BALANCED SKIP MULTIPLES
AND THEIR WRITING ALGORITHMS

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ABSTRACT

In order to avoid grade-of-service imbalance due to load imbalance among inlets, connection systems such as Erlang's ideal grading are desirable. This paper proposes a practical range realization of ideal grading using balanced incomplete block designs well-known in the field of "design of experiments" and its construction algorithms. It clarifies that the connection system absorbs load imbalance, is little influenced by concentration-ratio and increases traffic capacity. The connection system can be realized by a sparse matrix. It can also be realized by 1 x n strip-shaped matrices without wire crossing and n x n square matrices using mono-layer print boards. Besides, this paper clarifies wire density and wiring algorithm among boards which mount square matrices.

1. INTRODUCTION

As a practical version of A.K. Erlang's ideal grading (Ref.1), Random Slip Multiples (RSM) (Ref.2) is introduced to No.1 ESS and No.3 ESS. The RSM aimed to make the number R of inlet groups at most one, which are accessible to two arbitrary selected outlets, but this was realized only incompletely.

A USA patent (Ref.3) presented constructions of R=1 for the case of total outlets v=25 and accessibility k=5 as a special case without giving general construction method. Besides this, some other heuristic computer methods are reported (Ref.4,5).

This paper, first, describes the construction method and traffic characteristics of the connection system BSM. As the authors clarified (Ref.6), when the number of total outlets v is an even power of a prime, constructions of B=1 can be given by using Balanced Incomplete Block Designs (BIBD), which are easy to make. The connection system using BIBD is named Balanced Skip Multiples (BSM). The merits of BSM are the absorption of load imbalance among inlets, the reduction of concentration-ratio influence and traffic capacity increase, compared with usual connecting networks which have the same number of cross-points.

Second, BSM wiring algorithms are presented considering wire density. The BSM is constructed by giving contacts only to the cross-points corresponding to outlets, to which each inlet is accessible, in the matrix of (all inlets) x (all outlets). When constructing the equivalents of the above system using square or rectangular matrices, man-power-free wiring is possible, such as print-circuit-board wiring. This paper presents BSM wiring algorithms especially for the cases of 1 x n strip-shaped matrix and n x n square matrix (n is a power of a prime).

2. BSM CONSTITUTION CONCEPTS

The multiple-connection system, shown in Fig.1, which is given in reference 2 as an RSM example, does not satisfy the requirement that any two outlets are accessible together to at most one inlet-group (matrix). Such a connection that satisfies the requirement can be realized by using BIBD.

Here, BIBD are block designs of v treatments allocated over b blocks satisfying the following requirement:

\( R \) times together in the same block.

BSM CONSTRUCTION METHOD 1 is led from the BIBD PROPERTY 1 as follows.

BIBD PROPERTY 1: When the number \( v \) of treatments is an even power of a prime, BIBD exists where block size \( k=\sqrt{v} \) and the number of encounters \( R=1 \). It has the natures:

(1) BIBD can be divided into \( k+1 \) sets of blocks \( A_1, A_2, B_1, \ldots, B_{k-1} \), any of which contains \( k \) blocks and all treatments,

(2) \( A_1 \) is the transposition of \( A_2 \).

BSM CONSTRUCTION METHOD 1: When \( v \) is an even power of a prime, a BSM having

(1) number of inlet-groups : \( v \),

(2) number of outlets : \( v \),

(3) accessibility \( k=\sqrt{v} \) and

Cross-point matrix

\[
\begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9 \\
\end{array}
\]

Fig.1 Random slip multiple
(The same figure as Fig.1 in Ref.2)
(4) number of encounters: one for any two outlets of different outgoing-levels and zero for any two outlets of the same outgoing-level,
can be constructed through correspondence "D" to the above BIBD, except for A₂, such as:

Block : Set of outlet numbers accessible from each inlet-group,
Treatment : Outlet.

