The concept of effective availability generally has been applied to link systems providing point-to-group selection. Effective availability has been defined in terms of the access provided by a link system to a group of devices connected to its outlet terminals.

In this paper, the blocking characteristics of a link system providing point-to-point selection are considered in terms of the effective availability of a group of k devices, links or junctions between a pair of stages in the link system. The effective availability $k_g$ of the group of devices is expressible as the product of an average availability $M$ and an availability factor $g$.

The group of inter-stage devices of interest divides the link system into two subsystems. One subsystem consists of the links and stages providing access to the devices from an inlet terminal; the second subsystem consists of the links and stages providing access to the devices from an outlet terminal.

Each subsystem alone has blocking characteristics expressible in terms of an effective availability. These effective availabilities yield average availabilities $M_1$, $M_2$, and availability factors $g_1$, $g_2$. The average availability of the link system as a whole is given simply by

$$M = M_1M_2/k.$$  

The estimation of the value of $g$ from knowledge of the values $g_1$ and $g_2$, of the topological characteristics of the link system is the problematic part of the effective availability theory. Bounding values and a procedure for estimating the value of $g$ from a knowledge of the subsystem block characteristics are derived.

1.0 INTRODUCTION

The concept of effective availability generally has been applied to link systems providing point-to-group selection. Effective availability has been defined in terms of the access provided by a link system to a group of devices connected to its outlet terminals. A theory for this application of the effective availability concept has been provided in [3].

In this paper, the blocking characteristics of a link system providing point-to-point selection are considered in terms of the effective availability of a group of k devices, links or junctions between a pair of stages in the link system. The effective availability $k_g$ of the group of devices is expressible as the product of an average availability $M$ and an availability factor $g$.

The group of inter-stage devices of interest divides the link system into two subsystems. One subsystem consists of the links and stages providing access to the devices from an inlet terminal; the second subsystem consists of the links and stages providing access to the devices from an outlet terminal.

Each subsystem alone has blocking characteristics expressible in terms of an effective availability. These effective availabilities yield average availabilities $M_1$, $M_2$, and availability factors $g_1$, $g_2$. The average availability of the link system as a whole is given simply by

$$M = M_1M_2/k.$$  

The estimation of the value of $g$ from knowledge of the values $g_1$ and $g_2$, of the topological characteristics of the link system is the problematic part of the effective availability theory. Bounding values and a procedure for estimating the value of $g$ from a knowledge of the subsystem block characteristics are derived.

2.0 GENERAL THEORY

A two-sided multi-stage link system may be divided into two subsystems interconnected by a set of devices which may be links or junctions. One subsystem, say subsystem #1, provides access to the devices to a call appearing on an inlet of the link system and the other subsystem, say subsystem #2, provides access to the devices from an outlet to which the call is to be connected. Let $a_1$ be a measure of the traffic loading the links of subsystem #1, $a_2$ be a measure of the traffic loading the links of subsystem #2, and $p$ be the average occupancy per device. The blocking probability $P$ for the link system is expressible as

$$P = P(p; a_1, a_2).$$

Considering the two subsystems individually, let $P_1$ be the blocking probability for subsystem #1 achieved when no traffic loads the links of subsystem #2 and $P_2$ be the blocking probability for subsystem #2 achieved when no traffic loads the links of subsystem #1. The blocking probabilities $P_1$ and $P_2$ are expressible as

$$P_1 = P_1(p; a_1) = P(p; a_1, 0)$$
$$P_2 = P_2(p; a_2) = P(p; 0, a_2).$$

If the devices of interest are loaded with Bernoulli traffic in that each of them is independently made busy with probability $p$, and the generating function (mgf) $M(t)$ of the number of devices accessible for setting up a call through the link system is expressible as

$$M(t) = M(t; a_1, a_2) = P(e^{t}; a_1, a_2).$$

Considering the two subsystems independently, let $M_i(t)$
be the mgf of the number of devices accessible through subsystem #1 when no traffic loads the links of subsystem #2 and $M_2(t)$ be the mgf of the number of devices accessible through subsystem #2 when no traffic loads the links of subsystem #1. The mgf's $M_1(t)$ and $M_2(t)$ are expressible as

$$M_1(t) = M_1(t; \alpha_1) = P_1(e^{t \alpha_1}; \alpha_1)$$
$$M_2(t) = M_2(t; \alpha_2) = P_2(e^{t \alpha_2}; \alpha_2).$$

Let $k$ be the maximum number of devices available for access in setting up a call through the link system. The quantity $k$ is achieved when no traffic loads the links of both subsystems #1 and #2. Otherwise, the accessibility is less than the availability $k$. The average accessibility is called the average availability $M$ and has the form

$$M = \left. \frac{V(t)}{t} \right|_{t=0}$$

where $V(t)$ is a measure of the traffic loading the links of subsystems #1 and #2 and the quantities $V(t)$ and $M$ are constants of the link system.

