DERIVATIVES OF WILKINSON FORMULA
AND THEIR APPLICATION TO OPTIMUM DESIGN
OF ALTERNATIVE ROUTING SYSTEMS

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ABSTRACT

In the design of alternative routing systems, the equivalent random theory is widely applied in practice.

This paper presents the derivatives of Wilkinson formula which gives the mean and variance of non-random overflow traffic in the equivalent random theory.

As an application of the derivatives, an optimum design of alternative routing systems is proposed, which minimizes the system cost under given service criteria.

First, the optimum conditions for the basic triangular model are studied. Next, the optimizing method is expanded to the two-stage overflow model. The optimizing theory is presented as well as some examples of the optimum design for practical application.

1. INTRODUCTION

Communication networks can be efficiently utilized by introducing the alternative routing system, where overflow traffic from direct route is carried through alternative routes.

In dimensioning alternative routing systems, the equivalent random theory proposed by R. T. I. Wilkinson gives excellent accuracy and has been widely applied [1]. In the equivalent random theory, non-random traffic is evaluated by using two parameters, the mean and variance.

In applying the equivalent random theory, the optimum design of alternative routing systems is rather complicated because of the two parameters. So far, a number of relevant works on such optimum design have been studied, involving methods for determining a range of the optimum dimensions, and approximation theories for computer calculations [2][3][4].

One of the authors has presented an optimum design of switching systems by introducing the derivatives of Erlang B formula [5]. Expanding this method, the present paper proposes a new method to facilitate the optimum design of alternative routing systems.

The derivatives of Wilkinson formula, which gives the mean and variance of overflow traffic, are derived and a table of them has been prepared by computer. In the first step applying the derivatives, the optimum conditions for the basic triangular model are analysed. In the next step, based on the fundamental analysis the optimum design for the two-stage overflow model is developed.

Applying the theories proposed, some examples of optimum design are presented. Design charts prepared by this theory will provide a ready and accurate optimization of the alternative routing systems.

2. DERIVATIVES OF WILKINSON FORMULA

The mean b, and variance v of non-random traffic overflows from s trunk circuits with random traffic of a erlangs offered are given by Wilkinson formula

\[ b = b_E = b(a, s) \]
\[ v = v(b(1 - b + a/t) = v(a, s) \]

where \( t = s + l - a + b \), and \( E_s \) is Erlang B formula which is expanded for a continuous \( s \) as

\[ E_s = \frac{a^s}{s!} \sum_{r=0}^{s} \frac{a^r}{r!} = \frac{a^s e^{-a}}{\Gamma(s+1, a)} \]

where \( \Gamma(s+1, a) = \int_a^\infty x^{s-1} e^{-x} dx \) is an incomplete Gamma function of the second kind.

Partial derivatives of Eq.(1), denoted by \( b_a, b_s, v_a, v_s \), etc., are derived as

\[ b_a = tE_s \]
\[ b_s = -b \frac{a^s}{s!} \]
\[ v_a = v/t - t(b^2 - v)/a \]
\[ v_s = ab(b - l)/t^2 + (b^2 - v) \]

where \( \Psi = -a \log E_s / s \) for which evaluation formulas and a mathematical table have been presented by the authors [6][7]. In particular, for a non-negative integer \( s \), \( \Psi \) is given by

Fig.1 Derivatives of Wilkinson formula

(a)

\( a = 1 \) erl.

\( 3 \) 10

\( 30 \)

\( 5 \)

\( 7 \)

\( 10 \)

\( 1 \) 2 3 5 7 10 20 30 50 70 100

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The derivatives in Eq. (3) have been prepared in a mathematical table, a sample of which is shown in Table 1. Examples of the derivatives are illustrated in Fig. 1(a)-(d).

It can be seen from Fig. 1 that $b_\alpha$ and $b_\beta$ are monotone functions against $a$ and $s$, while $v_\alpha$ and $v_\beta$ have extremums around $a = s$. It is also noted that all the derivatives tend to vanish according when $a \approx s$.

3. BASIC TRIANGULAR MODEL

In a basic triangular alternative routing model shown in Fig. 2, the equivalent random theory is applied as follows.