Here, each block of A₂ is a set of outlet numbers, which belong to " an outgoing-level.
The above system is named Balanced Skip Multiples (BSM) from the construction using only skip multiples instead of slip multiples, from the load balancing characteristics and from the application of BIBD.
The BSM given above has multiplicity (number of inlet-groups accessible to an outlet) k and is called narrow-sense BSM, which is discriminated from wide-sense BSM. The wide-sense BSM uses a part of block sets A₁, B₁, ..., Bₖ₋₁ according to the multiplicity smaller than k.

EXAMPLE 1: Figure 3 shows a narrow-sense BSM in the case of one inlet per inlet-group led from the BIBD, v=9, k=3, l=1, given in Fig.2.

EXAMPLE 2: Figure 4 shows a BSM imaginarily applied to the first stage of DEX-1 or No.1 ESS connecting network.
The BSM can also be obtained by using another correspondence "D*":

Block : Set of inlet-group numbers accessible to each outlet,
Treatment : Inlet-group.
The "D*" is dual to the above mentioned correspondence "D".

EXAMPLE 3: Figure 5 shows the BSM obtained by the correspondence "D*" to the BIBD given in Fig.2. Three of the outgoing-levels in Fig.5 give a BSM equivalent to Fig.3.

<table>
<thead>
<tr>
<th>Inlet No.</th>
<th>Outgoing level</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 2 3 15 9</td>
<td>I 4 5 6 7 2 6 III</td>
</tr>
<tr>
<td>7 8 9 4 8 3</td>
<td>I 4 7 1 8 6</td>
</tr>
<tr>
<td>II 2 5 8 4 2 9 IV</td>
<td></td>
</tr>
<tr>
<td>3 6 9 7 5 3</td>
<td></td>
</tr>
</tbody>
</table>

Fig.2 Balanced incomplete block designs example (v=9, k=3, R=1)

In this section, wide-sense BSM with constant accessibility k and variable multiplicity r (r=q and number of inlet-groups = kq for concentration-ratio q:1) is called category A. Narrow-sense BSM (number of inlet-groups = v=k) is called category B.

Table 1. Call congestion probability

<table>
<thead>
<tr>
<th>*1</th>
<th>*2</th>
<th>Category A</th>
<th>Category B</th>
<th>*3</th>
</tr>
</thead>
<tbody>
<tr>
<td>.2</td>
<td>.00037±.00004</td>
<td>.00018±.00006</td>
<td>16</td>
<td>.00107</td>
</tr>
<tr>
<td>.3</td>
<td>.00382±.00033</td>
<td>.00175±.00021</td>
<td>x</td>
<td>.00827</td>
</tr>
<tr>
<td>.4</td>
<td>.01669±.00070</td>
<td>.00942±.00058</td>
<td>8</td>
<td>.02741</td>
</tr>
<tr>
<td>.07</td>
<td>.00027±.00009</td>
<td>.00017±.00006</td>
<td>40</td>
<td>.00254</td>
</tr>
<tr>
<td>.09</td>
<td>.00132±.00016</td>
<td>.00111±.00018</td>
<td>x</td>
<td>.00923</td>
</tr>
<tr>
<td>.15</td>
<td>.03429±.00153</td>
<td>.03151±.00127</td>
<td>8</td>
<td>.07202</td>
</tr>
<tr>
<td>.05</td>
<td>.00070±.00012</td>
<td>64</td>
<td>.00702</td>
<td></td>
</tr>
<tr>
<td>.075</td>
<td>.01227±.00079</td>
<td>.04198</td>
<td></td>
<td></td>
</tr>
<tr>
<td>.1</td>
<td>.05915±.00175</td>
<td>8</td>
<td>.10806</td>
<td></td>
</tr>
</tbody>
</table>

Values after sign ± are 95% confidence interval, by means of large sample method (41 samples, 5000 calls/sample).

