On Optimal Dimensioning of a Certain Local Network

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ABSTRACT

A group of m sources (telephone exchanges) is offering traffic to a group of n local exchanges. From each source $S_k$ ($k=1,2,\ldots,m$) there are $N_{ki}$ trunks leading directly to local exchange $E_i$ ($i=1,2,\ldots,n$). A call originating at $S_k$ is transmitted first to local exchange $E_i$ with probability $q_{ki}$ ($q_{ki}=1$). If the call’s destination is $E_j$ ($j\neq i$) rather than $E_i$, the call is transferred from $E_i$ to $E_j$.

For such a network, the optimal economic dimensioning of trunks (i.e., optimal allocation of the $N_{ki}$’s) is determined. It is shown that, for each source exchange, the optimal dimensioning is to direct all trunks to a single local exchange (which may differ for distinct sources). This single local exchange is determined as a function of the cost of trunks (i.e., distances) between the exchanges and the load offered by the sources to the various local exchanges.

A further complexity of the situation is the fact that, besides receiving calls originating at the various source exchanges, each local exchange offers traffic to its sister exchanges in the group. This traffic is routed directly according to its destination – but it is added to that part of the traffic which is generated by the source exchanges, and is rerouted among the local exchanges due to the random allocation of flow.

The objective of this study is to find the optimal dimensioning of trunks in the telephone network described above - i.e., the optimal partitioning and the optimal routing of trunks so as to achieve minimum investment costs while maintaining a required grade of service. It will be shown that the optimal economic allocation of trunks is to direct all trunks from a certain source exchange to a single local exchange. This local exchange may be different for distinct source exchanges; it is determined as a function of the cost of trunks (i.e., distances) between the exchanges and the load offered by the sources to the various local exchanges. This last result is an extension of an earlier study by the author [3] in which the measure of effectiveness to be minimized was taken to be the total flow in the network. It will be shown that the present framework reduces to the previous one if all costs of trunks (distances) between the exchanges are assumed to be alike.

In fact, the results obtained in [3] are qualitatively the same - namely, to direct all trunks from a given source exchange to a single local exchange. However, this local exchange is the one to which the offered load is maximal, and it may not coincide with the local exchange determined by the minimum-cost objective function.
The question of which objective function is most appropriate is usually left to be answered by the management. If one designs his network to minimize investment costs, he may not get the best possible grade of service. On the other hand, if he wants the best possible grade of service (i.e., with the minimum unnecessary flow in the system), which will be effective for a long period of time, he may find himself investing a little bit more.

2. THE MODEL

A group of \( m \) source exchanges offers traffic to a group of \( n \) local exchanges. Let \( A_{ki} \) \((k=1,2,\ldots,m; i=1,2,\ldots,n)\) be the offered traffic from source exchange \( S_k \) to local exchange \( E_i \). \( A_{ki} \) is given in Erlangs, and is assumed to be constant over the period under consideration. The total amount of traffic offered by source exchange \( S_k \) to all local exchanges is \( \sum_{i=1}^{n} A_{ki} \).

To assure a required grade of service, the total number of trunks leading from \( S_k \) to all local exchanges, \( N_{ki} \), is determined by standard teletraffic methods \([2]\). The problem we are concerned with is the optimal allocation of the \( N_{ki} \) trunks among the \( n \) local exchanges; that is, for each source exchange \( S_k \) \((k=1,2,\ldots,m)\), we wish to find the optimal partitioning of trunks \( N_{ki}=1,2,\ldots,n \) such that \( \sum_{i=1}^{n} N_{ki} = N_k \), where \( N_k \) is the number of trunks leading directly from \( S_k \) to local exchange \( E_i \) \((i=1,2,\ldots,n)\).

An optimal partitioning is one whose total investment cost in trunks is minimal (the cost of switching equipment is assumed to be almost the same for every partitioning of \( N_{ki} \)). It follows immediately that one should take into consideration the distances between the various exchanges. These distances may be expressed in terms of the cost of trunks between the various exchanges. Thus, we let \( q_{ki} \) \((k=1,2,\ldots,m; i=1,2,\ldots,n)\) be the cost of a single trunk from \( S_k \) to \( E_i \) (and, if one wishes, he may add to it the cost of the switching equipment).

There is another source of flow in the system. Every local exchange is offering traffic to each of its sister local exchanges. Denote \( Z_{ij} \) \((i,j=1,2,\ldots,n)\) the local traffic offered by \( E_i \) to \( E_j \). This traffic flows directly from \( E_i \) to \( E_j \) with no detours. Moreover, the \( Z_{ij} \) are independent of the \( N_{ki} \)’s. As a consequence, it follows that the \( Z_{ij} \) do not affect the optimal allocation of the \( N_{ki} \)’s.