Subsystem #1 provides access to the devices with an average availability $M_1$ achieved when no traffic loads the links of subsystem #1 and $M_2$ has the form

$$M_2 = v_2 m_2 (1-a_2)$$

where $V(t)$ and $m_i$ are constants of subsystem #1 and $v_2$ and $m_2$ are constants of subsystem #2.

The average availabilities are obtained from the mgf's by differentiation with respect to $t$ and setting $t$ equal to 0. Therefore

$$M_1 = M_1(1-a_1)$$
$$M_2 = M_2(1-a_2)$$

so that

$$M = (1-a_1)(1-a_2)$$

Therefore, $M$ is easily determined from $M_1$ and $M_2$ as

$$M = M_1 M_2 / k.$$

For specified values of $P$ and $a$, an effective availability $k_1$ with which the devices are accessed through the link system may be defined by

$$P = P^{k_1}$$

and $P^{k_1}$ is expressible as

$$k_1 = M_1(1-a_1)$$

By differentiation of $M(t; \alpha_1, \alpha_2)$, $M(t)$ will be found to have the form

$$M(t) = \left. \frac{V(t)}{t} \right|_{t=0}$$

and $k_1$ is obtained from $M_1$ and $M_2$ as

$$k_1 = M_1 M_2 / k_1.$$

The availability factors $k_1$ and $k_2$, and availability factors $g_1$ and $g_2$, may also be defined for subsystems #1 and #2 as follows:

$$P_1 = P_1^{k_1}$$
$$k_1 = M_1^E$$

so that

$$k_1 = M_1(1-a_1),$$

and

$$E_1 = \frac{1}{M_1};$$

Note that $g_1$ and $g_2$ are equal 1.0 for $P_1 = P_1^E$ and $P_2 = P_2^E$. Knowledge of the average and effective availabilities of the subsystems of a link system makes possible the determination of a lower bound for the blocking probability $P$ for the link system as a whole as follows:

$$P = \min(P_1, P_2)$$

Therefore

$$k_1 = \min(k_1, k_2)$$

or

$$k(1-a_1)(1-a_2) \geq \min(k_1, k_2) = (1-a_1)(1-a_2).$$

This is as far as the general theory has been developed for estimating the value of $P$. For knowledge of the gross conditions giving rise to the values of $P$, the effective availability $k_1$ is defined by

$$M = M_1(1-a_1)$$

where $M_1 = \min(M_1, M_2) = \min(M_1, M_2)$ and, since $g_1$ is not greater than 1.0, we have

$$P_2 = P_2^E = P^{k_2}.$$
3.0 THE VARIANCE OF THE ACCESSIBILITY

The theory in the preceding section yields an expression for the average availability \( M \) in terms of \( M_1 \) and \( M_2 \) and a lower bound \( P \) for the blocking probability of a multi-stage link system. These results are based on the use of the average availabilities \( M_1 \) and \( M_2 \) and subsystem blocking probabilities \( P_1 \) and \( P_2 \). Further refinements in obtaining estimates of \( P \) can be developed by considering the variance of the availability with which the centrally located group of devices is accessed through the link system and its subsystems. Before considering these refinements, a brief review of some properties of moment generating functions shall be made [3].

Let \( M(t) \) be the moment generating function (mgf) of a random variable (rv) \( n \) whose mean and variance are \( \mu \) and \( \sigma^2 \), respectively. Let \( X = \sum_{i=1}^{m} n_i \), each of which is scaled by a factor \( v \), i.e.,

\[
M = \sum_{i=1}^{m} n_i,
\]

and

\[
V = v^2 n.
\]

Then

\[
M(t) = M(v) = v^n.
\]

Note that, with \( M_0 \) and \( V_0 \) known, it is possible to infer the values of \( v \) and \( m \) from \( M \) and \( V \).

The above expressions are valid for independent identically distributed (iid) rv’s \( n \), each of which is scaled by a factor \( v \), i.e.,

\[
X = \sum_{i=1}^{m} n_i = v \sum_{i=1}^{m} n_i,
\]

where \( \rho \) is a correlation coefficient and \( \sigma^2 = \nu \). The value of \( \rho \) is limited to the range from \(-1/(m-1)\) to 1.0.

For the extreme values of \( \rho \), we have

\[
\rho = 1/(m-1). \quad \therefore \nu = 0. \quad \rho = 1.0. \quad \therefore \nu = v^2 m^2.
\]

In any event, the means and variances are obtained by differentiation of the mgf’s with respect to \( t \) at \( t = 0 \) as follows where a prime indicates first-order differentiation and a double prime indicates second-order differentiation:

\[
\begin{align*}
M_0 \quad & = \quad M'(0) \\
M_0 \quad & = \quad \sigma^2 = M''(0) - M^2(0) \\
M_0 \quad & = \quad M_x'(0) = M''(0) - M^2(0)
\end{align*}
\]

