GENERALIZED OPTIMAL CHANNEL GRAPHS
FOR SLIP CONNECTIONS IN A CLASS OF LINK SYSTEMS

K. TAKAGI
OKI Electric Industry Co., Ltd., Tokyo, Japan

ABSTRACT

A method to obtain optimal channel graphs (least blocking probability) is described, for a class of 4-stage link systems without "parallel" limitation. Slip notation, effective to classify and identify channel graphs, makes their comparison tractable. General blocking probability calculation procedure is utilized for this purpose. Generalized optimums are concretely determined for this purpose, and sometimes called "parallel" optimums. The authors have developed a representation and a general comparison method, utilizing this notation, for any channel graphs with a certain limitation of their structure. The optimal channel graphs which minimize the blocking probabilities and their application to a practical switching network design have been reported before. The structural limitation, closely related to the notation, may well be illustrated by "parallel" operations.

1. INTRODUCTION

Link systems have conveniently been represented by so-called multi-stage graphs, where a node stands for a (switch) matrix and an edge for a link. The blocking probability of a link system which determines its traffic capacity has effectively been calculated by an approximation adopted also in this paper, involving only the paths between an inlet and an outlet. The multi-stage graph expression of them has been introduced for this purpose, and sometimes called a channel graph. A comparison of channel graphs which minimizes the blocking probabilities and their application to a practical switching network design have been reported before. The structural limitation, closely related to the notation, may well be illustrated by "parallel" operations.

The author has developed a representation and a general comparison method, utilizing this notation, for any channel graphs with a certain limitation of their structure. The optimal channel graphs which minimize the blocking probabilities and their application to a practical switching network design have been reported before. The structural limitation, closely related to the notation, may well be illustrated by "parallel" operations.

The s-stage link systems dealt with in Refs. and are represented by such multi-stage graphs that can be derived by repeated applications of either of the operations: closed parallel, right open parallel or left open parallel; from the elementary (simplest) channel graph such as shown in Fig. 1. The closed parallel and the right open parallel are, respectively, illustrated in Figs. 2 and 3, clarifying the operation by (b), which derives each (c) from each (f). The left open parallel will easily be understood by reversing the left hand and the right hand of Fig. 3 (denoted by luo).

Notations lv named multiple number and lu0 sufficiently represent the operations, suffixes denoting the branching stages (except s:open), and the number of branches n being their values as lv=mn, etc. Any channel graphs obtained by repeated applications of these operations can be denoted by the assembly of the multiple numbers {lv}. Their commutativity and the identity of {luv=mn} and {luv=mn} will be evident, provided that isomorphic channel graphs are identified. (Note that matrix size nuxmu was adopted in Ref. instead of luo and lu0, considering the practical design applications. Since the both are identical from the relations lu0=nu/lu0 = nuluo/lu0.)

While the theoretical assumptions for the blocking probability calculation have been generalized and the optimal channel graphs have been ascertained in Refs. and , the above "parallel" limitation remained until recently. Generalizations to remove this restriction have lately reported, however, generalized optimums have not yet been clarified except a very simple four-stage case. This paper is an expansion of the latter for more general four-stage channel graphs, including concrete optimums for a range of small link sizes and some of necessary conditions without the "parallel" limitation.

2. REPRESENTATION AND CLASSIFICATION

2.1 SLIP REPRESENTATION OF 4-STAGE CHANNEL GRAPHS

The channel graphs to be dealt with throughout this paper assume the general form of Fig. 4, the channel connection is the one that has the relation J = 1 + Sj (mod m) between the 2nd node number I and the 3rd node number J to be connected by a link, when every 2nd and 3rd node is serially numbered from 0 to m-1 as in Fig. 5. Sj is called slip number. The slip connection of Fig. 4 that has n links per each 2nd and 3rd node is denoted by the combination of slip numbers Su, = u0, 1, ... , n-1. Node number 0 has no special meaning, so the notation (0, su1, ... , sn-1) is used for the general slip connection corresponding to Fig. 4, by letting S0=0 without losing generality.