Let \( d_{ij} \) \((i,j=1,2,\ldots,n)\) be the cost of a single trunk connecting local exchange \( E_i \) to local exchange \( E_j \). (In many cases, the \( d_{ij} \)’s and the \( q_{ki} \)’s are linear functions of the distances between \( E_i \) and \( E_j \) and \( S_k \) and \( E_i \), respectively. However, this assumption is not required for the analysis of the system).

Now, consider a call originating at \( S_k \). For ease of presentation and analysis, we assume that the grading is such that the traffic offered by \( S_k \) is distributed evenly among its \( N_k \) outgoing trunks. However, the final results remain the same if this assumption is relaxed and we assume that the traffic offered by \( S_k \) is distributed among the local exchanges according to some arbitrary distribution \( \{q_{ki}, i=1,2,\ldots,n\} \). Thus, assume that the probability of our call being directed to \( E_i \) is given by \( q_{ki} = N_{ki}/N_k \), \((k=1,2,\ldots,m; i=1,2,\ldots,n)\).

This probability is independent of the destination of the call. If it turns out that the call arrives at \( E_i \) but its destination is \( E_j \) \((j\neq i)\), then it is retransferred from \( E_i \) to \( E_j \).

We assume that the system is designed for small losses - i.e., the \( N_{ki} \)’s are determined such that, for each source exchange \( S_k \), the actual carried traffic from \( S_k \) to \( E_i \) is given by

\[
\sum_{j=1}^{n} Z_{ij} = \lambda_{ki} = \lambda_{ki} q_{ki} \quad (k=1,2,\ldots,m; i=1,2,\ldots,n).
\]

In other words, from the total amount of \( A_k \) Erlangs offered by \( S_k \) to all local exchanges, a proportion \( q_{ki} \) flows directly to \( E_i \). However, only \( \lambda_{ki} q_{ki} \) Erlangs of this amount have \( E_i \) as their destination. The rest of the traffic, amounting to \( \lambda_{ki} - \lambda_{ki} q_{ki} \) Erlangs, has to be retransferred to the various destinations. The amount of traffic to be transferred from \( E_i \) to \( E_j \) \((j\neq i)\) is \( \lambda_{ki} q_{ki} \) where, clearly, \( \lambda_{ki} = \lambda_{ki} q_{ki} + \sum_{j=1, j\neq i}^{n} \lambda_{kj} q_{kj} \). Since \( E_i \) receives calls from all source exchanges, the total amount of traffic which has to be retransferred from \( E_i \) to \( E_j \) is given by

\[
\lambda_{ij} = \sum_{k=1}^{m} \lambda_{ki} \lambda_{kj} \quad (i,j=1,2,\ldots,n; i\neq j).
\]

This retransferred traffic is added to \( Z_{ij} \), the load offered by \( E_i \) to \( E_j \), such that the total flow from \( E_i \) to \( E_j \) is

\[
\lambda_{ij} = \sum_{k=1}^{m} \lambda_{ki} \lambda_{kj} + Z_{ij} \quad (i,j=1,2,\ldots,n; i\neq j).
\]

Thus, every allocation of the \( N_{ki} \)’s determines the set of flows \( \{Z_{ij}\}\) through the \( N_{ki} \)’s - and, for any such set, the number of trunks required to carry the flow in each route is determined by standard teletraffic tables that guarantee a high grade of service. The network is illustrated in Figure 1.

3. OPTIMAL DIMENSIONING

A quick study of standard teletraffic tables - or, equivalently, an analysis of Erlang's loss formula such as in \([1]\) - reveals that if the offered traffic is large enough, the number of trunks required to carry the flow at a given grade of service is very closely proportional to the offered load. Let this proportion be denoted by \( a \). To carry one Erlang of offered or transferred traffic, \( a > 1 \) trunks are needed between the corresponding exchanges so that the network’s performance will meet the specified grade of service. In other words, the number of trunks required between \( S_k \) and \( E_i \) is \( a \lambda_{ki} \), and the number of trunks to be allocated to carry the traffic between \( E_i \) and \( E_j \) is \( a \lambda_{ij} \).
Our problem, therefore, is to find an allocation of trunks that will minimize the total investment cost in the network,

\[ C = \sum_{k=1}^{m} \sum_{i=1}^{n} C_{ki} a_{x_{ki}} + \sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij} (a_{x_{ij}}) . \]

Substituting the values of \( y_{ki} \) and \( x_{ij} \) from equations (1), (2) and (3) we get

\[ C = a_{x_{ki}} \sum_{k=1}^{m} C_{ki} a_{x_{ki}} + \sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij} (a_{x_{ij}}) . \]

