SUPERIOR CHANNEL GRAPHS

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ABSTRACT
We survey the literature of superior channel graphs and present them in an orderly way by classifying the results according to the graph structure and the probability model. By working with an explicit structure, we are able to unify, to simplify, and to generalize many existing results.

1. INTRODUCTION
A graph is called a multistage graph, or an s-stage graph, if its vertex set can be partitioned into s subsets, called stages,

\[ V_1, \ldots, V_s \]

and its edge set into \( s - 1 \) subsets

\[ E_1, \ldots, E_{s-1} \]

such that \( E_1 \) connects \( V_1 \) with \( V_1' \), etc. For \( v \in V_i \) the number of edges in \( E_i \) incident to \( v \) is called the indegree (outdegree) of \( v \) while the degree of \( v \) is simply the pair (indegree, outdegree), written as indegree \( \times \) outdegree. A link system is a switching network which can be represented by a multistage graph by taking switches as vertices and links as edges. The size of a switch \( v \), i.e., the numbers of inlinks and outlinks of \( v \), simply corresponds to the degree of the vertex \( v \).

Each link in a link system can either be in a blocking state, meaning it is carrying a call, or in a nonblocking state. Let a path between two switches (vertices) \( x \in V_i \) and \( y \in V_j \) denote a chain of \( j - i \) links (edges) connecting \( x \) and \( y \) (a path is also called a channel for \( x \in V_i \) and \( y \in V_j \)). Then a pair (input switch, output switch) is blocked if every channel between them contains at least one blocking link. A simple model to study the blocking probability of such a path is to look at its channel graph, originally called the linear graph by Lee [14], which is the union of all channels between this pair. Translating to the graph terminology, a channel graph is a connected multistage graph with a single vertex in both the first and last stage.

It is commonly assumed, though not without controversy, that the edges in \( E_i \) are independent and identical each having the probability \( p_i \) of being blocking.

Due to its independence of traffic load, the superiority comparison commands strong reliability and broad applicability. Recently there has been a surge of interest in studying the superiority of channel graphs. The current paper is an attempt to present the state of the art as well as giving some new results.

2. PROBABILITY MODELS
For a given \( E_i \), there are three important types of p.d.f.s, different in their degrees of generality, which have been given attention in the literature:

(i) Binomial. The edges in \( E_i \) are independent and identical each having the probability \( p_i \) of being blocking.

(ii) Combinatorial. The edges in \( E_i \) are exchangeable and the probability of a given subset of \( E_i \) being blocking depends only on the cardinality of that subset. Note that the binomial type is a special case of the combinatorial type.

(iii) General. No restriction at all. Clearly, both the binomial type and the combinatorial type are special cases of the general type.

For easy reference, a binomial model means that \( E_i \) is of the binomial type for every \( i = 1, \ldots, s - 1 \). We define the combinatorial model similarly.

In many cases we want to reduce a channel graph to a smaller graph for easier examination. There are two basic methods of reduction which preserve the blocking probability:

(i) Condensation is an interstage reduction which, when applied to the channel graph \( G \), replaces every nonempty path graph of a pair of vertices \( x \in V_i \), \( y \in V_j \) by an edge whose blocking probability is set to equal that of the replaced path graph.

(ii) Collapsing is an intrastage reduction which, when applied to the channel graph on \( E_i \), identifies a subset of \( E_i \) into a new edge (thus also identifying the two sets of vertices the edges are incident to into two new vertices). Such a new edge is called a multivariate edge which can have more than the two usual states, blocking and nonblocking. A state of a multivariate edge is a subset of the replaced edges. The probability of a state is set equal to the probability that the edges in...
the specified subset are all blocking and edges not in the subset are all nonblocking. Note that if the edges in $E_1$ are independent, then the new edges in $E_1$ after collapsings are also independent. Furthermore, if the edges in $E_1$ are partitioned into disjoint subsets of the same cardinality, then the union reduction preserves the "identical" and "exchangeable" properties. But a condensation reduction can preserve the "independent" and "identical" properties only if the replaced channel graphs are disjoint and isomorphic, respectively. It can preserve the "exchangeable" property only if the replaced channel graphs are disjoint paths.

The three types of p.d.f.'s we mentioned for the binary edge can be extended to the multivariate case. Thus we have:

(i) **Multinomial.** Edges in $E_1$ are independent and identical each having probability $p_{ij}$ of being in state $j$.