Each Su can assume an arbitrary number among 1 to m-1. So the total slip connections become µ-1, and this representation will enable one to examine every connection without omission. The whole slip connections are shown in Table 1 for the range n=24, m=8.
Two channel graphs are said to be isomorphic if one of them can be derived from the other by interchanging the nodes in a certain stage or stages together with the associated links. For example, Fig.5(c) is isomorphic to (d) which is the optimum of "parallel" channel graphs (we shall call this P-optimum) as seen from the attached node numbers. Isomorphic channel graphs need not be distinguished, since in general the corresponding link systems have the same blocking probabilities and the same scales (4). All the slip connections may contain many groups of isomorphic channel graphs. To avoid the overlap of comparisons, their classification and identification are to be made first.

2.2 CLASSIFICATION (NECESSARY CONDITION)

The necessary conditions that isomorphic channel graphs commonly possess are utilized not only for their classification but also as the measures of superiority closely related to the blocking probability. When n=2, the optimality involves the non-separability of the subgraph consisting of the whole 2nd and 3rd nodes and 2nd links(12). The slip representation is related to this property in the following way for general n.

**Condition S1**: Number of Separable Subgraphs: For the given m and (S1,...,Sn-1), if there is a common factor g, then the channel graph is isomorphic to the one that consists of g separate subgraphs (between the 2nd and 3rd nodes) each of which is represented by m/g and (S1/g,...,Sn-1/g).

For example, compare Fig.5(c) with (d). Also, Fig.5(c) is distinguished from Fig.5(a) and (b) by this property. In Table 1, P-optimums are marked by *, which correspond to g=m/n.

When n=2, it is intuitively obvious that all the isomorphic channel graphs of g=1 can be represented by Fig.6(a), or else the subgraph between the 2nd and the 3rd nodes is to be the one consisting of such separate cyclic connections as e.g. shown in Fig.6(b). Thus, in every slip connections of m/g, the isomorphism is determined only by g, with such g separate subgraphs being identical. On the contrary, with m/g Fig.5 (a) is not isomorphic to (b), even though they both have g=1. The next condition can classify them.

**Condition S2**: Common Nodes Pattern: Consider a 2nd node, e.g. node number O, and those n 3rd nodes which are connected to the 2nd node in consideration. Among these n nodes, when Ck nodes are also connected from No.k 2nd node (k=1,2,...,m-1), then 'ci1,c2,...,cm-1' is named common nodes pattern or, in short, common pattern (C.P.). Only the combination of Ck is taken in question, so each Ck is generally put in a frame of its value and ck=0 is omitted.

For example, Fig.7 is abstracted from Fig.6(b), where dashed lines show C=2 and dotted lines ck=1, thus, Fig.6(b) has C.P. '22111'. Generally, C.P. is obtained as a numbers Svk (including 0) and 2nd node numbers I, of one seen in Table 2. Obtain Svk+Svk-I (Slip and node numbers) always mean the numbers entered in the row of Ikk in the Table if it is equals anyone of the slip numbers. Then Ck is given as the number of Svk entered in the row of Ikk. Similarly, Fig.5(a) is known to have C.P. '21111', and assured not to be isomorphic to Fig.5(b).

2.3 IDENTIFICATION (SUFFICIENT CONDITION)

**Condition i**: Definition of Isomorphism: Two channel graphs are isomorphic, if and only if the 3rd nodes numbered i,j,...,j'-1 of one of them, connected from No.i 2nd node, can be one-to-one corresponded to the 3rd nodes numbered J', J',...J'-1, of the other, connected from No.'I' 2nd node, without conflict for every pair i and I' of m combinations.

The correspondence may be interpreted to give numbers of the former i,j etc. to the nodes of the latter numbered I', J' etc., which will easily be recognized from Figs.5(c) and (d). The slip connection is the case where both combinations (S1,...,Sn-1) and (S1,...,Sn-1) are respectively consistent for m pairs of I and I'. Especially, the following conditions are the cases where one-to-one correspondence of specific slip numbers S1 and S2 both located in the same position in (1), holds for all m pairs of I and I'.

**Condition S5**: Slip connections (0,S1,...,Sn-1) and (t0,...,Sn-1+t1,...,tn+t1,...,tn+t1,...,t0) are mutually isomorphic, where ty=S1, u=1,...,n-1.

Node Number Correspondence (marked by *)

Shift the start point of 3rd node numbers.