If \( n \) or \( n_i \) is the outcome of a Bernoulli trial with \( v = 0 \) occurring with probability \( \alpha \) and \( v = 1 \) occurring with probability \( 1 - \alpha \), then

\[
M = \alpha \quad \text{and} \quad \sigma^2 = \alpha(1 - \alpha).
\]

The group of devices of interest for estimating effective availability in a multi-stage link system is typically arranged in \( m \) subgroups of \( v \) devices. The quantities \( M \) and \( V \) may take different values depending on whether access to the group of devices is defined as being from end to end through the whole link system, 2) as being from a link system inlet through a subsystem in which case the quantities are \( M_1 \) and \( V_1 \), or 3) as being from a link system outlet through a complementary subsystem in which case the quantities are \( M_2 \) and \( V_2 \). In any case, the total number \( k \) of devices is

\[
k = \nu_1 m_1 = \nu_2 m_2 = \nu m.
\]

Any attempt to set up a call through a multi-stage link system may be considered as an attempt to access a selected group of \( k \) devices between a pair of stages in the link system. The devices are provided in, say, \( m \) subgroups of \( v \) devices. An attempt to access the devices corresponds to \( m \) Bernoulli trials made through the link system. Of these trials, \( m(1-\alpha) \) trials are successful on the average when the probability of success for a single trial is \( 1-\alpha \). Consequently, the average number \( M \) of accessible devices is

\[
M = \nu m(1-\alpha).
\]

The blocking probability \( P \) depends on the variance \( V \) of the number of accessible devices. If the number of accessible devices is constant such that, if

\[
\rho = 1/(m-1) \quad \text{and} \quad V = 0,
\]

then

\[
P = \rho M.
\]

If the number of accessible devices varies as the outcomes of \( m \) independent Bernoulli trials such that

\[
\rho = 0.0 \quad \text{and} \quad V = v^2 m^2,
\]

then

\[
P = (\alpha + (1-\alpha) \nu)^M.
\]

If the number of accessible devices varies in such a fashion that either all devices are accessible with probability \( \alpha \) or all devices are accessible with probability \( 1-\alpha \) such that

\[
\rho = 1.0 \quad \text{and} \quad V = k(\alpha^2 + (1-\alpha)^2),
\]

then

\[
P = (\alpha + (1-\alpha) \nu)^M.
\]

This last case yields another lower bound on the value of \( g \). It is

\[
g = 2 \ln(\alpha(1-\alpha) \nu) - \ln(\alpha(1-\alpha) \nu)^2.
\]

In the next section we investigate how the variance \( V \) depends on the variances \( V_1 \) and \( V_2 \) of the number of devices accessible through subsystems \#1 and \#2.

4.0 THE VARIANCES OF THE SUBSYSTEM ACCESSIBILITIES

Figure 2 is a representation in matrix form of the interconnection pattern of a group of devices connecting two subsystems of a link system. The dimension of the matrix is \( m_1 \times m_2 \). Entries in the cells of the matrix take the values of either 0 or \( v \). The matrix contains \( m \) entries where \( m_1 \) and \( m_2 \) occur.

The entries in a row of the matrix correspond to one of \( m_1 \) groups of devices accessible through subsystem \#1. The marginal sum of the numbers of devices in a row is \( v_1 \); therefore \( \sum v_1 \). The entries in a column of the matrix correspond to one of \( m_2 \) groups of devices accessible through subsystem \#2. The marginal sum of the numbers of devices in a column is \( v_2 \); therefore \( \sum v_2 \). The sum of all entries equals \( k \). Therefore

\[
k = \nu m \quad \therefore v_1 m_1 = v_2 m_2.
\]

If the probability of any row of \( v_1 \) devices being accessible through subsystem \#1 is \( (1-\alpha) \), then the average number \( M_1 \) of devices accessible through subsystem \#1 is

\[
M_1 = v_1 m_1 (1-\alpha).
\]

The variance \( V_1 \) of the number of devices accessible through subsystem \#1 is expressible as

\[
V_1 = v_1 m_1 (1-\alpha) \sigma^2(1-\alpha),
\]

where \( \sigma^2 \) is a correlation coefficient and \( \alpha(1-\alpha) \). Similarly, the probability of any column of \( v_2 \) devices being accessible through subsystem \#2 is \( (1-\alpha) \), then the average number \( M_2 \) of devices accessible through sub-

<table>
<thead>
<tr>
<th>GROUP NUMBER</th>
<th>1</th>
<th>2</th>
<th>\cdots</th>
<th>\text{m}_{2}</th>
<th>SUM</th>
</tr>
</thead>
<tbody>
<tr>
<td>\text{v}_1</td>
<td>\nu</td>
<td>\nu</td>
<td>\cdots</td>
<td>\nu</td>
<td>\text{v}_1</td>
</tr>
<tr>
<td>\text{m}_{1}</td>
<td>0</td>
<td>\nu</td>
<td>\cdots</td>
<td>\nu</td>
<td>\text{v}_2</td>
</tr>
<tr>
<td>SUM</td>
<td>\text{v}_2</td>
<td>\text{v}_2</td>
<td>\cdots</td>
<td>\text{v}_2</td>
<td>k</td>
</tr>
</tbody>
</table>
The variance \( V_2 \) of the number of devices accessible through subsystem \( #2 \) is expressible as
\[
V_2 = \sum_{i=1}^{m} \sum_{j=1}^{m} E \left( 1-\alpha_j \right) \left( 1-\alpha_i \right) \mu_i \mu_j.
\]
where \( \mu_i \) is a correlation coefficient and \( \alpha_i, \alpha_j = \alpha(1-\alpha) \).