*1 Concentration ratio. *2 Offered traffic per idle inlet. *3 Fully accessible matrix with n inlets and m outlets (n x m).
independently of concentration-ratio and number of inlets per inlet-group is $q$ for concentration-ratio $q:1$ is called category B. Table 1 shows the call congestion probabilities of both BSM categories, compared with those of fully accessible (inlets $kq$) x (outlets $k$) matrices obtained by Markov chain simulation. The Table shows that the BSMs have good traffic characteristics.

Moreover, the degree of load imbalance among inlets does not affect call congestion probability. For instance, the call congestion probability for category A BSM with concentration-ratio 5:1 and offered traffic per idle inlet $d_1=0.04erl$ for inlet groups 1–20 and $d_2=0.10erl$ for inlet groups 21–40, is 0.000026±0.000007, which is nearly the same value as the case of $d=0.07$ in Table 1. By using the following testing formula for population averages;

$$z_0 = \frac{x_1 - x_2}{\sqrt{\left(\sum_{j=1}^{n} \left(\frac{s_j^2}{a_j}\right) \cdot \frac{1}{n}\right)}}$$  \hspace{1cm} (1)

where

- $x_1, s_1^2$: sample average and sample variance for $d_1=d_2=0.07erl$,
- $x_2, s_2^2$: sample average and sample variance for $d_1=0.04erl$, $d_2=0.10erl$,
- $n$: number of samples,

the value $z_0=0.19 (<1.96)$. Therefore, the average values for the two cases cannot be concluded to be different at significance level 5%, assuming normal distribution for call congestion probability.

Here, relation between "inlet-usage" $a_i$ and "offered traffic per idle inlet" $d_i$ is approximately given by $a_i = d_i/(1+d_i)$, when blocking probability is small enough. Therefore, average inlet usages for the two cases are almost equal:

- $a_1 = 0.0654 ± 0.0013$ for $d_1=d_2=0.07erl$,
- $a_2 = 0.0647 ± 0.0013$ for $d_1=0.04erl$, $d_2=0.10erl$.

Table 2 shows outlet usage $b$ satisfying blocking probability 0.001 for each concentration-ratio. They are obtained from the approximation formula of distribution of number of occupied links $i$ (blocking probability in the case of $i=k$) for the BSM with total inlets $N$, total outlets $M$ and accessibility $k$ (Ref. 6):

$$\begin{align*}
\begin{array}{cccc}
\sum_{j=0}^{M} N_{j=0} M_{j} & \left(\frac{a_1}{a_2}\right)^{N_{j-1}} & \left(1-a\right)^{N_{j-1}} & a_{1}^{N_{j-1}} \sum_{j=0}^{M} N_{j=0} M_{j} \\
1 & 0 & 1 & 0 \end{array}
\end{align*}$$  \hspace{1cm} (2)

where $n=min(N-1, M-k+1)$, $a_i$: inlet usage ($qa=b$ when concentration ratio is $q:1$).

From the Table, BSM traffic capacity is little influenced by concentration-ratio.

### 3.3 Multi-BSM stages in connecting networks

Multistage connecting networks adopting BSM are considered taking 6 kinds of 3-stage networks shown in Fig. 6. Figure 6 also shows point-to-point internal blocking probabilities for "number of trunks per outgoing route" $B$, "number of outgoing routes" $8$ (8 outlets of each third-stage matrix belong to different 8 outgoing routes for (a), (b) and (f)), "first stage concentration-ratio" 3:1, and "offered traffic per idle inlet" 0.2erl. The figure says the order of internal blocking probability magnitude is

$$(f)<(c)<(a)<(b)<(e)<(d),$$

where the differences among (a), (b) and (c) are small, and the magnitudes of (d) and (e) are very large, so networks constructed only by BSM are proved to be undesirable.

In the case of one trunk per route, results are the same. That is, (c) with BSM on both ends is clearly better than (a) without BSM, from the internal blocking probabilities for "offered traffic per idle inlet" $d=0.4erl$ (in this case, blocking due to trunk busy is large, and the traffic carried through connecting network is nearly the same as the carried traffic in the case of 8 trunks per route and $d=0.2erl$):

- for $d=0.2erl$
  - (a) 0.0518±0.0011, (c) 0.0492±0.0011,
  - (a) 0.0074±0.0004, (c) 0.0055±0.0029,
  - (a) 0.000395±0.00003, (c) 0.000185±0.000060.