J=k + J+k*t = k+0,...,m-1

Node Number Correspondence is obtained by.

The restriction t0 results from the fact that S1 in the former must correspond to O in the latter.

**Condition S2**: Slip connections (0,S1,...,Sn-1) and (0,...,Sn-1,...,Sn-1) are isomorphic.

Node Number Correspondence: The latter corresponds to the relation I=S1, i.e. to obtain the 2nd node number from the 3rd node number in the former. So, even both the 2nd and 3rd nodes inversely from the same start point.

I+J-k = J+k=t = k+0,...,m-1

Node Number Correspondence is obtained by.

The correspondence of both 0 (the 1st slip number) gives the same node numbers for the same location of both the 2nd and 3rd nodes.

**Condition S3**: Slip connections (0,...,Sn-1,...,Sn-1) and (0,...,Sn-1,...,Sn-1) are isomorphic.

Node Number Correspondence: The correspondence of both 0 does not lose generality, if S1 is associated, which gives both the 2nd and 3rd node numbers at the same time as shown in S2.

I+J=k=S1 ↔ I+J=k=S1; k=0,1,2,...
If there is a common factor among \( m, S_1 \) and \( S_3 \), however, the above correspondence does not cover all the remaining. In another pair \( S_2 \) and \( S_4 \) which have another factor.

\[ I = k_1 S_1 + k_2 S_2 \iff I' = k_1 S_3 + k_2 S_4; \quad k_0, 1, 2, \ldots \]

Note that channel graphs are to be identified after the subgraph is separated by \( S \), i.e. there is no common factor for all the pairs. These procedures are equivalent to carry out \( I' \) for every pair of slip numbers, and the restrictions of \( S \) is the sufficient condition not to yield conflict among slip pairs, i.e. if \( k_0 S_1 = k_0 S_3 \), then \( k_0 S_1 = k_0 S_3 \) must hold simultaneously.

When \( n = 2, S_2 \) is modified to such that \((0, S_2)\) is isomorphic to \((0, S_3)\) unconditionally, provided that the subgraph separation by \( N_4^I \) is carried out and given to the both. The following relation thus gives the sufficiency of \( N_4^I \).

\[ I = k_2 S_2 \iff I' = k_2 S_3 \quad 0, 1, \ldots, m-1 \]

\[ \text{Fig.6(a) is the case of } S_3^I = 1. \]

2.4 CLASSIFIED SLIP CONNECTIONS

Table 1 is classified by \( N_4^I \), subdivided by \( N_2^r \), and arranged such that isomorphic ones are in a frame. Among them, those identified by \( S_4^I \) are shown in the rows whose node numbers to be compared to the preceding row by \( S_2^I \) or by \( S_3^I \) are denoted by \( 2^I \) or \( 3^I \), respectively, in the column 1 of the Table. For example, Fig.5(b) is identified to \( 2^I \) by the 3rd node numbers, Fig.5(a) to \( 2^I \) by altering the 2nd and 3rd node numbers \( I \) and \( J = 6 \) to \( 6-J \), respectively. It is only one case that is not satisfied with even condition and needed by itself (denoted by \( 1^I \) in the table). Note that \( S_2^I \) is logically included in \( S_3^I \). But the former is much simpler in application, and equally practical as \( S_3^I \).

Thus, the number of slip connections to be compared are remarkably reduced, e.g. \( \Omega \) combinations in \( n = 3, m = 6 \) becomes only three kinds in Figs.5(a)--(c). The total of 151 connections in Table 1 become 33 kinds. Among them, 27 will be compared, excluding the cases that consist of one kind per given condition \( n \) and \( m \).

3. COMPARISON OF 4-STAGE CHANNEL GRAPHS WITH \( n = 2 \)

3.1 OPTIMAL CHANNEL GRAPH(12)

A channel graph is said to be superior to another if the blocking probability of the former is not greater than that of the latter under any link occupancies. It has already been proved that Fig.6(a) is superior to (b); if every possible busy-idle combination occurs with the same probability under the given number of idle links (so-called random hunting) in the whole 1st links and the 3rd links, respectively; if each 2nd link is occupied independently of each other; and if mutual independence is assumed between the links in every stage.