In order to obtain an expression for \( V \) in terms of \( V_1 \) and \( V_2 \), let us begin by considering a derivation for \( V \). Let \( \left( 1-\alpha_i \right) \) be the indicator function for the \( i \)th of \( m \) subgroups of \( \nu \) devices being accessible through the link system. If \( \left( 1-\alpha_i \right) \) equals 1.0 with probability \( \alpha \) and equals 0.0 with probability \( 1-\alpha \), then the expectation of \( \left( 1-\alpha_i \right) \) is
\[
E \left( 1-\alpha_i \right) = \alpha(1-\alpha).
\]
Let \( \left( 1-\alpha_i \right) \) be the indicator function for an arbitrary pair \( i \) and \( j \) of the \( m \) subgroups of \( \nu \) devices being accessible through the link system. Let the expectation of \( \left( 1-\alpha_i \right) \) be defined by
\[
E \left( 1-\alpha_i \right) = \rho_2^{(1-\alpha)^2}(1-\alpha_i)^2
\]
where \( \rho_2 \) is a correlation coefficient and \( \alpha_i = \alpha(1-\alpha) \).

Let \( V_0 \) be defined by
\[
V_0 = \sum_{i=1}^{m} \sum_{j=1}^{m} E \left( 1-\alpha_i \right) \left( 1-\alpha_j \right) \mu_i \mu_j.
\]
Then
\[
V_0 = m \sum_{i=1}^{m} \sum_{j=1}^{m} E \left( 1-\alpha_i \right) \left( 1-\alpha_j \right) \mu_i \mu_j = m \sum_{i=1}^{m} \sum_{j=1}^{m} E \left( 1-\alpha_i \right) \left( 1-\alpha_j \right) \mu_i \mu_j = \rho_1^{(1-\alpha)^2}(1-\alpha_i)^2
\]
where \( \rho_1 \) is a correlation coefficient and \( \alpha_i = \alpha(1-\alpha) \).

Let \( \left( 1-\alpha_i \right) \) be the indicator function for the \( i \)th of \( m \) subgroups of \( \nu \) devices being accessible through subsystem \( #1 \). If \( \left( 1-\alpha_i \right) \) equals 1.0 with probability \( \alpha \) and equals 0.0 with probability \( 1-\alpha \), then the expectation of \( \left( 1-\alpha_i \right) \) is
\[
E \left( 1-\alpha_i \right) = \alpha(1-\alpha).
\]
Let \( \left( 1-\alpha_i \right) \) be the indicator function for an arbitrary pair \( i \) and \( j \) of the \( m \) subgroups of \( \nu \) devices being accessible through subsystem \( #2 \). Let the expectation of \( \left( 1-\alpha_i \right) \) be defined by
\[
E \left( 1-\alpha_i \right) = \rho_2^{(1-\alpha)^2}(1-\alpha_i)^2
\]
where \( \rho_2 \) is a correlation coefficient and \( \alpha_i = \alpha(1-\alpha) \).

If the \( i \)th of the \( m \) groups of \( \nu \) devices is in the cell at the intersection of row \( u \) and column \( v \) of the matrix in Figure 2, then its indicator function \( \left( 1-\alpha_i \right) \) is
\[
\left( 1-\alpha_i \right) = \left( 1-\alpha_i \right)_{uv}.
\]
This is so because, if any group of \( \nu \) devices is to be accessible through the link system, then it must be accessible through both of subsystems \( #1 \) and \( #2 \). Furthermore, if
\[
E \left( 1-\alpha_i \right) = E \left( 1-\alpha_i \right)_{uv} E \left( 1-\alpha_i \right)_{uv} = E \left( 1-\alpha_i \right)_{uv},
\]
then
\[
E \left( 1-\alpha_i \right) = \left( 1-\alpha_i \right) = \left( 1-\alpha_i \right)_{uv}.
\]
Any distinct pair of groups of \( \nu \) devices has one of three possible arrangements in the matrix in Figure 2. They may lie at either 1) the intersection of one row and one column, or 2) the intersection of two rows and one column, or 3) the intersection of one row and one column and the intersection of another row and another.