![Fig. 6 3-stage connecting networks and their internal blocking probabilities B](image-url)
3.4 Multistage connecting networks with BSM concentration stage

Table 3 shows traffic capacity of multistage connecting networks given in Fig. 7, satisfying the internal blocking probability 0.001. The probability is calculated by the following procedure:

1. Distribution of number of occupied links on each stage is assumed to be mutually independent.
2. Distribution of number of occupied links on the first stage is assumed to be according to Eq. (2) in the case of BSM and Engset distribution in the case of fully accessible matrices.
3. Distribution of number of occupied links on the last stage is Engset distribution and on the other stages is Bernoulli distribution.
4. Reference outgoing route is composed of 8 trunks and is offered 3.12756erl (trunk blocking 0.01), number of link matching trials is at most 5 times.

Table 3. Traffic capacity satisfying blocking probability 0.001

<table>
<thead>
<tr>
<th>No.</th>
<th>No. of stages</th>
<th>Parameter (A)</th>
<th>(B)</th>
<th>(B)/(A)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>16</td>
<td>8</td>
<td>2</td>
<td>20.54</td>
</tr>
<tr>
<td>2</td>
<td>16</td>
<td>8</td>
<td>3</td>
<td>35.58</td>
</tr>
<tr>
<td>3</td>
<td>16</td>
<td>2</td>
<td>4</td>
<td>41.34</td>
</tr>
<tr>
<td>4</td>
<td>40</td>
<td>2</td>
<td>5</td>
<td>35.33</td>
</tr>
<tr>
<td>5</td>
<td>40</td>
<td>16</td>
<td>6</td>
<td>32.38</td>
</tr>
<tr>
<td>6</td>
<td>40</td>
<td>1</td>
<td>7</td>
<td>67.20</td>
</tr>
</tbody>
</table>

(A): Connecting networks given in Fig. 7
(B): Same as (A), except adopting BSM on the first stage
*1: 1/2 concentration at the second stage

In order to confirm the calculation method given above, some simulation experiments were made. The simulation result corresponding to the case No.1 in Table 3, n=16, m=8, k=1 (2:1 concentration only on the first stage) shows traffic capacity (A)=21.1erl for usual connecting network, (B)=24.9erl for the connecting network adopting BSM on the first stage, and (B)/(A)=1.18, which almost equal to the calculation result.

From the above study, it is proved that traffic capacity can be increased by adopting BSM on one concentration end or both ends of any connecting network, regardless of inlet load balance degree.

4. BSM WIRING ALGORITHM

Replacing a part of a usual multistage connecting network by BSM stages improves its traffic characteristics. To introduce BSM into real systems, simple wiring methods are required, which are acceptable to existing multistage connecting networks without disturbance. Adopting a sparse matrix (as shown in Figs. 3 and 4) for BSM stage results in two different matrix kinds in a network. This is undesirable. The authors found out that the BSM can be realized by using rectangular cross-point matrices, which are also basic elements for usual connections. In the following sections, simple and practical BSM wiring algorithms are explained for \( \times n \) strip-shaped and \( n \times n \) square matrices, respectively.

4.1 Wiring algorithm for strip-shaped matrices

Usually, when constructing a switch with concentration-ratio 1:1 by \( \times n \) strip-shaped matrices, all outlets belonging to a same horizontal level of \( n \times n \) matrices are multiple-connected, as shown in Fig. 8. Such a connection is simple, but does not make the most of the merit of small unit size \( \times n \).