The optimal channel graphs which have minimum blocking probabilities under the given \( m \) are immediately known to be of subtypes as Fig.6(a), i.e. \( g^I \) in Table 1, (cf. \( N_4^I \))

The above random hunting assumption in the whole 1st and 3rd links implies that it holds also in any specified part of these links. So it follows that the channel graph with \( g \) separate subgraphs is superior to the one with \( g+1 \) separate subgraphs, provided that \( g-1 \) subgraphs of the both are identical and that the other one subgraph of the former corresponds to the two of the latter just as in Fig.6.

Thus, all the slip connections in Table 1(a) can be compared, and the one with smaller \( g \) is known to be superior to that with larger \( g \). An exception is the comparison of \( g = 2 \) and \( 3 \) in \( m = 6 \), which does not satisfy the above premise. It will be shown later that unconditional superiority does not hold for these. It is also clarified that the generalized optimum with \( g^I = 1 \) is superior to the \( P \)-optimum with \( g^I = m/n \).

3.2 BLOCKING PROBABILITY CALCULATION FOR \( m = 2 \)

In Fig.6(a), when idle 1st links are specified, the remaining channel graph can be represented by series-parallel one. If \( i \) consecutive links are idle and the both adjacent links (when \( i-1 \)) and \( i \) by the blocking probability of this part is determined merely by Fig.8(a) independently of the state of the other links. Fig.8(b) shows the mutually independent elements in this part, which consist of \( i = 1 \) type A and one type B paths, when all of them are simultaneously blocked, the part becomes busy. Similarly, when the numbers of runs \( r \) and idle links \( i \) are given, it follows that there appear \( i-r \) type A and \( r \) type B paths.

If each pattern of \( m_1 \) combinations under \( i \) idle 1st links occurs with the probability \( P_1 \), the following number of cases that the number of runs \( r \) is \( r \) then the following probability \( P_1 \) under \( i \) idle 1st links is given by

\[ P_1 = \sum_0^i (N(m_1,i+r) + (i-r)N(m_2,i) + i) \]

where \( P_1 \) and \( P_2 \) are blocking probabilities of path A and B, respectively, in Fig.8(b) (independence of each path is assumed). \( N(m,i+r) \) is easily obtained by combinatorial theory(11).

\[ N(m,i+r) = \binom{m}{i} \]

where \( m_1 = i \). It is required to define \( N(m_0,0) \) and \( N(m_0,1) \), which necessitates \( \binom{m_0}{0} = 1 \). The other illegal combinations must be 0.

Let \( N(m_1,2) \) denote the number of cases with \( i \) idle links and \( r \) runs for the channel graph which consists of two separate subgraph as shown in Fig.6(b). Then letting \( i = 1 \) and \( r = 1 \),

\[ N(m_1,2) = \sum_0^i N(m_1,i+1) \]

Also, Eq.(3) can be substituted in Eq.(1), which enables one to obtain blocking probabilities for an arbitrary separate subgraph case, not necessarily of slip connections, and similarly for more than two separate subgraphs.

3.3 GENERALITY OF CALCULATION FORMULA

As seen from the derivation of Eq.(1), it holds when the 2nd and the 3rd links are not of single links but substituted by multi-stage graphs. The generalizations of paths A and B in Fig.8(b) are shown in Fig.9. Multi-stage graphs C and D, respectively, corresponding to a 2nd and a 3rd link. However, left hand end stage of C must consist of one node connected to a 1st link (cf. Fig.10).

Thus, the calculation of the blocking probability holds under the random hunting in 1st links and the mutual
independence of each of the multi-stage graphs C and D (throughout this paper). Note that the proof in Ref. (12) differs in the random hunting in the 3rd links which can not be of multi-stage graph D but of single links (cf. 3.1).

3.4 EXAMPLE OF COMPARISON FOR n=2

The values of Eqs. (2) and (3) are exemplified in Table 3, where the number of 2nd or 3rd nodes m is denoted by e.g. \( m = 2 \) for the two separate subgraph case consisting of \( m_1 \) and \( m_2 \) node cyclic connections as in Fig. 6(b), and similar notations for more subgraphs for simplicity. These values can be utilized for optimality confirmation and for the ready comparison of separations such as unsolved in 3.1. A useful relation is prepared.