The number of ways of arranging a pair of distinct groups of \( \nu \) devices so that they lie in the same row or column is
\[
\sum_{i=1}^{m} \sum_{j=1}^{m} \left( 1-\alpha_i \right)_{uv} \left( 1-\alpha_j \right)_{uv} \mu_i \mu_j.
\]
The indicator function \( \left( 1-\alpha_i \right) \) for such an arrangement is
\[
E \left( 1-\alpha_i \right) = \left( 1-\alpha_i \right)_{uv} \mu_i \mu_j.
\]
Its expectation is
\[
E \left( 1-\alpha_i \right) = \left( 1-\alpha_i \right)_{uv} \mu_i \mu_j.
\]
The number of ways of arranging a pair of distinct groups of \( \nu \) devices so that they lie in the same column of the matrix is
\[
\sum_{i=1}^{m} \sum_{j=1}^{m} \left( 1-\alpha_i \right)_{uv} \left( 1-\alpha_j \right)_{uv} \mu_i \mu_j.
\]
The indicator function \( \left( 1-\alpha_i \right) \) for such an arrangement is
\[
E \left( 1-\alpha_i \right) = \left( 1-\alpha_i \right)_{uv} \mu_i \mu_j.
\]
Its expectation is
\[
E \left( 1-\alpha_i \right) = \left( 1-\alpha_i \right)_{uv} \mu_i \mu_j.
\]

The number of ways of arranging a distinct pair of \( \nu \) devices so that they do not lie in the same row or column is
\[
\sum_{i=1}^{m} \sum_{j=1}^{m} \left( 1-\alpha_i \right)_{uv} \left( 1-\alpha_j \right)_{uv} \mu_i \mu_j.
\]
The indicator function \( \left( 1-\alpha_i \right) \) for such an arrangement is
\[
E \left( 1-\alpha_i \right) = \left( 1-\alpha_i \right)_{uv} \mu_i \mu_j.
\]
Its expectation is
\[
E \left( 1-\alpha_i \right) = \left( 1-\alpha_i \right)_{uv} \mu_i \mu_j.
\]

Using the relationship
\[
\sum_{i=1}^{m} \sum_{j=1}^{m} \left( 1-\alpha_i \right)_{uv} \left( 1-\alpha_j \right)_{uv} \mu_i \mu_j = \rho_1^{(1-\alpha)^2}(1-\alpha_i)^2 \rho_2^{(1-\alpha)^2}(1-\alpha_j)^2
\]
with \( \rho_1, \rho_2 \) correlation coefficients and \( \alpha_i, \alpha_j = \alpha(1-\alpha) \), we get
\[
V = m \sum_{i=1}^{m} \sum_{j=1}^{m} \left( 1-\alpha_i \right)_{uv} \left( 1-\alpha_j \right)_{uv} \mu_i \mu_j = \rho_1^{(1-\alpha)^2}(1-\alpha_i)^2 \rho_2^{(1-\alpha)^2}(1-\alpha_j)^2
\]
with \( \rho_1, \rho_2 \) correlation coefficients and \( \alpha_i, \alpha_j = \alpha(1-\alpha) \). The variance \( V \) of the number of accessible devices equals \( \rho_1 \rho_2 \). Therefore, the desired expression for \( V \) in terms of \( V_1 \) and \( V_2 \) is
\[
V = \rho_1 \rho_2 \mu_1 \mu_2.
\]

5.0 LIMITING VALUES OF THE BLOCKING PROBABILITY

It has been pointed out in [1] that knowledge of \( M \) and \( V \) is, in general, insufficient for the determination of \( P \) and, therefore, of \( g \). However, for specific values of \( M \) and \( V \), specific values of \( P \) may be uniquely defined and these particular values of \( P \) may be used to define a curve of \( P \) versus \( V \) for the purpose of obtaining a value of \( P \) for specified \( V \) by interpolation. An alternative method to working with \( P \) and \( V \) is to work with \( g \) and \( \rho \) which is what we shall do here.

We shall consider a number of cases determined by specific values of \( \alpha, \rho \), and \( \rho \), using the formulas derived in the preceding sections. Those cases yielding values of \( P \) that provide values of \( g \) that are lower bounds on \( g \) are marked with an asterisk; those that provide upper bounds on \( g \) are marked with double asterisks.

Considering \( \rho \), we have three special cases:

Case 1. \( \rho = 1/(m-1) \), \( \alpha = 0.0 \). In this case, the number of devices accessible through the link system is constant. Therefore \( \mu = \rho \mu_\mu \) and \( \rho = 1.0 \).

