(75)
After insertion of (75) in (67) and summing up the series we get

\[ \text{M} \cdot \text{Var} \left\{ R_m \right\} = E_{1c} (1 - E_{1c}) + 2 E_{1c} \sum_{i=2}^{\infty} \frac{C_i}{Z_i(Z_i-1)} - \frac{2 E_{1c}}{M} \sum_{i=2}^{\infty} \frac{1 - \left( \frac{1}{Z_i} \right)^M}{(Z_i-1)^2} (-C_i) \]  

(76)

2.8 THE VARIANCE OF OBSERVED TIME CONGESTION

We consider the general loss system and assume that we make observations of the time congestion by scanning. This means that at times \( t_1, t_2, \ldots \) we observe whether all servers are occupied or not. Assume that the intervals \( t_i - t_{i-1} \) form a renewal process with distribution function \( B(t) \) and that the observations go on until we have made \( m \) scans. Define

\[ T_i = \begin{cases} 1 & \text{if all servers are occupied at } t_i \\ 0 & \text{not occupied at } t_i \end{cases} \]

\( i = 1, 2, \ldots, m \)

We estimate the time congestion by

\[ D_m = \frac{1}{m} \sum_{i=1}^{m} T_i \]  

(77)

Define the Laplace-Stieltjes transform of \( B(t) \)

\[ b(s) = \int_0^\infty e^{-st} dB(t) \]  

(78)

By \( p^{(n)}(t) \) we denote the probability that at two scans at distance \( n \) there will be a transition from state \( c \) to state \( c \). We have

\[ p^{(n)}(t) = \int_0^t p_{cc}(t) \cdot dB^{(n)}(t) \]  

(79)

where \( p_{cc}(t) \) is given by (28) and \( dB^{(n)}(t) \) means the \( n \)th convolution of \( dB(t) \) with itself. Insertion of (28) in (79) gives

\[ p^{(n)}(t) = p_{cc} \int_0^t dB^{(n)}(t) + \frac{1}{\lambda_c} \sum_{j=1}^{c} A_j \int_0^t e^{\delta_j t} \cdot dB^{(n)}(t) \]

Now

\[ \int_0^t dB^{(n)}(t) = 1 \]

and

\[ \int_0^t e^{\delta_j t} \cdot dB^{(n)}(t) = \left[ b(-\delta_j) \right]^n \]

so

\[ p_{cc}^{(n)} = p_{cc} + \frac{1}{\lambda_c} \sum_{j=1}^{c} A_j \left[ b(-\delta_j) \right]^n \]  

(80)

The variance of \( D_m \) may now be calculated as follows

\[ E \left\{ D_m \right\} = p_{cc} \]

and

\[ \text{Var} \left\{ D_m \right\} = \frac{1}{m^2} \sum_{i=1}^{m} \sum_{j=1}^{m} \text{Cov} \left\{ T_i, T_j \right\} \]

We have

\[ \text{Cov} \left\{ T_i, T_j \right\} = p_{cc} \cdot p_{cc}^{(n-1)} - p_{cc}^2 \]

so

\[ \text{Var} \left\{ D_m \right\} = \frac{p_{cc}(1-p_{cc}) + 2 \frac{p_{cc}}{\lambda_c} \sum_{j=1}^{c} A_j \frac{b(-\delta_j)}{1-b(-\delta_j)} - \frac{2 \frac{p_{cc}}{m^2 \lambda_c} \sum_{j=1}^{c} A_j \cdot b(-\delta_j) \cdot 1 - \left[ b(-\delta_j) \right]^m}{(1 - b(-\delta_j))^2}} \]  

(81)

Insertion of (80) in (81) gives after summing up the series

\[ m \cdot \text{Var} \left\{ D_m \right\} = p_{cc}(1 - p_{cc}) + 2 \frac{p_{cc}}{\lambda_c} \sum_{j=1}^{c} A_j \frac{b(-\delta_j)}{1-b(-\delta_j)} - \frac{2 \frac{p_{cc}}{m^2 \lambda_c} \sum_{j=1}^{c} A_j \cdot b(-\delta_j) \cdot 1 - \left[ b(-\delta_j) \right]^m}{(1 - b(-\delta_j))^2} \]  