<table>
<thead>
<tr>
<th>Table 3</th>
<th>( m(n,i,r) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>m=4</td>
</tr>
<tr>
<td>(b)</td>
<td>m=6</td>
</tr>
<tr>
<td>(c)</td>
<td>m=8</td>
</tr>
</tbody>
</table>

**Theorem 1:** If \( q_0 + q_2 > 0 \) (or \( q_0 + q_4 > 0 \)) and \( q_2 + q_4 > 0 \) (or \( q_0 + q_2 > 0 \)), then \( q_0 + q_3 > 0 \) (or \( q_0 + q_2 > 0 \)).

**Proof:** Self-evident, since the values in the both of the following Eq. are non-negative.

**EQ.(1)**

For idle 1st link \( m-1 \) that can be represent­ed generally by Fig. 10 (cf. 3.3), the condition in this paper (cf. 3.3) (or)

The blocking probability of the path element and if the mutual independence of each path element is assumed, then the conditional blocking probability of the whole remaining channel graph is given by

\[
P_i = \sum_{r=0}^{m-1} P_{ij}(n|m_i,j) = \frac{n_i}{1}.
\]

Since all the values of \( P_{ij} \) that can be obtained by any interchange of \( (1,2,4), \ldots, (3,5,7) \) are the same, merely their combination is taken in question and is named path pattern (P.P.). Different P.P.s under the given \( i \) are discriminated by the 2nd link suffix \( j \). Since random hunting in the 3rd links as in 3.2, besides, let \( N_i,j \) denote the number of cases that yield the P.P. numbered \( j \), then the blocking probability \( P_i \) (cf. Eq.(1)) becomes

\[
P_i = \sum_{j} \{| N_i,j(n|m_i,j)| P_{ij} \}.
\]

4.2 PATH PATTERN AND ITS SIZE

**Property 1:** In slip connections, the 1st link combinations \( \{ i_j = i, j \} \) and \( \{ i, j, t \} \) yield the same P.P. for \( i = 1, \ldots, m-1 \).

This makes the derivation of \( N_i,j \) fairly simple. Table 5 shows typical \( i_j \) for each pair of \( i \) and \( j \). The number of P.P.s per \( i_j \) denoted by \( m \), becomes less than \( m \) when the same combination happens according to E.G. \( (2,4) \) of \( m=6 \). \( i=3 \) gives \( \{ 1, 2, 3 \} \). \( \{ 2, 4 \} \), \( \{ 3, 4 \} \), \( \{ 1, 3 \} \). The Table enables one to obtain \( N_i,j \), e.g. only by P.P.s instead of \( 8 \times 56 \) combinations. All P.P.s and \( N_i,j \) for \( n=3 \) to \( m=6 \) are shown in Tables 6 and 7.

**Property 2:** When \( i=0, 1, m-1 \) and \( m \), their P.P.s always become one kind, for all link connections.

**Property 3:** For a P.P. numbered \( j \) and denoted by \( \{ n, r_1, \ldots, m \} \) under idle 1st links \( i \), there exists such a P.P. for idle 1st links \( m-1 \) that can be denoted by \( \{ n-r, k \} \) with the same size. The conditional number \( j \) is designated for the latter, thus \( N_{i,j} = N_{i-1,j} \). (cf. Fig. 11)

Examples are in Tables 6 and 7. The procedure to obtain \( N_i,j \) is sufficient to be carried out for \( 1 \times m/2 \) if Property 3 is associated. Besides, P.P.s in Table 2 need not be compared. So, the comparison process is fairly reduced. The values of \( i \) in Tables 5 and 8 with succeeding ones respectively are thus selected. The above method can be applied to any \( m \) and \( n \), however, e.g. for \( m=12 \), the kinds of P.P. reaches as many as \( 2 \times 12 \times 56 \) (total combinations 2497), which will need a
computer. The algorithm is sufficiently simple as described above.

Condition 2: The coincidence of path pattern size combinations \( \{ n_j \} \) is a necessary and sufficient condition for isomorphic channel graphs.

Proof: Its necessity is obvious. When the coincidence occurs, the blocking probabilities coincide at infinite points, which means the identity of structure is determined by about \( m^2 \) parameters at most. The classification of \((0124)\) and \((0125)\) in \( m=4, m=8 \) will be difficult except for \( 2^m \) (e.g. by \( m^2 \) of 2.3).