Case 2. \( \rho = 0.0 \), \( \alpha = 0.0 \). In this case, the \( m \) group of \( \nu \) devices are accessed by
a sequence of independent Bernoulli trials. Therefore
\[ P = \binom{m}{n} \rho^{n}(1-\rho)^{m-n} \]

**Case 3.** \( \rho = 1.0 \)

In this case, the group of \( k \) devices are either inaccessible with probability \( \alpha \) or accessible with probability \( (1-\alpha) \). Therefore,
\[ P = (1-\rho)^{m} + \rho = k^2 \alpha^2 \]

In this case, the number of devices accessible through subsystem \#1 is constant and they are either inaccessible through subsystem \#2 with probability \( \alpha \) or accessible with probability \( (1-\alpha) \). Therefore,
\[ P = \alpha + (1-\rho)\rho \]

**Case 12.** \( \rho_{1} = -1/(m_{1}-1), \rho_{2} = 1.0 \)

In this case, the number of devices accessible through subsystem \#1 is constant and they are either inaccessible through subsystem \#2 with probability \( \alpha \) or accessible with probability \( (1-\alpha) \). Therefore,
\[ P = \alpha + (1-\rho)\rho \]

In this case, the number of devices accessible through subsystem \#2 by a sequence of independent Bernoulli trials is constant. Therefore,
\[ P = \alpha + (1-\rho)\rho \]

**Case 14.** \( \rho_{1} = 0.0, \rho_{2} = -1/(m_{2}-1) \)

In this case, the number of devices accessible through subsystem \#1 is constant and they are either inaccessible through subsystem \#2 with probability \( \alpha \) or accessible with probability \( (1-\alpha) \). Therefore,
\[ P = \alpha + (1-\rho)\rho \]

In this case, the number of devices accessible through subsystem \#2 by a sequence of independent Bernoulli trials is constant. Therefore,
\[ P = \alpha + (1-\rho)\rho \]

**Case 16.** \( \rho_{1} = 1.0, \rho_{2} = -1/(m_{2}-1) \)

In this case, the number of devices accessible through subsystem \#1 is constant and they are either inaccessible through subsystem \#2 with probability \( \alpha \) or accessible with probability \( (1-\alpha) \). Therefore,
\[ P = \alpha + (1-\rho)\rho \]

**Case 20.** \( \rho_{1} = 1.0, \rho_{2} = 0.0 \)

In this case, the number of devices accessible through subsystem \#1 is constant and they are either inaccessible through subsystem \#2 with probability \( \alpha \) or accessible with probability \( (1-\alpha) \). Therefore,
\[ P = \alpha + (1-\rho)\rho \]

By considering particular values of \( \rho_{1} \) and \( \rho_{2} \) jointly, we have nine special cases.

**Case 10.** \( \rho_{1} = -1/(m_{1}-1), \rho_{2} = -1/(m_{2}-1) \)

In this case, the number of devices accessible through each of subsystems \#1 and \#2 is constant. In particular, if \( m_{1} = m_{2} \), then \( V = 0.0 \) and this is the same case as Case 1.

The value of \( \rho \) is given by
\[ \rho = -1/(m_{1}-1)(m_{2}-m_{1})a_{1}a_{2} \]

In this case, the number of devices accessible through each of subsystems \#1 and \#2 is constant. In particular, if \( m_{1} \neq m_{2} \), then \( V = 0.0 \).

In this case, the number of devices accessible through subsystem \#1 is constant and they are either inaccessible through subsystem \#2 with probability \( \alpha \) or accessible with probability \( (1-\alpha) \). Therefore,
\[ P = \alpha + (1-\rho)\rho \]

In this case, the number of devices accessible through subsystem \#2 by a sequence of independent Bernoulli trials is constant. Therefore,
\[ P = \alpha + (1-\rho)\rho \]

A summary of the blocking probability equations obtained for all of the above cases is presented in Table 1. The cases provide the following lower and upper bounds on \( g \) in terms of the access to the set of links between stages 2 and 3. An estimate of \( P \) is provided by

**ITC-9 PEDESEN-5**
TABLE 1. Blocking probability equations for unique values of correlation coefficients $P_1, P_2$ and $p$.

<table>
<thead>
<tr>
<th>Case</th>
<th>$P_1$</th>
<th>$P_2$</th>
<th>$p$</th>
<th>Probability</th>
<th>Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1*</td>
<td>-1/(m-1)</td>
<td>0.0</td>
<td>1.0</td>
<td>$P=P^M$</td>
<td>$P=\alpha+(1-\alpha)pk$</td>
</tr>
<tr>
<td>2</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>$P=\alpha+(1-\alpha)pk$</td>
<td></td>
</tr>
<tr>
<td>3,18**</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>$P=\alpha+(1-\alpha)pk$</td>
<td></td>
</tr>
<tr>
<td>4*</td>
<td>-1/(m-1)</td>
<td>0.0</td>
<td>1.0</td>
<td>$P=\alpha+(1-\alpha)pk$</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>$P=\alpha+(1-\alpha)pk$</td>
<td></td>
</tr>
<tr>
<td>6**</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>$P=\alpha+(1-\alpha)pk$</td>
<td></td>
</tr>
<tr>
<td>7*</td>
<td>-1/(m-1)</td>
<td>0.0</td>
<td>1.0</td>
<td>$P=\alpha+(1-\alpha)pk$</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>$P=\alpha+(1-\alpha)pk$</td>
<td></td>
</tr>
<tr>
<td>9**</td>
<td>-1/(m-1)</td>
<td>0.0</td>
<td>1.0</td>
<td>$P=\alpha+(1-\alpha)pk$</td>
<td></td>
</tr>
<tr>
<td>10*</td>
<td>-1/(m-1)</td>
<td>0.0</td>
<td>1.0</td>
<td>$P=\alpha+(1-\alpha)pk$</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>-1/(m-1)</td>
<td>0.0</td>
<td>1.0</td>
<td>$P=\alpha+(1-\alpha)pk$</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>-1/(m-1)</td>
<td>0.0</td>
<td>1.0</td>
<td>$P=\alpha+(1-\alpha)pk$</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>0.0</td>
<td>1.0</td>
<td>1.0</td>
<td>$P=\alpha+(1-\alpha)pk$</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>0.0</td>
<td>1.0</td>
<td>0.0</td>
<td>$P=\alpha+(1-\alpha)pk$</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>1.0</td>
<td>0.0</td>
<td>1.0</td>
<td>$P=\alpha+(1-\alpha)pk$</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>1.0</td>
<td>0.0</td>
<td>1.0</td>
<td>$P=\alpha+(1-\alpha)pk$</td>
<td></td>
</tr>
</tbody>
</table>

