ABSTRACT

The mathematical treatment of random processes often affords the calculation of moments \( M_j \) by means of the appropriate derivatives of a transformed distribution e.g. the Laplace transform \( L(s) \) of a p.d.f. \( p(t) \). The application of this well known method might become complicated, if the transform is a fractional function \( L(s) = \frac{A_o(s)}{B_o(s)} \) and shows at \( s=0 \) the indetermined form \( 0/0 \) which must be evaluated by the rule of l'Hospital.

The indeterminateness is assumed to be of \( m \)-th order \((m=0,1,2,\ldots)\). Based on a Maclaurin's series expansion for nominator \( A_o(s) \) and denominator \( B_o(s) \) a general formula and also a recursion formula are developed which allow the exact calculation of the \( j \)-th derivative \( L^{(j)}(0) \) and thereby of the moment \( M_j \) \((j=1,2,\ldots)\).

The general formula makes use of Faa di Bruno's differentiation formula. The recursion formula can be performed by a universal computer program and is used preferable for moments \( M_j \) of higher order, e.g. \( j > 3 \).

All results can be transferred to other transforms like characteristic function \( C(w) \) and generating function \( G(z) \).

In order to demonstrate the recursion method in an uncomplicated case the moments \( M_j \) of the waiting time and delay time distribution of queueing system \( M/G/1 \) are calculated for orders \( j \geq 8 \).

A typical example could be the following situation: as result of extensive calculations the p.d.f. \( p(t) \) for a certain time random variable \( T \in [0, \infty) \) is given by its L-transform as a fractional function

\[
L(s) = \int_0^\infty p(t) \cdot e^{-st} dt = \frac{A_o(s)}{B_o(s)}
\]

It is intended to gain the moments

\[
M_j = \int_0^\infty t^j \cdot p(t) \cdot dt
deq. (2)
\]

by means of the recommended formula

\[
M_j = (-1)^j \cdot L^{(j)}(0)
\]

where \( L^{(j)}(0) \) is the \( j \)-th derivative of \( L(s) \) at \( s=0 \).

Frequently at \( s=0 \) the fractional function \( L(s) \) shows to be of the indetermined form

\[
\lim_{s \to 0} L(s) = \frac{0}{0}
\]

of the \( m \)-th order \((m=1,2,\ldots)\).

This phenomena requires the repeated use of the rule of l'Hospital in order to verify \( L(0)=1 \) and to obtain the derivatives \( L^{(j)}(0) \) in eq. (3). This operation though in principle straightforward becomes in practice quite often complicated and frustrating time consuming: the first moment \( M_1 \) and perhaps the second one \( M_2 \) is usually all that can be achieved. The formulae involved might become very long e.g. [5] and even powerful computer languages for formula manipulation like ALPAK [6] or SCRATCHPAD [7] might have difficulties to handle them [8].

1.2 In a previous paper [9] methods were developed to gain the moments \( M_1, M_2 \) from indetermined transformed distributions. The results in [9] applied to L-transformed distributions like eq. (1) can be summarized as follows. The indetermined form is of \( m \)-th order which means for denominator \( B_o(s) \) and nominator \( A_o(s) \) in eq. (1) at \( s=0 \).
Once the order \( m \) has been determined, the first two derivatives of \( L(s) \) at \( s=0 \) are given by the equations

\[
\begin{align*}
(1) \quad L(O) &= \frac{(m+1)}{(m+1)!} B(O) \\
(2) \quad \frac{L(O)}{(m+2)} &= \frac{(m+1)}{(m+2)!} \frac{(m+1)!}{D(O)} \cdot D(O)
\end{align*}
\]

where the \( D \)-notation abbreviates the difference of derivatives

\[
\frac{(m+i)!}{A(O) - B(O)}
\]

These formulae can be used without further application of the rule of l'Hospital and other complications.

1.3 In the present paper in extension of [9] methods will be developed to obtain the general moment \( M_j \) resp. the \( j \)-th derivative \( L(j-O) \) according to eq.(3) under the assumption that \( L(O) \) is of the indetermined form \( \frac{(m+i)}{(m+i)!} \) of the \( m \)-th order \( (m=1,2,\ldots) \). This will of course include the proof for eq. (6) and eq. (7) as a special case. The moments \( M_j \) can be used for example to construct approximative distribution functions e.g. [10], [11] resp. bounds on a distribution function [12] in all cases where the exact solution can not be obtained.

2. GENERAL FORMULA FOR \( L(j-O) \)

In this section we make use of methods developed by L. Takács in [13], see section 6.2.