The matrix made of \( \times n \) matrices, shown in Fig. 8, is called pseudo-matrix to distinguish it from the basic \( n \times n \) matrix. Multistage connecting networks are usually mounted with a unit of \( n \) pseudo-matrices, as seen in the switching system. When \( n \) is a power of a prime, the number \( v(=n^2) \) of total outlets of an equipment unit (hereinafter referred to as grid) composed of \( n \) pseudo-matrices is an even power of a prime. The number of \( \times n \) matrices in a grid is \( n^2 \). Therefore, narrow-sense BSM can be constructed by applying CONSTRUCTION METHOD 1. For instance, matrices may be multiple-connected, as shown in Fig. 3(b), for the case of \( n=3 \). Though Fig. 3(b) is expressed with crossing wire for the help of understanding, it is constructed without crossing wire in real systems (see APPENDIX 1).

Figure 9 shows a BSM example without crossing wire by means of correspondence "D" given in...
Section 2, which is equivalent to correspondence "D" in the case of narrow-sense BSM. For the case of concentration-ratio k:1, all outlets which belong to a horizontal level of k grids should be multiple-connected.

The wiring algorithm illustrated in Fig.9 is briefly explained in that, by selecting the uppermost outlet-line U and the lowest outlet-line V of one outgoing-level, the other outlet-lines go through the gap between them. (At outgoing-level ii' of Fig.9, e goes through the gap between d and f). When the number of outlets belonging to an outgoing-level is n, (n-2) outlet-lines go through the gap between U and V. If the uppermost outlet-line terminal A and the lowermost outlet-line terminal B are adjacent, (n-2) outlet-lines go through the gap between the two terminals A and B, as shown in Fig.10(a). On the other hand, if the uppermost outlet-line terminal and the lowermost outlet-line terminal are not adjacent in anywhere, the number of outlet-lines in the gap between two arbitrary adjacent matrices can be made less than (n-2), as shown in Fig.10(b).

THEOREM 1 shows the sufficient conditions where the number is less than (n-2). (The proof is given in APPENDIX 3.)

THEOREM 1 : When BSM is constructed using \( n^2 \) i x n matrices, where \( n^2 = (m^2) \), the number of wires in the gap between two adjacent outlet-terminals at a same outgoing-level is at most \( n-3 \), by mapping the position numbers of element \( 0 \) and the first position numbers of element \( w \) and \( v \) for a Latin square given in APPENDIX 2 to the matrix numbers connected by the uppermost wire U and the matrix numbers connected by the lowermost wire V at one outgoing-level. Here, q is an integer which satisfies \( 1+q = w \), \( 2+q = n \) (equation is valid in mod 2) and mod f(w). The f(w) is an irreducible polynomial of degree m, whose coefficients are elements of GF(2) and "position number" is a serial number \( (1,2,\ldots,n^2) \) assigned to each cell of a Latin square.

The distribution of the number of wires in the gap between two adjacent outlet-terminals of a matrix is determined as the following theorem. (The proof is given in APPENDIX 4.)

THEOREM 2 : For BSM there exists k (p: a prime, \( k: \) a positive integer), \( n \times n \) matrices number of matrices =number of outlets = \( n^2 \) in the gap between their outlet-terminals on the two adjacent outgoing-levels under consideration is

\[ m+1 \quad \text{for} \quad 0 \leq m \leq n-1 \] \[ 2n-1-m \quad \text{for} \quad n \leq m \leq 2n-2. \]

In conclusion, the maximum number of wires between outlet-terminals on an outgoing-level of two adjacent matrices may be reduced by applying THEOREM 1, but the maximum number of wires in the gap between two adjacent outlet-terminals of a matrix cannot be reduced.

4.2 Wiring algorithms for square matrices

A narrow-sense BSM composed of a unit of \( n^2 \) \( n \times n \) matrices results in too many inlets (\( n^2 \)) per unit. Therefore, this section presents a wiring algorithm for wide-sense BSM, which has variable multiplicity and \( n^2 \) inlets per unit. Let \( A_1, A_2 \) and \( B_1 (i=1,2,\ldots,n-1) \) be block sets derived by procedures in APPENDIX 2. When \( n \geq m \) (p: a prime, \( m: \) a positive integer), wide-sense BSM with maximum multiplicity \( n \) can be constructed by the following procedures:

1. Each grid \( M_i (i=1,2,\ldots,n-1) \) consists of \( n \) matrices.
2. Outlet-terminal-numbers (OTN) of matrices in a grid are given by \( n \times n \) matrices and are common to all grids.
3. "Outlets in each grid are rearranged in order of serial OTN 1,2,...,n \( \) by noncrossing wirings on a mono-layer-board.
4. Outlet-numbers accessible to each matrix of \( M_i \) are given by each row of \( B_1 (i=1,2,\ldots,n-1) \).
5. Terms of \( M_i \) and \( B_{i+1} \) are multiple-connected by \( n \) outlets; OTN 1,2,...,n of \( M_i \) with OTN 1,2,...,n of \( M_{i+1} \). OTN \( n+1,n+2,...,2n \) of \( M_{i+1} \) with OTN \( n(n-1)+1,n(n-1)+2,...,n^2 \) of \( M_i \) and OTN \( j \) of \( B_{i+1} \) with OTN \( (i-j) \) of \( M_i \) (\( n \leq j \leq 2n \)).

When a basic grid structure and an extension grid structure are permitted to be different, wide-sense BSM with maximum multiplicity \( n \) can be constructed. The outlet-terminal-numbers of the basic grid \( M_0 \) match outlet-numbers, i.e. \( M_0 \) does not need to be rearranged in serial numbers and outlet \( k \) of \( M_0 \) is multiple-connected with outlet \( k \) of the \( j \) extension grid \( M_j \) given above (\( k=1,2,\ldots,n^2 \); \( j=1,2,\ldots,n-1 \)).

EXAMPLE : Figure 11 gives an (extension) grid for \( n=4 \) derived by the above procedure (1)-(4), using the following block sets:

\[
B_1 = \begin{bmatrix} 1 & 14 & 7 & 12 \\ 12 & 5 & 10 & 3 \\ 3 & 10 & 5 & 16 \\ 16 & 9 & 6 & 15 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & 10 & 15 & 8 \\ 15 & 6 & 10 & 9 \\ 6 & 3 & 12 & 13 \\ 10 & 9 & 11 & 16 \end{bmatrix}
\]

\[
B_3 = \begin{bmatrix} 1 & 6 & 11 & 16 \\ 6 & 2 & 5 & 12 \\ 2 & 15 & 12 & 5 \\ 11 & 12 & 13 & 7 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \\ 3 & 1 & 2 & 4 \\ 4 & 2 & 1 & 3 \end{bmatrix}
\]
n = 8, the authors estimate the maximum number may be always more than n - 4.

For strip-shaped matrices, the maximum number of wires in the gap between 2 outlet-terminals on an outgoing-level for more general cases is left for further study. From the detailed study of n = 8, the authors estimate the maximum number may be always more than n - 4.

5. CONCLUSION

This paper has explained that BSM can be constructed when accessibility k is a power 2 of a prime and number of total outlets is k'. By applying BSM to one or both end stages of multi-stage connecting networks, the following merits are obtained:

1. Decrease in call congestion probability, i.e. increase in traffic capacity.
3. Traffic capacity insensitivity to concentration-ratio.

Even if concentration-ratio is 1:1, BSM is superior to usual structures when load among matrices is unbalanced, though it is inferior in the case of a balanced load. In multistage networks, BSM should not be applied to two successive stages.

BSM wiring algorithms for 1 x n strip-shaped matrices and n x n square matrices were presented for practical construction.

The following items were clarified.

For 1 x n strip shaped matrices:

1. The maximum number of wires in the gap between 2 outlet-terminals at an outgoing-level of 2 adjacent matrices in case of n = 2 m (m ≥ 2).
2. The distribution of the number of wires between 2 outlet-terminals of each matrix.

For n x n square matrices:

3. Wide-sense BSM with multiplicity from 2 to (n - 1) can be constructed with a kind of mono-layer-board (i.e. grid) and with simple subgroup slip wiring between grids. By applying BSM to one or both end stages of multi-stage connecting networks, the following merits are obtained:

Two wiring algorithms are applicable to any rectangular matrix. The easier one of them may be adopted, according to matrix shape, concentration-ratio and so on.