where $a_2$ is the average loading per link between stages 1 and 2, $a_1$ is the average loading per link between stages 4 and 5, $b$ is the average loading per link between stages 3 and 4, and $p$ is the average loading per link between stages 2 and 3.

Figures 3A and 3C are the Takagi graphs of the two subsystems providing access to the devices of interest, i.e., the links between stages 2 and 3. Subsystem 1 is obtained by letting $a_2=b=0.0$ and subsystem 2 is obtained by letting $a_1=0.0$. Blocking probability estimates and mgf's for the subsystems are

$P_2(p;a_2)=\sum_{j=0}^{n_2} a_2^{j-1}b_2^{j-1}p^{n_2-j}a_2^{j-1}b_2^{j-1}p^{n_2-j}$

$M_2(t)=e^{-a_2} \sum_{j=0}^{n_2} a_2^{j-1}b_2^{j-1}p^{n_2-j}a_2^{j-1}b_2^{j-1}p^{n_2-j}$

By further differentiation of the mgf's, the variances of the availabilities are found to be

$V_1=n_1 n_2 \alpha_1^2 \alpha_2^2$

$V_2=n_1 n_2 \alpha_1^2 \alpha_2^2$.

The correlation coefficients in the above variance expressions are

$\rho_1=0.0$ $\rho_2=(a_2+(1-a_2)b_2^2)\alpha_1^2 \alpha_2^2$.

Inspection of the Takagi graph in Figure 3A reveals that the multiple number of the edge between nodes 2 and 3 equals 1.0. Therefore $k=1$, $\alpha_1=a_1$, $\alpha_2=a_2+(1-a_2)b_2^2$.

From the expressions for $P_2(p;a_2)$ and $P_1(p;a_1)$ we also have

$P_2(p;a_2,J) = (a_2+(1-a_2)p)\alpha_2^2$

$P_2(p;a_2,J) = (a_2+(1-a_2)p)\alpha_2^2$.
Therefore $v_1=8$, $v_2=8$, $v=8$, $v=1$

$M_1=38.40$, $M_2=62.1986$, $M=37.3191$

$\alpha_1=0.40$, $\alpha_2=0.2815$, $\alpha=0.41689$

$\epsilon_1=8$, $\epsilon_2=8$, $\epsilon=1$

$\Pi_1=0.0$, $\Pi_2=0.10495$, $\Pi=0.11220$

$P_1-P_1(p;\alpha_1)=0.000653$, $P_2-P_2(p;\alpha_2)=0.0006606$

$k_{E_1}=7.99142$, $k_{E_2}=7.99135$

$g_1=0.20811$, $g_2=0.12848$

From the results in Section 2, an upper bound $g_0$ on the value of $g$ is

$g_0=\min \left( g_1/(1-\alpha_1), g_2/(1-\alpha_2) \right) = 0.21413$.

This yields the following lower bound $P_0$ on the value of $P$:

$P_0=0.000661$

The use of $P_0$ as a lower bound on the value of $P$ yields

$P=0.41689$

which corresponds to $g=1.0$. This is the same as Case 1 treated in Section 5.

The results in Section 3 yield a lower bound on $g$ given by

$g=\min \left( P_1(p;\alpha_1), P_2(p;\alpha_2) \right)=0.000662$

which corresponds to an upper bound on $P$ of $0.41689$.

This is the same as Case 3 treated in Section 5.

Considering all other cases in Section 5 and Table 1, the pairs of cases, Cases 7 and 13, 8 and 14, and $g$ and 15 are identical because $r_1=0.0$. The condition, $m_1\leq v$, is satisfied for Case 10, therefore Cases 1 and 10 are identical.

Furthermore, since the conditions, $m_1\leq v$, $m_2\leq v$, $m_1\leq v_2$, and $m_2\leq v_1$, are all satisfied, all cases in Table 1 are applicable to the example.