4.3 THEOREMS FOR COMPARISON

Theorem 1: For any possible number of path elements that satisfy \( \sum r_i = n \), Theorem 1 for \( i \) and \( m-i \). A typical example is seen in Table 7 above holds for not only for \( \{ n_j \}, \) the corresponding blockings \( \sum b_i \), and the number of \( \sum b_i \) parameters.

4.4 EXAMPLE OF COMPARISON FOR \( 2^3 \)

In Tables 7-12, P.P.s are arranged by Theorem 2 such that \( P_1, P_2, P_3 \) if \( i=1 \); e.g. \( P_1, P_2 \) and \( P_{22} \) in Table 7 are discriminated by \( b_2 b_3 b_4 b_5 \), \( P_{34} \) and \( P_{35} \) by \( b_4 b_5 b_6 b_7 \). However, \( P_{32} \) and \( P_{32} \) cannot be distinguished by Theorem 2, so equal numbers are given for such cases. The P.P. with \( j=1 \) is the case where \( \sum b_i = m \) is distributed as equally as possible into \( m \) parts.

The superiority is defined in 3.1. The method of comparison is quite similar to 3.4, while \( 4 \) and \( N(m, i, r) \) are replaced here by \( P_{ij} \) and \( m \), respectively. An example is shown below. A mentioned the difference of Eq. (5) instead of Eq. (1), multiplied by \( j \) in 3.4.

Comparison of \((012)\) and \((012)\) in Table 7:

(a) \( i=2 \) \( \Delta =-3P_2 + 6P_{22} - 5P_{23} \) \( \not< 0 \)
(b) \( i=3 \) \( \Delta =-5P_3 + 6P_{32} - 6P_{33} + 6P_{34} \) \( \not< 0 \)
(c) \( i=4 \) : From Table 7, the validity of Theorem 3 is assured.

The results show that in every comparison the left hand channel graph is superior to the right hand one unconditionally, with only one exception (a comparison in Table 8) that depends on link occupancies. Thus, the optimum is determined for all the conditions, when Property 2 and Theorem 3 are considered.

4.5 NUMERICAL COMPARISON

For simplicity, independence of every link is
assumed (so-called Lee's model), with the occupancy 0.5 per each link. When \( m = 5 \), Eq. (4) is obtained by Eq. (4) together with Tables 6-12. Then, by Eq. (5), the blocking probability of each channel graph can be calculated, with an appropriate distribution for 1 being associated. When \( m = 2 \), although the similar procedure is available by using Table 14 and 5, and also alternative, it is necessary to be used for it is suitable for the present condition.

Let \( B_n \) denote the blocking probability of Fig. 6 (a), then that of Fig. 6 (b) is obviously \( B_n B_m \) according to the Lee's model. 

\[
B_n = \frac{B_n + \Delta \chi}{2} \quad (m = 2x) 
\]

\[
B_n = \frac{B_n B_{n+1} + \Delta \chi}{(m = 2x+1)} 
\]

\[
\Delta \chi = -2 B_n (1 - B_n) X (1 - B_n) 2 \beta (1 - B_n) X 
\]

\[
\Delta \chi = \frac{B_n B_{n+1}}{2} \quad B = B_n + (1 - B_n) (B_{n+1} + (1 - B_n) B_n^2) 
\]

Where \( B_n \) is the occupancy per an i-th link. Recursive use of Eq. (6) makes it easy to apply to any link connections, if the above referred Fig. 6 (b) cases are added. \( \Delta \chi \) and \( \Delta \) are the differences of blocking probabilities between Fig. 6 (a) and (b), when \( m = 2x \) and \( m = 2x + 1 \), respectively (simplified by the general expression given in Ref. (12)).

The blocking probabilities are compared in Table 13. The difference between the P-optimum and the non-slip optimum connections are so small that the link system design confined to the "parallel" channel graphs will be sufficient for practical purposes, for their influences to the traffic capacities are usually further slight.