(ii) **Multivariate combinatorial.** Let $(E_{ij})$ be a partition of $E_1$ such that edges in $E_{ij}$ are in state $j$. Then the probability of $(E_{ij})$ depends only on the cardinalities of $E_{ij}$.

(iii) **Multivariate general.** No restriction at all.

We also define the multinomial model and the multivariate combinatorial model as straightforward generalizations of the binomial model and the combinatorial model. Furthermore, we now introduce a new model called the Takagi model which assumes importance in later discussions. A Takagi model satisfies the following conditions:

(i) It has four stages.

(ii) $E_1$ is multivariate general while $E_2$ and $E_3$ are multinomial.

(iii) The p.d.f. of an edge in $E_3$ can depend on the states of the edges in $E_1$ and $E_3$ to which it is connected.

It is easily seen that the Takagi model includes the 4-stage multinomial model as a special case but not the 4-stage multivariate combinatorial model.

3. SOME NECESSARY CONDITIONS

Lee [14] suggested that one way to compute the blocking probability $B(G)$ of a channel graph $G$ is to use the inclusion-exclusion principle. Namely, let $T$ denote the set of all channels in $G$ and let $t_1, t_2, \ldots$ denote the graph consisting of the union of channels in $T \subseteq T$. Define $B(G) = 1 - B(G)$. Then Lee [14] gave the Inclusion-Exclusion Channel Graph Theorem.

$$B(G) = \sum_{t_1 \in T} B(t_1) - \sum_{t_1 \times t_j \in T} B(t_1 \cup t_j) + \sum_{t_1 \times t_j \times t_k \in T} B(t_1 \cup t_j \cup t_k) - \ldots$$

Consider the binomial model where $p_1 = p + 1$. Then $B(G)$ is approximated by the first term on the right-hand side since any union of channels contains more edges than a single channel. The following two theorems, first noticed by Timperi and Grillo [20], then easily follow.

**THE STAGE THEOREM**

If a channel graph $G$ has more stages than another channel graph $G'$, then $G$ is not superior to $G'$ under the binomial model (hence any model containing the binomial model as a special case).

**THE CHANNEL THEOREM**

If an s-stage channel graph $G$ has fewer channels than another s-stage channel graph $G'$, then $G$ is not superior to $G'$ under the binomial model.

Note that if two channel graphs have different numbers of stages or different number of channels, then usually the corresponding link systems cannot be synthesized with the same set of hardware, i.e., the number of switches, the sizes of switches and the number of edges. Therefore we will be comparing blocking probabilities of two link systems with different specifications and different costs. Furthermore, other things held constant, then more stages mean more channels. Hence the Stage Theorem will work against the Channel Theorem and more often ends in noncomparability. Therefore most of the literature on superior channel graphs compare channel graphs which have the same number of stages and the same number of channels. Link systems correspond to these channel graphs can be synthesized with exactly the same set of hardware. The following theorem, again first noticed by Timperi and Grillo [20], gives a useful necessary condition for superiority in such cases.

**THE EDGE THEOREM**

Let $G$ and $G'$ be two s-stage channel graphs. Suppose $E'_1$ contains more edges than $E_1$. Then $G$ is not superior to $G'$ under the binomial model.

**Proof:** Assume $p_j = 0$ for all $j \neq 1$ and $1 > p_1 > 0$. Then

$$B(G') = p_1^{-1} B(G) = B(G)$$

where $|S|$ is the cardinality of the set $S$.

4. REGULAR SERIES-PARALLEL CHANNEL GRAPHS

Let $G$ be an s-stage channel graph and $G'$ an s'-stage channel graph. A series combination of $G$ and $G'$ is an $(s + s' - 1)$-stage channel graph obtained by identifying $V_1$ with $V'_1$. A parallel combination of $G$ and $G'$, defined only when $s = s'$, is an s-stage channel graph obtained by identifying $V_1$ with $V'_1$ and $V_2$ with $V'_2$. A channel graph is series-parallel if either it is an edge or it can be obtained by a series or a parallel combination of two smaller series-parallel channel graphs. A multistage graph is regular if the degrees of the vertices in a same stage.
are the same. It is simple if no multiple edges are allowed between two vertices. Note that if $E_i$ contains multiple edges, we can insert a vertex in the middle of each edge of $E_i$ to split $E_i$ into two sets both without multiple edges.