Calculations were made of $P$ and $g$ for all of the cases listed in Table 1. All results of the calculations except for Case 5 are presented in Table 2. The results of Case 5 are not presented because the case corresponds to the problem in the example.

Bounds on $g$ are obtained from

$\min \left( \mathrm{Case~1,~Case~4,~Case~7,~Case~10} \right) = \min \left( 1.00, 0.2141, 0.2141, 1.00 \right) = 0.2141$

$\max \left( \mathrm{Case~3,~Case~6,~Case~9} \right) = \max \left( 0.0256, 0.0268, 0.1038 \right) = 0.1038$

Similarly, bounds on $P$ are obtained from

$\max \left( 1.41x10^{-15}, 0.000661, 0.000662, 1.41x10^{-15} \right) = 0.000662$

$\min \left( 0.41689, 0.0039, 0.02879 \right) = 0.02879$

Finally, bounds on $P$ are obtained from

$\max \left( -0.0159, -0.000695, 0.1025, -0.0159 \right) = 0.1025$

$\min \left( 1.00, 0.9463, 0.1471 \right) = 0.1471$

Thus, $g$ and $P$ lie within bounds given by

$0.2141 \leq g \leq 0.02879$

$0.1025 \leq P \leq 0.1471$

For the example, calculations yield $P=0.11220$. Interpolation of $g$ versus $P$ between their bounds yields the estimate $g=0.1937$. Therefore, the effective availability $k_{E_2}$ is $M_k=7.0944$ and the estimated value of $P$ is $0.00150$. The calculated value of $P$ is $0.00133$ which corresponds to $g=0.1937$.

7.0 SUMMARY

A theory has been presented for estimating the blocking characteristics of a multi-stage link system providing point-to-point selection using a novel application of effective availability theory to a group of inter-stage devices such as links or junctors. The group of devices of interest divide the link system into two subsystems, each of which provides point-to-group access to the devices.

Throughout the paper the devices of interest have been assumed to be loaded with smooth traffic. If the devices are to be assumed to handle random traffic, then the effective availability obtained by use of the smooth traffic assumption may be used with the Palm-Jacobaeus formula or the modified-Palm-Jacobaeus formula to estimate the blocking characteristics of the link system when the devices of interest are loaded with random traffic. The method is illustrated in [3].

REFERENCES


<table>
<thead>
<tr>
<th>Case</th>
<th>$\pi_1$</th>
<th>$\pi_2$</th>
<th>$\rho$</th>
<th>$k_E$</th>
<th>$g$</th>
<th>$P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1,10</td>
<td>-0.1429</td>
<td>-0.1429</td>
<td>-0.0159</td>
<td>37.3191</td>
<td>1.00</td>
<td>1.41x10^{-15}</td>
</tr>
<tr>
<td>2</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>30.0746</td>
<td>0.8059</td>
<td>1.08x10^{-12}</td>
</tr>
<tr>
<td>3,18</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>0.9549</td>
<td>0.0256</td>
<td>0.41689</td>
</tr>
<tr>
<td>4</td>
<td>-0.1429</td>
<td>0.1049</td>
<td>-0.000695</td>
<td>7.9901</td>
<td>0.2141</td>
<td>0.000661</td>
</tr>
<tr>
<td>6</td>
<td>1.0</td>
<td>0.1049</td>
<td>0.9463</td>
<td>0.9989</td>
<td>0.0268</td>
<td>0.40039</td>
</tr>
<tr>
<td>7,13</td>
<td>0.0</td>
<td>-0.1429</td>
<td>0.1025</td>
<td>7.9895</td>
<td>0.2141</td>
<td>0.000662</td>
</tr>
<tr>
<td>8,14</td>
<td>0.0</td>
<td>0.0</td>
<td>0.1081</td>
<td>7.9880</td>
<td>0.2140</td>
<td>0.000663</td>
</tr>
<tr>
<td>9,15</td>
<td>0.0</td>
<td>1.0</td>
<td>0.1471</td>
<td>3.8719</td>
<td>0.1038</td>
<td>0.02879</td>
</tr>
<tr>
<td>12</td>
<td>-0.1429</td>
<td>0.0</td>
<td>-0.0107</td>
<td>28.0816</td>
<td>0.7525</td>
<td>6.69x10^{-12}</td>
</tr>
<tr>
<td>16</td>
<td>1.0</td>
<td>-0.1429</td>
<td>0.9314</td>
<td>1.0000</td>
<td>0.0268</td>
<td>0.40000</td>
</tr>
<tr>
<td>17</td>
<td>1.0</td>
<td>0.0</td>
<td>0.9399</td>
<td>1.0000</td>
<td>0.0268</td>
<td>0.40000</td>
</tr>
</tbody>
</table>

**TABLE 2.** Blocking probabilities, effective availabilities and availability factors for cases in Table 1 for example.
August 1975.