Since \( \sum_{j=1}^{n} d_{ij} z_{ij} \) is a constant term independent of the allocation of the \( N_k \)'s, and we are using the relations \( A_{k} = C_{ki} a_{x_{ki}} \) and \( q_{ki} = N_{ki}/N_k \), our problem is finally stated as: find non-negative integer-valued variables \( N_k \)'s (\( k=1,2,\ldots,m \); \( i=1,2,\ldots,n \)), so as to minimize

\[ \left( \frac{m}{k=1} A_{k} (q_{ki} + z_{ij}) \right) N_k . \]

subject to

\[ \sum_{i=1}^{n} N_{ki} = N_k \quad (k=1,2,\ldots,m) . \]

It follows immediately that (6) and (7) can be separated into m independent problems. Each such problem corresponds to a single source exchange, and is an n-variable Integer Linear Programming problem with a single constraint. For each source exchange \( S_k \) (\( k=1,2,\ldots,m \)), the problem is

\[ \text{(8) minimize } \sum_{i=1}^{n} \sum_{j=1}^{n} A_{k} (C_{ki} a_{x_{ki}} + d_{ij}) N_{ki} \]

subject to

\[ \sum_{i=1}^{n} N_{ki} = N_k \quad (k=1,2,\ldots,m) . \]

The solution of (8) and (9) follows readily: find the local exchange \( E_i(k) \) for which \( \sum_{i=1}^{n} A_{k} (C_{ki} a_{x_{ki}} + d_{ij}) \)

\[ \text{minimize } \sum_{i=1}^{n} \sum_{j=1}^{n} A_{k} (C_{ki} a_{x_{ki}} + d_{ij}) N_{ki} \]

subject to

\[ \sum_{i=1}^{n} N_{ki} = N_k \quad (k=1,2,\ldots,m) . \]

The qualitative advantage that it is independent of the actual value of \( N_k \).

Before proceeding to a special case where \( C_{ki} = 1 \) for all \( k \) and \( i \), and \( d_{ij} = 1 \) for all \( i,j \), \( d_{ii} = 0 \), we want to give an interpretation of our result. The term

\[ \sum_{i=1}^{n} \sum_{j=1}^{n} A_{k} (C_{ki} a_{x_{ki}} + d_{ij}) N_{ki} \]

in equation (8) has the dimension of Erlangs x cost. If all of the flow from \( S_k \) is directed to \( E_k \), then \( A_{k} C_{ki} a_{x_{ki}} \) is the investment cost of the \( N_k \) trunks leading from \( S_k \) to \( E_k \). In such a case, \( A_{k} \) is the amount of traffic to be transferred from \( E_k \) to \( E_j \). Thus, the investment cost for trunks between \( E_k \) and \( E_j \) is \( A_{k} C_{k} a_{x_{ki}} z_{ij} \).

Equation (5) may be interpreted in the same way. It should be noted that because of the proportionality role of \( a \), the \( z_{ij} \)'s do not affect the (optimal) solution.

If the costs \( C_{ki} \) and \( d_{ij} \) are proportional to the corresponding distances between the exchanges, then the optimal partition is the one which minimizes the summation of products of Erlangs x distances. That is, the optimal solution is the one for which the overall distance of total traveled traffic is minimal. This last objective
function may serve as a measure of effectiveness in other network studies.

4. A SPECIAL CASE

Suppose that $c_{ki} = 1$ for all $k$ and $i$, $d_{ij} = 1$ for $i 
eq j$, and $d_{ii} = 0$ for all $i$. That is, we assume that all distances between the various exchanges are alike, and we are interested only in minimizing the flow of traffic in the network. In such a case, each of the problems (8) becomes

\[
\text{(10) minimize } \frac{1}{N_k} \left[ \sum_{i=1}^{n} \left( \sum_{j=1}^{n} A_{ij} N_{ki} \right) \right]
\]

subject to (9).

Since $\frac{1}{N_k}$ is a given constant, the problem is equivalent to

\[
\text{(11) minimize } \sum_{i=1}^{n} \left( \sum_{j=1}^{n} A_{ij} N_{ki} \right) - \sum_{i=1}^{n} A_{ik} N_{ki}.
\]

Since $\sum_{j=1}^{n} A_{ij} N_{ki} = A_k N_k$, the problem is equivalent to

\[
\text{(12) maximize } \sum_{i=1}^{n} A_{ik} N_{ki}
\]

subject to (9).

This case was considered in [3].

The optimal solution of (12) under (9) is, for each source exchange $S_k$, to find the local exchange $E_i(k)$ for which $A_{ki}(k) = \max \{A_{ki}\}$, and to direct all $N_k$ trunks to that local exchange. In other words, all the traffic is directed to the local exchange to which the offered load is maximal.

As was pointed out in the Introduction, one should decide whether he wishes to design his network according to objective (8), which emphasizes the cost function, or according to equation (12), which emphasizes the traffic point of view.

REFERENCES