2.1 If we assume an indeterminateness of \( L(O) \) of order \( m \) as expressed by eq.(5) the functions \( A(O) \) and \( B(O) \) in eq.(1) can be expanded into Maclaurin's series

\[
L(s) = \sum_{i=0}^{m} \frac{a_i s^i}{i!} = \frac{A(O)}{B(O)}, \quad s=0
\]

where the coefficients \( a_i \) are given by

\[
\begin{align*}
(m+1)! & \quad A(O) = \frac{a_0}{0!} \\
(m+1)! & \quad B(O) = \frac{a_m}{m!}
\end{align*}
\]

For \( j=1 \) and \( j=2 \) we get from eq.(12b) without difficulty the explicit formula for \( L(1-O) \) resp. \( L(2-O) \) eq.(7) which were obtained in [9] by another approach. For \( j=3 \) in eq.(12b) we get the subsequent explicit formula

\[
\begin{align*}
\gamma_0(s) &= \frac{1}{\gamma(s)} = \gamma_0[\gamma(s)] \\
\gamma_0(s) &= \left\{ \begin{array}{ll}
1, & n=0 \\
\sum_{n=1}^{n} \frac{(-1)^{i+1} \cdot i!}{b_{i+1}^n} \cdot \gamma_n, & n=1,2,\ldots
\end{array} \right.
\end{align*}
\]
\[ L(0) = \frac{6}{(m+1) \cdot B_0(0)} \left[ \frac{1}{B_0(0) \cdot D} \right] \] (13)

where the D-notation is given by eq. (8).

The expressions eq. (6), (7) and (15) are the tool to calculate straightforward the moments \( M_1, M_2 \) and \( M_3 \) according to eq. (3). These expressions and the general formula eq. (12b) are also useful in the case \( m=0 \), that is if \( L(s) \) eq. (1) does not have an indeterminateness at \( s=0 \).

3. RECURSION FORMULA FOR \( L(0) \)

The general formula eq. (12b) can be used to calculate any derivative \( L^{(j)}(0) \) resp. moment \( M_j \), of course also for \( j=3 \). But since the mechanism of Faa di Bruno's formula becomes for high \( j \) resp. \( n \) more and more complicated it might be preferable to calculate \( L^{(j)}(0) \) for \( j=3 \) on the basis of a simple recursion formula which shall be established now.

3.1 The \( j \)-th derivative of \( L(s) \) eq. (9) can be expressed as a fractional function

\[ L(s) = \frac{a_j(s)}{B_j(s)}, \quad j = 0, 1, 2, \ldots \] (16)

For \( j=1 \) we get

\[ L(s) = \frac{a_1(s)}{B_1(s)} = \frac{a_0(s) \cdot B_0(s) - a_0(s) \cdot B_0(s)}{B_0(s)} \] (17)

If the operation eq. (17) is repeated \( j \)-times, it is found that the denominator \( B_j(s) \) in eq. (16) can be expressed as the \((j+1)\)-th power of \( B_0(s) \)

\[ L(s) = \frac{a_j(s)}{B_0(s)} = \frac{\frac{a_{j-1}(s)}{B_0(s)} \cdot \frac{a_0(s)}{B_0(s)} - \frac{a_{j-1}(s)}{B_0(s)} \cdot \frac{a_0(s)}{B_0(s)}}{B_0(s)} \] (18)

By means of eq. (19) we can develop the \( k \)-th derivative of nominator \( a_j(s) \) resp. more general- of a nominator \( a_p(s) \) \((p=1, 2, \ldots j)\)

\[ a_p(s) = \sum_{n=0}^{k} \left[ \frac{(k-n+1)}{(k-n)!} a_{p-1}(s) \cdot B_0(s) - \frac{(k-n+1)}{(k-n)!} a_{p-1}(s) \cdot B_0(s) \right] \] (19)

If we set \( s=0 \) in eq. (18) and (19) and make use of eq. (10b) we get

\[ L(0) = \frac{a_j(0)}{b_j+1} \] (20)

and the recursion formula for derivatives of the nominator

\[ a_p(0) = \sum_{n=0}^{k} \frac{k!}{(k-n)!} a_{p-1}(0) \cdot b_n \] (21)

3.2 The formula eq. (21) is useful to gain the nominator \( a_j(0) \) in eq. (20) and thereby the desired function \( L^{(j)}(0) \) by recursion. This can be done by the following systematic procedure.