Chung and Hwang [5] showed that a simple regular $s$-stage series-parallel channel graph can be uniquely represented by a $\lambda$-vector $(\lambda_1, \ldots, \lambda_{s-1})$ where $\lambda_1$ is defined as the ratio of the outdegree of a vertex in $E_i$ to the indegree of a vertex in $E_{i+1}$, which is either an integer or the reciprocal of an integer. It can be easily verified that the number of paths for such a channel graph is simply the product of the $\lambda_i$ which are integers.

A regular series-parallel graph is called a regular SP-canopy if no series combination is allowed except when one graph is an edge. The $\lambda$-vector for a regular SP-canopy is characterized by the properties that it starts with all integers and ends with all reciprocals of integers and that the product of any sequence of adjacent elements is either an integer or the reciprocal of an integer. Chung and Hwang [2, 5] proved that a regular series-parallel graph is always a Takagi graph.

A rigorous proof that $\{m_{ij}\}$ uniquely represents a Takagi graph is due to Cattermole [1] and Hull [7].

Consider a Takagi graph represented by the multiplex numbers $M$. A demultiplex operation, namely, setting the value of certain $m_{ij}$ in $M$ to one, is a set of collapsing reductions on $E_i, E_{i+1}, \ldots, E_j$ which, together, remove $m_{ij}$ copies of a path graph from stage $i$ to stage $j$ by a single copy. Therefore the demultiplexed Takagi graph $M' = M - \{m_{ij}\}$ has multivariate edges for $E_i, E_{i+1}, \ldots, E_{j-1}$ and preserves the independent and exchangeable properties. Thus we have

THE DEMULTIPLEX THEOREM

Let $M_1$ and $M_2$ be two sets of multiplex numbers representing two Takagi channel graphs under either the binomial model or the combinatorial model. Suppose $M_1 \cap M_2 = M$. Let $M_1 = M - M_2$ and $M_2 = M - M_1$ also represent two Takagi channel graphs. Then $M_1$ is superior to $M_2$ if and only if $M_1'$ is superior to $M_2'$.

Proof: Because the blocking probabilities of $M_1$ and $M_2$ under any given $F_1, \ldots, F_{s-1}$ which are binomial or combinatorial can be translated to the blocking probabilities of $M_1'$ and $M_2'$ under some $F_1', \ldots, F_{s-1}'$ which are multinomial or multivariate combinatorial.

Takagi [18] proved:

THE TAKAGI THEOREM

$M = \{m_{13} = n, m_{23} = m\}$ is superior to $M' = \{m_{13} = m, m_{23} = n\}$ under the Takagi model if $n \geq m$.

Corollary. $M = \{m_{13} = n, m_{23} = m\}$ is superior to $M' = \{m_{13} = m, m_{23} = n\}$ under the multivariate model if $n \geq m$. (The current version of the Corollary strengthens the original version given by Takagi which says $M = \{m_{13} = m\}$ is superior to $M' = \{m_{23} = n\}$.)

The Takagi Theorem was first proved by le Gall [12] under the binomial model and later under a slight generalization [13]. It was also proved by van Bosse [21] under the combinatorial model. However, among these models, only the Takagi model deals with multivariate edges and therefore can take advantage of the Demultiplex Theorem to extend the Takagi Theorem to the comparisons of $s$-stage Takagi graphs. Namely,

THE TAKAGI EXTENSION THEOREM

Let $M$ and $M'$ be two sets of multiplex numbers representing two s-stage Takagi graphs under the binomial model. Suppose $M$ and $M'$ are the same except for $i + 1 \leq j < x \leq y$, $m_{ij} = m'_{ij}$ for $i + 1 \leq j < x \leq y$. Then $M$ is superior to $M'$. Takagi then claimed that for a given degree sequence $(d_1, \ldots, d_s)$ where $d_1$ is the degree of a vertex in $V_i$, and a given number of channels, an optimal set of multiplex numbers must satisfy the constraint that it does not contain two numbers $m_{ij} > 1$, $m_{kz} > 1$ with $i < j < x \leq y$. Since there is a unique set of multiplex numbers satisfying the above constraint, Takagi claimed
that his method determines the optimal Takagi channel graph (one superior to all other Takagi channel graphs). However, these claims are wrong on two accounts. The first is that since \( M = \{m_{13} = n, m_{24} = m\} \) to \( M' = \{m_{14} = m, m_{23} = n\} \) when \( m > n \), the constraint is not a necessary condition for optimality. For example, setting \( n = 2 \) and \( m = 3 \) in the above \( M \) and \( M' \) under the binomial model, then \( B(M) < B(M') \) when \( p_3 \) is large and the reverse is true when \( p_3 \) is small.