(82)

If we specialize \( B(t) = e^{-Yt} \), i.e. exponential scanning with intensity \( Y \), we have to put

\[ b(-\delta_j) = \frac{Y}{Y - \delta_j} \]

in (82)

If the scanning is made at regular intervals at distance \( h \) we have

\[ B(t) = \begin{cases} 0 & \text{when } t < h \\ 1 & \text{when } t \geq h \end{cases} \]

and

\[ b(-\delta_j) = e^{h \delta_j} \]

and (82) takes the form

\[ \text{Var} \left\{ D_m \right\} = \frac{p_{cc}(1-p_{cc}) + 2 \frac{p_{cc}}{m^2 \lambda_c} \sum_{j=1}^{c} A_j \frac{e^{h \delta_j}}{1-e^{h \delta_j}} - \frac{2 \frac{p_{cc}}{m^2 \lambda_c} \sum_{j=1}^{c} A_j \cdot e^{h \delta_j} \cdot 1 - e^{h \delta_j} \cdot \sum_{j=1}^{c} A_j \cdot e^{h \delta_j}}{(1-e^{h \delta_j})^2}} \]  

(83)
This variance has in the Erlang case been derived in [2]. In (83) we replace $m$ by $t/h$ and let $h \to 0$ and $m \to \infty$ with $t$ constant. We then get the formula valid for continuous measurement

$$\text{Var}\{D_t\} = \frac{2}{\lambda^2} \sum_{i=1}^{m} A_i - \frac{2}{\lambda^2} \sum_{i=1}^{\infty} A_i \left(1 - e^{-\lambda t}\right)$$

The corresponding formula for the Erlang case has been derived in [1].

An asymptotic formula of $\text{Var}\{D_t\}$ may be obtained in the following way.

Consider the alternating renewal process, which is defined by the start and the end of busy periods. The length of a busy period is exponentially distributed with mean $\mu^{-1}$ and variance $\mu^{-2}$. The length of a non busy period is given by the variable $X^{-1}$, the two first moments of which can be calculated from the recurrence relations (61) and (62). Now we can use formula (12) p. 90 of [6] and get

$$t \cdot \text{Var}\{D_t\} \approx \frac{\mu_c^3 E\{X^{-1}\} + \mu_c^2 E^2\{X^{-1}\} \lambda_c}{\mu_c + E\{X^{-1}\}^2}$$

Using the relation

$$E\{X^{-1}\} = \frac{\lambda_c E\{X^{-1}\}^{-1}}{\mu_c}$$

we can write the denominator of (85) as

$$\left(\frac{\lambda_c E\{X^{-1}\}^{-1}}{\mu_c}\right)^3$$

As

$$\rho_c = \frac{1}{\lambda_c E\{X^{-1}\}}$$

we finally get

$$t \cdot \text{Var}\{D_t\} \approx \mu_c \cdot \rho_c^3 \cdot E\{X^{-1}\}^2$$

In the Erlang case

$$\mu_c = c$$

and

$$\rho_c = \rho_c$$

so

$$t \cdot \text{Var}\{D_t\} \approx c \cdot E\{X^{-1}\}^2 E\{X^{-1}\}$$

where $E\{X^{-1}\}$ can be calculated with the recurrence relations (61) and (62) specialized to the Erlang case, i.e. with

$$\mu_r = r$$

and

$$\lambda_r = \lambda$$

3. THE VARIANCES OF SOME ESTIMATES IN AN M/M/c WAITING SYSTEM WHEN THE OBSERVATIONS COMPRISE A FIXED NUMBER OF CALLS

We consider an M/M/c waiting system with a mean service time that we select as the time unit. The arrival intensity is denoted $\lambda$.

In [4] are given asymptotic formulas for the calculation of the variance of observed mean values of waiting times, queue lengths and the proportion of delayed calls in an interval of fixed length. In the paper mentioned has been derived a functional equation for the transform of the joint distribution of the length of a busy period, the number of incoming calls in such a period and the accumulated waiting time during that period. From this equation there has been derived formulas for the first and second order moments of the variables considered. In the paper are also given recurrence relations for calculation of the first and second order moments of the length of a non busy period and the number of calls in such a period. Use of these moment formulas in a formula for the variance of a so called multidimensional process gave the variances wanted. By using the same set of formulas we can get asymptotic expressions for the variance of the observed mean waiting time and the proportion of delayed calls, when we observe a fixed number of calls.