3.2.1 Take the expressions for nominator \( A_0(s) \) and denominator \( B_0(s) \) as given by eq. (1) and obtain \( j+1 \) derivatives \( a^{(j+1)}(0) \) and \( j+1 \) coefficients \( b_i \) as defined by eq. (10a,b):

\[ a_0(0) = B_0(0); \quad b_0 = B_0(0) \]

3.2.2 In order to obtain the nominator \( a_j(0) \) in eq. (20) calculate all derivatives \( a^{(j+1)}(0) \) \((k=0, 1, 2, \ldots j)\) by repeated use of recursion formula eq. (21) in upward sequence:

\[ a_1(0), a_2(0), a_3(0), \ldots a_j(0) \] (22)

3.2.3 Calculate

\[ L(0) = \frac{a_j(0)}{b_j+1} \] (23)

All subsequent steps in this procedure especially the recursion part 3.2.2 can be performed by a relative simple universal computer program.

4. OTHER TRANSFORMS

Beside the Laplace-transform the following common transforms\([2,3,4]\) are often fractional functions. The generating function \((z\text{-transform})\) \( G(z) \) in case of a discrete non-negative random variable \( \omega=0 \) described by a p.f. \( p(\omega) \)
G(z) = \sum_{a=0}^{\infty} P(a) \cdot z^a = A_0(z)/B_0(z) \quad (22a)

The discrete characteristic function \( C(\Psi) \) in case of a discrete random variable \( \Omega = \{0, 1, 2, \ldots \} \) described by a p.f. \( P(a) \)

\[ C(\Psi) = \sum_{a=-\infty}^{\infty} P(a) \cdot e^{i\Psi a} = A_0(\Psi)/B_0(\Psi) \quad (22b) \]

The continuous characteristic function \( C(\omega) \) in case of a continuous random variable \( \Omega \) described by a p.d.f. \( p(\tau) \)

\[ C(\omega) = \int_{-\infty}^{\infty} p(\tau) \cdot e^{i\omega \tau} \, d\tau = A_0(\omega)/B_0(\omega) \quad (22c) \]

All these transforms may show at \( z=1 \) resp. \( \Psi = 0 \) resp. \( \omega = 0 \) the indetermined form \( 0/0 \) of the \( m \)-th order (\( m=1, 2, \ldots \)). Obviously all considerations and results for the Laplace-transform \( L(s) \) in section 1, 2 and 3 can be applied to obtain the \( j \)-th derivative \( G^{(j)}(1) \) resp. \( C^{(j)}(0) \). Only in case of the generating function \( G(z) \) in various formulae the notation \( "(0)" \) for \( s=0 \) must be replaced by the notation \( "(1)" \) for \( z=1 \).

5. APPLICATIONS

The following applications are chosen so that the use of eqs. (6), (7), (15) and of section 3.2 can be demonstrated in an uncomplicated environment.

5.1 MOMENT SPECTRUM OF THE RAISED-COSINE FUNCTION

We consider the raised-cosine p.d.f. \( p(\tau) \) as shown in fig.1

\[ p(\tau) = \begin{cases} \frac{1}{T} (1 - \cos \frac{\tau}{T}), & 0 \leq \tau \leq T \\ 0, & \tau > T \end{cases} \quad (23) \]

which can be useful to describe approximately a certain type of randomness.

The L-transform of eq. (23)

\[ L(s) = \frac{1 - e^{-st}}{st[1 + (\frac{st}{2\pi})^2]} = \frac{A_0(s)}{B_0(s)} \quad (24) \]

exhibits at \( s=0 \) the indetermined form \( 0/0 \) of order \( m=1 \). The functions \( A_0(s) \) and \( B_0(s) \) have at \( s=0 \) the derivatives

\[ A_0(0) = (-1)^i \cdot \frac{T^{i+1}}{(2\pi)^i}, \quad i = 0, 1, 2, \ldots \quad (25a) \]

\[ B_0(0) = \frac{6 \cdot T^3}{(2\pi)^2}, \quad i = 2 \quad (25b) \]

From eqs. (6), (7) and (15) we get straightforward the first three moments

\[ M_1 = \frac{1}{2} \cdot T; \quad M_2 = \frac{1}{3} \cdot (1 - \frac{6}{(2\pi)^2} \cdot T^2); \quad M_3 = \frac{1}{4} \cdot (1 - \frac{12}{(2\pi)^4}) \cdot T^3 \quad (26) \]

The moments \( M_j \) for \( j = 4, 5, \ldots \) are obtained by use of a computer program which performs all steps as described in section 3.2. The numerical values for the moments up to the 9-th order are given in the following table and are displayed in fig.2.