Let $a_1$ and $s_1$ be the random traffic and the number of trunk circuits, respectively, in the direct route $A-C$. The mean $b_1$ and variance $v_1$ of the overflow traffic from the direct route are calculated by Eq. (1) and can be represented as

$$
\begin{align*}
    b_1 &= b(a_1, s_1) \\
v_1 &= v(a_1, s_1).
\end{align*}
$$

The primary random traffic in the alternative route $A-B$ and $B-C$ is denoted by $a_{0i}$, with $i = 2$ for $A-B$ and $i = 3$ for $B-C$. Thus, the mean $a_i$ and variance $v_i$ of the resultant non-random traffic offered to the alternative route is given by

$$
\begin{align*}
a_i &= a_{0i} + b_1 \\
v_i &= a_{0i} + v_1,
\end{align*}
$$

The equivalent random traffic $a^*_i$ and the number of fictive trunk circuits $s^*_i$ are determined by the equation system

$$
\begin{align*}
b(a^*_i, s^*_i) &= a_i \\
v(a^*_i, s^*_i) &= v_i,
\end{align*}
$$

Therefore, the mean and variance of the overflow traffic from the alternative route are given by, respectively,

$$
\begin{align*}
b_1 &= b(a^*_1, s^*_1) \\
v_1 &= v(a^*_1, s^*_1),
\end{align*}
$$
where \( s_i \) represents the number of actual trunk circuits in the alternative route.

The total cost of the alternative routing system shown in Fig. 2 is represented as

\[
f = c_1s_1 + c_2s_2 + c_3s_3
\]

(9)

where \( c_1 \) is the cost of a trunk circuit in the respective route. The cost of the switching facility in point B may be included in \( c_2 \) or \( c_3 \).

The optimum value of \( s_i \), minimizing the system cost, is determined by differentiating \( f \) with respect to \( s_i \) and equating the result to zero. Hence we have the optimum condition

\[
c_i = b_{s_1}(c_2s_2a_2 + c_3s_3a_3) - v_{s_1}(c_2sw_2 + c_3sw_3)
\]

(10)

where \( b_{s_1} = b_s(a_i, s_i) \) and \( v_{s_1} = v_s(a_i, s_i) \)

which can be calculated by Eq. (3) or found in the mathematical table. On the other hand, \( a_i = g_i/a_i \) and \( s_i = g_i/s_i \) are determined by the service criteria applied in the alternative route, as will be described in the following section.

In the case of \( a_{02} = a_{03} = a_0 \), which is referred to as the symmetric model, Eq. (10) is reduced to

\[
k = -1/(b_{s_1}s + v_{s_1}s)
\]

(11)

where \( k = c_2 + c_3/c_1 \) is the cost ratio of alternative to direct route, and \( s_1 \) and \( s_2 \) are the derivatives corresponding to \( a_2 = a_3 = a \), and \( s_2 = w_3 = w \).

In the general case where \( a_{02} \neq a_{03} \), it is required to solve Eq. (10) by using the iteration method in order to get an exact solution. However, that by introducing a weighted average

\[
\tilde{a}_0 = (c_2a_{02} + c_3a_{03})/(c_2 + c_3),
\]

(12)

the general case is reduced approximately to the symmetric model by substituting \( \tilde{a}_0 \) for \( a_0 \). Thus, Eq. (11) can be also applied to the general case.

4. OPTIMIZATION UNDER CONSTRAINT

The following two service criteria for the alternative route are considered:

- Criterion 1: The mean of overflow traffic from the alternative route is maintained at a predetermined value \( b_0 \). That is

\[
b_i = b(a_i, s_i) \quad \text{for} \quad i = 2, 3
\]

(13)

- Criterion 2: The call loss of the alternative route, or the ratio of \( b_i \) to \( a_i \), is maintained at a predetermined value \( b_0 \). That is

\[
b_i/a_i = b_i/b(a_i, s_i) \quad \text{for} \quad i = 2, 3
\]

(14)

According to the equivalent random theory as described in the previous section, we have the constraints of Eq. (7). Therefore, for Criterion 1, there are the following constraints, with the suffix \( i = 2, 3 \) omitted:

\[
\begin{align*}
g_1 &= b(a_*, s*) - a = 0 \\
g_2 &= v(a_*, s*) - w = 0 \\
g_3 &= b(a_*, s+s*) - b_0 = 0
\end{align*}
\]