For strip-shaped matrices, the maximum number of wires in the gap between 2 outlet-terminals on an outgoing-level for more general cases is left for further study. From the detailed study of n = 8, the authors estimate the maximum number may be always more than n - 4.

APPENDIX 1: Non-crossing wiring between strip-shaped matrices

Figure A1 shows the principle of non-crossing wiring, taking outgoing-level ii' in Fig. 9 as an example. In outgoing-level ii', 9 outlet-terminals, each of which is taken from different 9 matrices, are multiple-connected to outlets d, e and f. The 9 terminals are able to be arranged on 3 straight-lines, as shown in Fig.A1(A). In order to place terminals multiple-connected to outlets d and e on one straight-line:

(1) Basic wirings among terminals to outlets d and e stay at the original places as in Fig.A1(A).
(2) Terminals to outlet d are lowered onto the straight-line of terminals to outlet e.
(3) Being induced by (2), wiring among terminals to d and e have falling parts I and II shown in Fig.A1(B), respectively.

Terminals to d, e and f can be placed on one straight-line by repeating the above procedure (1) - (3). This procedure can be applied for any number of outlets.

APPENDIX 2: Complete system of orthogonal Latin squares and BIBD induced from it

Elements of GF(p m) are only 0, 1, w, w 2, ... , w n-2 (n = p m) when w is a primitive root of Galois field GF(p m) (p: a prime, m: a positive integer). So, Latin square L i (i = 0, 1, ... , n - 2), orthogonal to each other, is given by

\[
L_i = \begin{bmatrix}
0 & 1 & w & \cdots & w^{n-2} \\
1 & w & w^2 & \cdots & w^{n-3} \\
w & w^2 & w^3 & \cdots & w^{n-4} \\
\vdots & \vdots & \vdots & & \vdots \\
w^{i-1} & w^{i+1} & \cdots & w^{i+n-2} & w^{i+n-1}
\end{bmatrix}
\]

where w n-1 = 1.

That is, the first row of L i+1 is the same as the first row of L i , the last-1 row of L i+1 is the same as the second row of L i and the j-th row of L i+1 is the same as the (j+1)-th row of L i (j = 2, 3, ... , n).

Elements of each row can be represented by only 0, 1, w, w 2, ... , w n-2, where equality is valid in mod p and mod f(w) (f(w)) : an irreducible polynomial of degree m, whose coefficients are elements of GF(p m)), and Latin squares L i which have elements 0, 1, 2, ... , n-1 are obtained by placing w j by (j + 1) = 1, 2, ... , n-2. In the following paragraphs, only L i are discussed, but the same discussion is possible for L i by considering mapping w j to (j + 1).

BIBD, which has parameters b = n 2, v = n, k = n, r = n-1 and k = n, is derived by the following procedures (1) - (5), where n cells of L i are...
given serial position numbers 1, 2, ..., n² (w)
(1) To make n blocks by grouping position-
numbers of each row of L₁.
(2) To make n blocks by grouping position-
numbers of each column of L₁.
(3) To make n blocks by grouping position-
numbers of each row of L₁, where the elements are the same, i.e., 1, 2, ..., n². Treatments in each block are "n-1" arranged in order of columns of L₁.

Thus, square-matrices, having n blocks made by above procedures as rows, are obtained. Matrices made by procedures (1) and (2) are called A₁ and A₂, respectively. Matrices by procedure (3) are called B₁, B₂, ..., B n-1.

The following relations hold between B₁ and B₁+1:
* Treatments 1, 2, ..., n of B₁+1 and B₁ are placed on the same position.
* Treatment (n-2)-q of B₁+1 is placed on the position of treatment k of B₁, 2n ≥ k ≥ n.
* Treatment (k'-n) of B₁+1 is placed on the position of treatments k' of B₁, n² ≥ k'> 2n.