The second source of incomparability is due to the fact that the unique set of multiple numbers satisfying the constraint may contain nonintegral \( m_{ij} \) and hence does not correspond to a Takagi graph. Since the proof of the Demultiplex Theorem depends critically on the integrality of the multiplex numbers, the Takagi Extension Theorem no longer follows for nonintegral multiplex numbers.

Recently, van Bosse [22] showed for a few cases with small parameters that the Takagi Extension Theorem also holds under a combinatorial model. Pedersen [16] attempted to define multiplex numbers for non-Takagi graphs. However, the unique representation property is then lost.

Furthermore, one cannot expect the Demultiplex Theorem and the Takagi Extension Theorem to hold for these new multiplex numbers which may be nonintegral.

6. REGULAR CHANNEL GRAPHS

It is easily seen that a degree sequence does not represent a regular channel graph uniquely. Thus there are two types of comparisons with respect to regular channel graphs depending on whether the compared graphs have the same degree sequence or not. We first discuss the type of different degree sequences.

Chung and Hwang [4] proved that a regular channel graph \( D \) with degree sequence \((d_1 = 0x, d_2 = 1xn, d_3 = n'x1, d_4 = m'x0)\), where the \( n \) edges of \( d_1 \) and the \( n' \) edges of \( d_3 \) are connected to distinct vertices, is superior to the Takagi graph with multiplex numbers \((m_{ij} = m_{23} = n)\) under the Takagi model if \( m' > m \). When \( m' = m \), \( D \) is the Takagi graph with multiplex numbers \((m_{ij} = m_{23} = n)\) and the above theorem is reduced to the Takagi Theorem. (A careful examination of the proof in [4] reveals that the condition of "distinct vertices" can actually be dropped.) Hwang [8] showed that the Takagi graph \((m_{14} = m_{23} = n)\) is not only a special case of \( D \) with \( m \) and \( n \) being fixed, but also the worst among all \( D \) with \( m' > n \) under the model \( E_1 \) is general, \( E_2 \) and \( E_3 \) are binomial.

For the comparisons of regular channel graphs with the same degree sequence, Takagi [19] proved the following theorem with an ingenious but involved argument.

THE TAKAGI CYCLE THEOREM

Among all regular channel graphs with the degree sequence \((d_1 = 0xm, d_2 = m'x2, d_3 = 2xn, d_4 = n'x0)\), the optimal graph is the one where \( E_2 \) forms a single cycle assuming \( E_1 \) and \( E_3 \) are combinatorial and \( E_4 \) is binomial.

Hwang [9] recently proved the above theorem under the probability model that \( E_2 \) and \( E_3 \) are multivariate and \( E_1 \) is combinatorial.

Ikeno [11] studied a \( t \)-ary class of regular channel graphs under the binomial model such that a \((t,2k)\)-graph has the degree sequence \((d_1 = 0xt, d_2 = 1xt, ..., d_k = 1xt, d_{k+1} = tx1, ..., d_{2k-1} = tx1, d_{2k} = tx0)\). He advocated the Takagi graphs with multiplex numbers \((m_{1k} = k' \text{for } k = 1, ..., 2k)\) (we will call these graphs the Ikeno graphs) as the optimal \((t,2k)\) graphs and purported to have proved the optimality asymptotically. The question that whether the Ikeno graphs are optimal for small \( k \) had been left unanswered until recently Chung and Hwang [6] showed that for \( t = 2 \), no optimal \((t,2k)\)-graphs exist for \( k > 5 \). Chung and Hwang also gave a construction for \((2,2k)\)-graphs which uniquely possess some local optimal properties.

For \( k = 4 \), this locally optimal graph is optimal and superior to the Ikeno graph. Work is also underway to study the \((t,2k)\) graph for \( t > 2 \).

7. WEIGHTED CHANNEL GRAPHS

In this section we study channel graphs which can have different blocking probabilities for edges in a single stage. We call them weighted channel graphs. Though interesting in their own account, the motivation behind the comparisons of such graphs is to have applications for unweighted channel graphs.