(15)

Differentiating \( g_1 \), \( g_2 \) and \( g_3 \) by \( a \), respectively, yields simultaneous equations

\[
\begin{align*}
b^*a^* + b^*s^* &= 1 \\
v^*a^* + v^*s^* &= 0 \\
b^*a^* + b^*s^* + b^*s &= 0
\end{align*}
\]

(16)

where \( a^* = 2a*/a^0, s^* = 2s*/a^0. \) And the derivatives defined by

\[
\begin{align*}
b^* &= b((a^*, s^*), v^* = v((a^*, s^*) \\
b^* &= b((a^*, s+s*), v^* = v((a^*, s+s*)
\end{align*}
\]

(17)

Simultaneously, differentiating Eq. (15) by \( w \), we obtain

\[
\begin{align*}
s_w &= J(b, v^*)/b_sJ(b^*, v^*)
\end{align*}
\]

(18)

Now, using Eqs. (17) and (18), we can calculate Eq. (10) for Criterion 1. In particular, for the symmetric model, we have from Eq. (11)

\[
k = b_sJ(b^*, v^*)/[b_{s1}J(b, v^*) - v_{s1}J(b, b^*)].
\]

(19)

In the case of Criterion 2, we have the constraints, omitting suffix \( i = 2, 3 \),

\[
\begin{align*}
g_1 &= b(a_*, s*) - a = 0 \\
g_2 &= v(a_*, s*) - w = 0 \\
g_4 &= b(a_*, s+s*) - b_0b(a_*, s^*) = 0
\end{align*}
\]

(20)

In the similar process as Criterion 1, we can derive the derivatives subject to Eq. (20)

\[
\begin{align*}
s_a &= [b_0 - J(b, v^*)/J(b^*, v^*)]/b_s
\end{align*}
\]

(21)

It should be noted that \( s_w \) in Eq. (21) is the same as that in Eq. (18) for Criterion 1.

Using Eq. (21), we can calculate the optimum condition for Criterion 2 by Eq. (10). For the symmetric model, we have

\[
k = b_s/\{[b_{s1}J(b, v^*) - v_{s1}J(b, b^*)]/J(b^*, v^*) - b_{s1}b_0]\}
\]

(22)
By using the derivatives in Eq. (3) which are tabulated, we can calculate the numerical relation between the cost ratio \( k \) and the optimal \( s_1 \) from Eq. (19) or (22) in a straightforward method for the symmetric model.

An example of calculated results for Criterion 2, with \( B_0 = 0.01, a_1 = 5 \text{ erlangs}, \) and \( a_0^2 = a_0^3 = a_0 \) being a parameter, is shown in Fig. 3. It will be seen from the figure that the optimal \( s_1 = 5 \) for \( a_0 = 15 \text{ erlangs} \) and \( k = 1.5 \), for example. It is also shown that the direct route should not be provided when \( k < 1.2 \).

In the conventional practice, an approximate method is being used for determining the optimum number of direct route trunk circuits, from the following relation:

\[
k = \frac{\text{Additional Trunk Capacity (ATC) of A.R.}}{\text{Last Trunk Capacity (LTC) of D.R.}}
\]  

(23)

A calculated example by Eq. (23) with ATC = 0.83 erlang is also drawn as a dashed line in Fig. 3. It will be recognized that the approximate method by Eq. (23) presents a good agreement with the exact values for the range of \( k > 1.4 \) and \( a_0 > 20 \) erlangs, but the error becomes significant for \( k < 1.3 \).

If the design charts such as Fig. 3 are provided for various values of \( a_1 \), the optimum number of the direct route trunk circuits can be readily and exactly determined for a given cost ratio \( k \).

5. TWO-STAGE OVERFLOW MODEL

Let us proceed to a more complex model based upon the optimization method developed in the previous section. Assume a two-stage overflow model shown in Fig. 4. The alternative routes are assumed to be symmetrical, and notations are denoted as illustrated.

Overflow traffic from the direct route is offered to the intermediate route A-B-C, like as the basic model. On the other hand, from the intermediate route, the traffic with mean \( b_2 \) and variance \( v_2 \) overflows to the final route A-D-C. For the final route, Criterion 2 is applied with a constant \( B_0 = b_3/a_3 \). The equivalent random theory is applied to the overflow traffic as described in the basic model.