Assume that we have observed $M$ successive waiting times, $W_i$, $i = 1, 2, \ldots, M$ and put

$$\bar{W} = \frac{1}{M} \sum_{i=1}^{M} W_i$$

Then the use of the formulas mentioned above gives

$$M \cdot \text{Var}\{\bar{W}\} \approx$$

$$\approx \frac{E_w^2}{p \cdot c^2} \cdot \left[ \frac{\rho^2 + 7\rho + 2 - 2E_{w}(3\rho + 1) + E_{w}(1 + \rho)}{(1 - \rho)^5} \right] +$$

$$+ \frac{E_w^2}{1 - \rho} \cdot \text{Var}\{R_{c-1}\}$$

In this formula

$E_w$ = the probability that a call is delayed according to Erlangs delay formula, $\rho = \lambda / \mu$, i.e. the mean load per server, Var $\{R_{c-1}\}$ = the variance of the number of incoming calls in a non busy period.

In the special case $c = 1$ we have

$$E_{2,1} = \rho$$

and

$$\text{Var}\{R_{c-1}\} = 0$$

and we get

$$M \cdot \text{Var}\{\bar{W}\} \approx \frac{\rho(\rho^3 - 4\rho^2 + 5\rho + 2)}{(1 - \rho)^5}$$

This formula has with a quite different technique been derived in [11].

Put

$$\alpha_1 = 0 \text{ if } W_i = 0$$

$$\alpha_1 = 1 \text{ if } W_i > 0$$
\[ \sum_{i=1}^{M} a_i \] then is the number of delayed calls and
\[ E = \frac{1}{M} \sum_{i=1}^{M} q_i; \]
is the proportion of calls, which are delayed.
Using the formulas mentioned above gives
\[ M \cdot \text{Var}\{E\} \approx \frac{E \cdot (1 - p)}{p} \left[ (1 - q_{2, c}) \cdot \frac{p(1 + p)}{(1 - p)^2} + E_{2, c} \cdot \text{Var}\{R_{2, 1}\} \right]; \]
(90)
When \( c = 1 \) we get
\[ M \cdot \text{Var}\{E\}_{c=1} \approx \frac{E (1 + p)}{p} \]
(91)
a formula that is also derived in [11].

Now suppose that we observe the queue length at each incoming call and denote by \( q_i \) this length when call number \( i \) arrives. We estimate the mean queue length with
\[ \bar{q} = \frac{1}{M} \sum_{i=1}^{M} q_i; \]
We denote the number of calls arriving in a busy period by \( L \) and the sum of the queue lengths at the corresponding arrival instants by \( Q_c \).
We define the joint transform
\[ \psi(\omega, u) = E \left\{ \omega \cdot u^{Q_c} \right\}; \]
By considering the two possibilities for the first event that occurs in a busy period (call or termination) and using the fact that the birth and death intensities in all states \( c \) are the same (\( \lambda \) and \( c \) respectively) we can derive the functional equation
\[ \psi(\omega, u) = \frac{c}{\lambda + c} \cdot 1 + \frac{\lambda}{\lambda + c} \cdot \omega \cdot \psi(\omega, u) \psi(\omega, u) \cdot \psi(\omega, u); \]
(92)
From this equation we can get formulas for the two first moments of \( L \) and \( Q \). Using these formulas, the formulas for the mean and variance of the number of calls in a non-busy period and the formulas for the mean and variance of a cumulative process in [8] we get
\[ E\{Q\} = \frac{p}{1 - p} \cdot E_{2, c}; \]
(93)
and
\[ M \cdot \text{Var}\{Q\} \approx \]
\[ = E_{2, c} \cdot \frac{p}{1 - p} \left[ (1 + p) \cdot \left( 4p + 1 - p E_{2, c} (4 - E_{2, c}) \right) \right] + \]
\[ + E_{1, c} \cdot \frac{1}{1 - p} \cdot \text{Var}\{R_{2, 1}\}; \]
(94)
The usefulness of this formula has been confirmed by simulations.

REFERENCES