We first introduce some classical concepts in mathematics which provide a simple but forceful framework in unifying many results of this section.

A set \((w_1, ..., w_n)\) is said to majorize another set \((w_1', ..., w_n')\) if

\[
\sum_{i=1}^{k} w_i' \geq \sum_{i=1}^{k} w_i \quad \text{for every } k = 1, ..., n-1,
\]

\[
\sum_{i=1}^{n} w_i' = \sum_{i=1}^{n} w_i.
\]

A function \(f(x_1, ..., x_n)\) is called a Schur function if it satisfies the conditions

\[
\left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right) (x_1, ..., x_n) \geq 0 \quad \text{for all } x_1, x_2.
\]

The two notions are connected by the following theorem:

THE SCHUR-OSTROWSKI THEOREM

\(f(w_1, ..., w_n) > f(w_1', ..., w_n)\) for all \((w_1, ..., w_n)\) majorizing \((w_1', ..., w_n)\) if and only if \(f\) is a Schur function.

From the Schur-Ostrowski Theorem, we derive the following two theorems in comparing weighted channel graphs.

Consider a \(4\)-stage channel graph \(G\) with \(m\) disjoint channels. We assume that \(E_1\) and \(E_3\) are unweighted and have combinatorial p.d.f.s but \(E_2\) contains weighted edges with blocking probabilities \(W = (w_1, ..., w_n)\). Let \(G'\) be a channel graph exactly like \(G\) except the set of
THE 4-STAGE MAJORIZATION THEOREM

B(G) \geq B(G') \text{ if the set } R_nW \text{ majorizes the set } R_nW'.

Proof: We will consider an edge in E_1 and an edge in E_3 as a pair if they are on the same path. Let B(G|m) denote the blocking probability of G given m pairs are nonblocking. Let S_m denote the set of all m-subsets of the set \{1, \ldots, n\}. Then

B(G|m) = \sum_{s_m \in S_m} \prod_{i=1}^n p_i^{/n} = \sum_{s_m \in S_m} \prod_{i=1}^n p_i^{/n}.

It is straightforward to verify that B(G|m) is a Schur function. Therefore B(G|m) \geq B(G'|m) by the Schur-Ostrowski Theorem. Since m is arbitrary, the 4-Stage Majorization Theorem is proved.

Next consider two sets of 2-stage channel graphs H = \{H_1, \ldots, H_m\} and H' = \{H'_1, \ldots, H'_m\}. Suppose each H_j(H'_j) consists of m disjoint channels and the j^{th} channel of H_j(H'_j) has blocking probability \pi_{ij} (\pi'_{ij}). Define

F_i = \sum_{j=1}^m \ln \pi_{ij} \text{ and } F'_i = \sum_{j=1}^m \ln \pi'_{ij}.

and B(H) and B(H') be the average blocking probabilities over H_j and H'_j respectively.

THE 2-STAGE MAJORIZATION THEOREM

B(H) \geq B(H') \text{ if the set } \{F_1, \ldots, F_m\} \text{ majorizes the set } \{F'_1, \ldots, F'_m\}.

Proof:

B(H) = \frac{1}{n} \sum_{i=1}^n B(H_i) = \frac{1}{n} \sum_{i=1}^n \prod_{j=1}^m \pi_{ij} = \frac{1}{n} \sum_{i=1}^n \prod_{j=1}^m \pi'_{ij} = \frac{1}{n} \sum_{i=1}^n F_i.

is easily seen to be a Schur function. Therefore the 2-Stage Majorization Theorem follows immediately from the Schur-Ostrowski Theorem.

We now show that many results available in the literature are immediate corollaries of the above theorems.

(i) Timperi and Grillo [20] proved two special cases of the 2-Stage Majorization Theorem. In the first case, m = n, \pi_{ij} = x_{j-i+1}y_j and \pi'_{ij} = x_{j-i+1}y_j where [z]_n is the smallest integer not less than z, and the subscripts of x and y are computed modulo n. In the second case, n = \binom{c}{k}, m = ck, and there are c distinct blocking probabilities for the edges. Each H_j has the collection of blocking probabilities which consists of c copies of a distinct k-subset of the c probabilities. H_j has the collection consisting of k copies of the c probabilities.