The system cost is represented by

\[
f = c_1s_1 + 2c_2s_2 + 2c_3s_3.
\]  

(24)

The optimum values of \( s_1 \) and \( b_2 \) which minimize the cost \( f \) are determined by the following relations:

\[
\frac{\partial f}{\partial s_1} = 0, \quad (b_2 = \text{constant})
\]  

(25)

\[
\frac{\partial f}{\partial b_2} = 0, \quad (s_1 = \text{constant}).
\]  

(26)

Putting the cost ratios \( k_1 = 2c_2/c_1 \) and \( k_2 = c_3/c_2 \), and solving Eqs. (25) and (26) simultaneously as shown in APPENDIX, we get

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**Fig. 3** Example of optimum design chart

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**Fig. 4** Two-stage overflow model
The authors wish to express their appreciations to the members concerned in Electrical Communication Laboratories for their kind cooperations and discussions on this work.

REFERENCES


APPENDIX

Derivation of Eq. (27)

From Eq. (26), it follows for a constant \( s_1 \) that

\[
c_2 \left( \frac{\partial s_2}{\partial b_2} \right) + c_3 \left( \frac{\partial s_3}{\partial b_2} \right) = 0. \quad (A1)
\]

Noting that

\[
\gamma s_3 / \partial b_2 = \left( s_{a3} b_2 + s_{w3} v_{a2} \right) / b_2,
\]

we get from Eq. (A1)

\[
k_2 = - 1 / \left( s_{a3} b_2 + s_{w3} v_{a2} \right), \quad (A2)
\]
where $s_3$ and $w_3$ are given by Eq. (21), since Criterion 2 is applied for the final route. Thus, we get $k_2$ in Eq. (27) from Eq. (A2).

Next, from Eq. (25) we get for a constant $b_2$

$$c_1 + 2c_2(s_2^2/s_1) + 2c_3(s_3^2/s_3) = 0. \quad (A3)$$

Noting that

$$2s_2^2/s_1 = s_2^2 + s_2 w_1^2$$

$$2s_3^2/s_3 = [s_3(b_2^2) + w_3(v_2^2)](s_2^2/s_3),$$

we get from Eq. (A3)

$$k_1 = -1/(s_2^2 b_1 + s_2 w_1^2)(1 + r k_2). \quad (A4)$$

Table 1 Sample of Derivatives of Wilkinson Formula

<table>
<thead>
<tr>
<th>$s = 5$</th>
<th>$a$</th>
<th>$b$</th>
<th>$v$</th>
<th>$b_a$</th>
<th>$v_a$</th>
<th>$-b_s$</th>
<th>$-v_s$</th>
<th>$J(b,v)$</th>
<th>$a$</th>
</tr>
</thead>
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<tr>
<td>1</td>
<td>0.00307</td>
<td>0.00367</td>
<td>0.01355</td>
<td>0.01905</td>
<td>0.00524</td>
<td>-0.06673</td>
<td>0.00000</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.07339</td>
<td>0.1094</td>
<td>0.14984</td>
<td>0.27684</td>
<td>0.07587</td>
<td>-0.11015</td>
<td>0.10721</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.35016</td>
<td>0.51859</td>
<td>0.56650</td>
<td>0.61034</td>
<td>0.22646</td>
<td>0.35001</td>
<td>0.19944</td>
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<td></td>
</tr>
<tr>
<td>4</td>
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<td>1.30127</td>
<td>0.55664</td>
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<td>0.38680</td>
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</tr>
<tr>
<td>5</td>
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<td>1.18537</td>
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<tr>
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<td>0.4262</td>
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<td></td>
</tr>
</tbody>
</table>

In Eq. (A5), \((b_2^2)v_2^2\) represents \(v_2^2\) under a constant \(b_2\) and of course vanishes in this case.

On the other hand, \((v_2^2)b_2\) represents \(v_2^2\) under a constant \(b_2\), and can be derived by differentiating \(v\) with respect to \(s\) subject to a constant \(b\) in Eq. (1), thus we have

\[
(v_2^2)b = -(a + t)b_s + a_t/b^2.
\]

$s_3$ is given by Eq. (21), for Criterion 2, and finally we have $k_1$ in Eq. (27) from Eq. (A4).