<table>
<thead>
<tr>
<th>( j )</th>
<th>( M_j/T^j )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0.5</td>
</tr>
<tr>
<td>2</td>
<td>0.282673</td>
</tr>
<tr>
<td>3</td>
<td>0.174009</td>
</tr>
<tr>
<td>4</td>
<td>0.114078</td>
</tr>
</tbody>
</table>

Fig.2 Moment spectrum of the raised-cosine function

5.2 QUEUEING SYSTEM M/G/1

The well known queueing system M/G/1 with infinite storage capacity is characterized by a negative exponential input process with intensity \( \lambda \). The service time \( TA \) is described by a general p.d.f. \( p(TA) \) resp. its L-transform \( L_T(s) \). It is assumed that \( L_T(s) \) is a known function. Therefore the moments

\[ M_j = \int_{0}^{\infty} t^j \cdot p(tA) \, dtA \quad (27a) \]

\[ M_j = \frac{1}{\lambda} \cdot \frac{1}{(2\pi)^j} \cdot (2\pi)^j \cdot \Gamma(j+1) \quad (27b) \]

\[ M_j = \frac{1}{\lambda} \cdot \frac{1}{(2\pi)^j} \cdot (2\pi)^j \cdot \Gamma(j+1) \quad (27c) \]
can be calculated for any order \( j = 1, 2, \ldots \) The intensity of the service time process is

\[
\lambda = \frac{1}{\bar{T}_A}
\]

(28)

and the ratio of intensities

\[
\rho = \frac{\lambda}{\bar{T}_A}
\]

(29)

means the offered traffic.

5.2.1 WAITING TIME (FIFO)

Assuming the FIFO-queue discipline and the stationary case the waiting time \( T_W \) of system M/G/1 is described by a p.d.f. \( p(T_W) \) whose L-transform e.g. \([14], [15]\)

\[
L_T(s) = \frac{L_{T_A}(s)}{1-L_{T_A}(s)} = \frac{L_T(s)}{1-L_{T_A}(s)}
\]

(30)

shows at \( s=0 \) the indeterminate form \( 0/0 \) of order \( m=1 \). The functions \( A_0(s) \) and \( B_0(s) \) have at \( s=0 \) the derivatives

\[
A_0(0) = \begin{cases} 
1 - \rho, & i = 0 \\
0, & i = 1, 2, \ldots 
\end{cases}
\]

(31)

\[
B_0(0) = \begin{cases} 
1 - \rho, & i = 0 \\
(-1)^i \lambda M_{i+1}(\bar{T}_A), & i = 1, 2, \ldots 
\end{cases}
\]

(32)

If we use the abbreviation

\[
n = \frac{1}{1-\rho}
\]

(33)

the application of eq.(6), (7) and (15) yields the first three moments of \( p(T_W) \) in the form

\[
M_1(T_W) = \bar{T}_W = \frac{1}{2} \cdot n \cdot M_2(T_A)
\]

(34a)

\[
M_2(T_W) = \sigma^2(T_W) + \bar{T}_W^2 = \frac{1}{2} \cdot n^2 \cdot M_2^2(T_A) + \frac{1}{2} \cdot n \cdot M_3(T_A)
\]

(34b)

\[
M_3(T_W) = \frac{3}{2} \cdot n^3 \cdot M_2(T_A) + \frac{1}{2} \cdot n \cdot M_4(T_A) + \sum_{i=4}^{\infty} \frac{1}{i} \cdot n \cdot M_i(T_A)
\]

(34c)

Eq.(34a) is the well known formula of Khintchine and Pollaczek cf. \([16]\); eq.(34b) and eq.(34c) have been derived in \([13], [17]\). If we assume as an example the general service time p.d.f. \( p(T_A) \) to be the raised-cosine function according to section 5.1 we get the moments \( M_1(T_W), \ldots, M_3(T_W) \)

as a function of \( \rho \), see fig. 3a. Fig. 3a shows also the moments \( M_4(T_W), \ldots, M_8(T_W) \) which were calculated by recursion, see section 3.2. In all calculations the values for the moments \( M_j(T_A) \) were taken from the table in section 5.1. The higher moments \( M_j(T_W) \) \((j=4, 5, \ldots)\) could also be calculated on the basis of a general formula in \([13]\) which can be easily obtained by introducing eq.(31) and (32) into eq.(12b).

Note. In \([13]\) the term "delay time" is used for what is "waiting time" in the present paper.