(ii) Theorem 3, 5 and 6 in [3] are various special cases of the 4-Stage Majorization Theorem for n = 2.

(iii) The 4-Stage Majorization Theorem can also apply to spiderweb channel graphs. For example, the result reported in Sec. 6 that the Takagi graph with multiplex numbers \{m_1m_2m_3m_4\} is the worst among all regular channel graphs with the degree sequence \(d_1 = Ox_{m}, d_2 = 1x_{m}, d_3 = n'x_1, d_4 = m'x_1\) follows from the 4-Stage Majorization Theorem in the following way. Assuming k edges in E_2 are nonblocking, we can delete the blocking edges and unite the nonblocking edges. The edges in E_2 are then grouped into m' disjoint sets which can be replaced by m' weighted edges. The 4-Stage Majorization Theorem now applies by setting the probability of any edge in E_2 being blocking to be zero.

Next we discuss a comparison which has the least constraints on the graph structure. Let G be an s-stage channel graph and let x and y be two vertices of V_1. Suppose x and y are connected to the same set of vertices in V_{s-1} and the edges \([z,x] \text{ and } [z,y] \) have the same blocking probability for all \(z \in V_{s-1}\). Let G' be a graph obtained from G by merging x and y, i.e., uniting \([z,x] \text{ and } [z,y] \) for all z.

THE MERGE THEOREM

Proof: We prove the theorem by induction on s. For s = 3, V_1 must be V_2. We give an argument which is good for the case V_1 = V_{s-1} for all s. Let e_x and e_y be the two edges connecting x and y to the single vertex in V_s. Then B(G) is the same as B(G') except when both e_x and e_y are nonblocking. Therefore we can reduce G and G' by uniting e_x and e_y. But the only difference between the reduced G and G' is that for every edge \([z,x] \text{ in } G'\), there are two edges \([z,x] \text{ in } G\). Clearly B(G) \leq B(G').

For s > 3 and 1 < s-1, assume that V_{s-1} is the subset of edges in V_{s-1} which are nonblocking. Reduce G and G' by deleting edges in V_{s-1} - V_{s-1} and uniting edges in V_{s-1}. Then the reduced G and G' can be treated as two (s-1)-stage channel graphs.
graphs such that the latter is obtained from the former by merging x and y. The Merge Theorem now follows by induction.

Special cases of the Merge Theorem have appeared in the literature in various places. When the graphs before and after merging are both Takagi graphs, then the Merge Theorem follows from the Takagi Theorem. For non-Takagi graphs, the Merge Theorem was first studied by Hwang and Odlyzko [10] and later generalized by Chung and Hwang [3]. Our proof is similar to the proof of Theorem 7 in [3] which deals with nonweighted edges.

8. CONCLUSIONS

Our goal is to provide some simple structure into the flourishing literature on superior channel graphs. The foundation of the structure is the classifications on graphs and on probability models. By working with an explicit structure we are able to unify, to simplify, to reinforce and to generalize many existing results. In particular, we feel that we have helped in clarifying the important results obtained by Takagi with respect to Takagi graphs in the following ways:

(i) We show that for a given degree sequence and a given number of channels, an optimal channel graph is often a regular non-Takagi channel graph. This fact is worth pointing out since Takagi referred to optimal Takagi channel graphs as optimal channel graphs in his writing and thus left an impression that his results are applicable to all s-stage channel graphs. For example, Neiman [15] wrote that "the problem of optimal graph construction was solved in (18,20)".

(ii) By bringing the Demultiplex Theorem and the Takagi Extension Theorem to the front, we highlight the importance of the Takagi model in the Takagi Theorem. Namely, the Takagi model is the only existing model dealing with multivariate edges on which the validity of the Demultiplex Theorem hinges.

(iii) We point out for the first time that the Takagi Extension Theorem is not sufficient to determine an optimal Takagi channel graph for a given degree sequence and a given number of channels.

A few problems also surface for immediate future research:

(i) Find a necessary and sufficient condition for the comparability of simple regular series parallel channel graphs.

(ii) Prove the Takagi Theorem under the multivariate combinatorial model.

(iii) Prove the superiority of a regular channel graph D mentioned in Sec. 6 over the Takagi graph with multiplex numbers \(m_1 \geq m_2 \geq m_3 \geq \ldots \geq m_n\) under a multivariate model when \(m' > n\).

REFERENCES