APPENDIX 3: Proof of THEOREM 1
If n=2k (k: integer), a Latin square
L₀= [0 1 w ... w n-2 ]
    [1 0 ... w 1 w n-2 ]
    [w 1 w ... w n-2 ]
     ... ... ...
    [w n-2 1 w w n-2 ]

is derived from APPENDIX 2.

Elements adjacent to element 0 are 1 and w h(1+w), h=0,1,2, ..., n-3. Equality has the same meaning as APPENDIX 2.

There is such a q that
1+w=w q+1
because elements of L₀ are only 0, 1, w, ..., w n-2.
Using the relation w q+1=1,

w h(1+w)≡1 for j≡(n-1)-q.

Elements adjacent to element 0 are
w h(1+w), h=0, 1, ..., n-3,
i.e., w 0, w 1, ..., w n-2.

Therefore, element w q-1 is not adjacent to element 0. From this fact, any matrix on U is not adjacent to any matrix on V, if matrix numbers on U are equal to position-numbers for element 0 of L₀ and matrix numbers on V are equal to position-numbers for element w q-1 of L₀.

The maximum number of wires in the gap between two outlet-terminals in an outgoing-level of at least two adjacent matrices cannot be reduced to less than (n-3), because w q-1 is only one element not adjacent to 0.

The fact given above is true for any outgoing-
level because elements adjacent to element 0 are the same in any Latin square L₁.

APPENDIX 4: Proof of THEOREM 2
Carry out the correspondence "D*" for B₁ derived from APPENDIX 2. Let D₁ be matrix obtained by replacing rows of B₁ in wiring order from U to V. Let members (treatments) of j-th row in D₁ be

\[ d(i,j) , d(1,j), \ldots, d(i,n) \]

Any pair \( d(i,j) , d(i,j) \) where i₁, 2, ..., n does not appear in any other block, because BIBD in APPENDIX 2 has parameter R=1. Accordingly, each row of D₁ contains just one member of j-th row of B₁. The order of members in each block, i.e. in each row of D₁ and D₁+1, is arbitrary.

Therefore, transposition of new D₁ can be new D₁+1 by rearranging in order of members.

Let E₁ be a matrix with new serial numbers 1, 2, ..., n given to members of D₁

\[
E₁ = \begin{bmatrix}
1 & 2 & \ldots & n \\
2 & n+1 & \ldots & 2n \\
\vdots & \vdots & \ddots & \vdots \\
(n-1) & n & \ldots & 2n-2 \\
\end{bmatrix}
\]

then

\[
E₁+1 = \begin{bmatrix}
1 & n+1 & \ldots & n(n-1)+1 \\
2 & n+2 & \ldots & n(n-1)+2 \\
\vdots & \vdots & \ddots & \vdots \\
\end{bmatrix}
\]

because mapping D₁ to E₁ is on a one-to-one basis.

The numbers of wires in the gap between 2 adjacent terminals in matrix k (the new matrix number in accordance with E₁) is equal to the number of rows existing between member number k of E₁ and member number k of E₁+1 because wires are arranged in order of rows of E₁ and then E₁+1.

Accordingly, (n-1) wires go through a gap between 2 terminals in matrix k, n wires in matrix 2, (n+1) wires in matrix 3, \ldots, \ldots, \ldots, (n-k+1)+(j-1) wires in matrix (kn+j), \ldots, \ldots, \ldots, (n-1) wires in matrix n².

Thus THEOREM 2 is proved.

The following COROLLARY is also clear.

COROLLARY: In BSM derived from THEOREM 2, the maximum number of wires in the gap between 2 adjacent terminals of each matrix is (2n-2) and the matrix number, having the value, is given by member number common among first row of B₁ and \((q+1)th row of B₁+1\) where q satisfies \(1+w=w q+1\) (see APPENDIX 2).

REFERENCES
(2) E.M.JOHNSON, "Inherent load balance in link systems with random slip multiples", 7th ITJ, 236 (1973).