Fig. 3 Queueing system M/G/1: moments \( M_j(T_W) \)

resp. \( M_j(T_B) \) as a function of the offered traffic \( \rho \). The assumed service time p.d.f. \( p(T_A) \) is the raised-cosine function, see eq.(23).
5.2.2 DELAY TIME (FIFO)

The delay time $\tau_B = \tau_W + \tau_A$ ("waiting and being served") of system M/G/1 is described by a p.d.f. $p(\tau_B)$ whose L-transform

$$L_x(s) = L_x(s), L_Y(s) = \frac{A_Y(s)}{s - \lambda [1 - L_Y(s)]}, L_Z(s) = \frac{B_Y(s)}{s - \lambda}$$

shows again at $s = 0$ the indeterminate form $O(0)$ of order $m = 1$.

While the derivatives $B_Y(0)$ are identical represented by eq. (32) we find now for the derivative of nominator $A_Y(s)$ at $s = 0$

$$A_Y(0) = (-1)^i (1 + i)^{-1} \frac{p}{(1 - \lambda)}$$

The application of eq. (6), (7) and (15) yields the first three moments of $p(\tau_B)$ in the form

$$M_1(\tau_B) = M_1(\tau_W) + \frac{1}{\mu} \text{ resp. } \tau_B = \tau_W + 1/\mu$$

In these formulae the moments concerning waiting time $\tau_W$ are given by eq. (34a,b,c) and the intensity $\mu$ is defined by eq. (33). It should be noted that eq. (37b) is in accordance with the addition law for variances of independent random variables

$$\sigma^2(\tau_B) = \sigma^2(\tau_W) + \sigma^2(\tau_A)$$

The higher moments $M_j(\tau_B)$, $j > 3$ can be calculated by recursion, see section 3.2.

6. APPENDIX: DIFFERENTIATION FORMULAE

In the present paper two general differentiation formulae are used. The following notation for integer variables has been chosen such that the formal correspondence to the application of these formulae in section 2 is as close as possible.

6.1 The $j$-th derivative of a function of two functions $f(s) = g(s) \cdot h(s)$ is

$$f(s) = \sum_{n=0}^{j} \binom{j}{n} g(s)^n h(s)^{j-n}, j = 0, 1, 2, \ldots$$

This formula corresponds to the well known binomial expansion if the powers are replaced by the orders of derivatives.

6.2 The following differentiation formula of Faa di Bruno [18] has been introduced successfully by L. Takács as a tool for the determination of moments in queueing system M/G/1 [13]. This useful formula being unknown in most formula collections describes the $n$-th derivative of a function of a function

$$f(s) = f[g(s)]$$

and can be written in the form

$$f(s) = n! \sum_{i=1}^{n} \binom{n}{i} \frac{g(s)^{n-i}}{(n-i)!}$$

where the notation $f[g]$ stands for the $i$-th derivative of $f[g]$ with respect to $g$. The function $Q_{n,i}(s)$ is given by the expression

$$Q_{n,i}(s) = \sum_{q_1,q_2,\ldots,q_n} \frac{g(s)^{q_1} g(s)^{q_2} \cdots g(s)^{q_n}}{q_1!q_2!\cdots q_n!}$$

where the sum has to be taken over all $n$-tupels $(q_1, q_2, \ldots, q_n)$ which for every pair $n, i$ are determined by the two equations

$$q_1 + q_2 + \ldots + q_n = n$$

$$q_1 + 2q_2 + \ldots + nq_n = i$$

For $n = 4$ we obtain the following values [13]

<table>
<thead>
<tr>
<th>$n$</th>
<th>$i$</th>
<th>$q_1$</th>
<th>$q_2$</th>
<th>$q_3$</th>
<th>$q_4$</th>
</tr>
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<tbody>
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<td>1</td>
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<td>4</td>
<td>0</td>
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<td>0</td>
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</tr>
</tbody>
</table>

For $n = 4$, $i = 2$ we observe the case that eq. (40c) yields for a pair $n, i$ more than one $n$-tupel $(q_1, q_2, \ldots, q_n)$.

Both formulae eq. (39) and eq. (40a,b,c) can be verified by repeated differentiation of the function $f(s)$ in each case. No doubt the formula mechanism of Faa di Bruno is far away from being obvious.

ACKNOWLEDGEMENT

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ABBREVIATIONS

- p.f. = probability function
- p.d.f. = probability density function
- L-transform = Laplace-transform
- FIFO = First In First Out
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