5. COMPARISON WITH NON-SLIP CHANNEL GRAPHS

5.1 TOTAL NUMBER OF CONNECTIONS

When \( m = 2 \), the kinds of isomorphic channel graphs can be determined by the subgraph separation. The optimum has been determined on the basis of Ref. (12), and some examples in 3.4 have illustrated them. The both cases include any non-slip connections. When \( m = 5 \), e.g. for \( m = 5 \) in Table 14 (b) which consists of one kind, isomorphic to Fig. 12 (a), there exists such a non-slip one as in Fig. 12 (b). So, further comparisons will be required to obtain unconditional optimums.

For this purpose, the total number of connections is obtained under the given \( m \) and \( n \). The number of connections from a 2nd node, e.g. No. 1 to 0 is obviously \( g_n \). From 1, two, three, and so forth, such exclusions getting greater, results in the total to be obtained. It includes many isomorphic duplicates and also slip cases. Their classified results for \( n = 2, m = 3 \) are shown in Table 14.

Since \( S^8 = -S^9 \) can not be applied, the classification is confirmed, in such a "connection allocated chart" as Fig. 12, by interchanging two rows or two columns repeatedly (cf. 2.1). So it is extremely troublesome compared with that of slip.

The kinds of non-slip connections are not so many than that of slip, denoted by ( ) in the Table. For example, for \( m = 5, n = 5 \) there are no overlaps shown in Fig. 12. The other non-slip connections are in Table 15, giving one representative per each kind. For \( m = 2 \), the notation in 3.4 is utilized in Table 14, which has already been proved sufficient for the classification.

5.2 COMPARISON BETWEEN SLIP AND NON-SLIP

Comparisons are carried out as in 4.4 from the F.P. size combinations \( \{ i j \} \). However, property 1 cannot be utilized for non-slip cases, and P.P. must be obtained for every possible combination of idle 2nd node numbers directly from Fig. 12 or Table 15. So the process becomes very lengthy.

The comparison results are shown in Table 16-18. Since Property 2 and Theorem 3 hold in general, the Tables sufficiently fix the superiority. It is shown that the left hand one is unconditionally superior to the right hand one (exclusive of one case in Table 18), and that slip connections are always optimal. The number of subgraphs \( g \) and \( C.P. \) are shown in Table 19, which are directly obtained from Fig. 12 and Table 15, for the method in 2.2 can not be utilized for non-slip cases neither. Note that C.P. do not coincide when different nodes are taken in consideration for these cases.

Property 4: Slip connections yield "identical C.P.\( \)m" (quoted hereafter) whichever node is taken in consideration. But the inverse is not true. Counterexample: Typical one for the inverse is shown in Ref. (11) cited as NEASIM link system.

5.3 COMPLEMENTARY CHANNEL GRAPH

In Table 14, two pairs of cases: \( n = 2 \) and 3 of \( m = 5 ; m = 2 \) and 4 of \( m = 6 \) will be noticed that each pair shows quite the same number of connections, respectively, together with their subdivisions.

Property 5: The kinds of channel graphs, the numbers of connections in every kind and their total under the given \( m \) and \( n \) are respectively coincident with those obtained under \( m = n \) and \( n = m \).
ing the 2nd and 3rd nodes, always such a complementary channel graph can be defined that on the contrary it has \( m(m-n) \) link connections corresponding to those pairs of nodes which are not connected in the given original one. For example, in Fig.12, \( o \) denotes the given link connections and \( x \) those of the complementary channel graph. The correspondence of the both will be clear. Since all the nodes in the original channel graphs are obtained by interchanging two rows or two columns repeatedly (cf. 2.1), it follows that they correspond as a whole to the isomorphic ones of the complementary channel graphs'. Also clearly:

Property 6 : The complementary channel graph of a slip connection is of slip connection.

Property 7 : When one of the P.P. of a channel graph under \( i \) idle 1st links is given by \( \{ r_k ; k=0,...,m-1 \} \), then the corresponding P.P. becomes \( \{ i-r_k ; k=1,...,m \} \) in the complementary channel graph, as seen from Fig.13. The size \( N_{ij} \) of the former is applicable to the latter.

Theorem 4 : The superiority between channel graphs holds as it is between their complementary channel graphs.

Proof: In Theorem 3, some characters related to \( \{ r_k \} \) are preserved in \( \{ n-r_k \} \). Consider the property of \( r_k \) and the interchangeability of \( n \) and \( i \). Then, with Property 7 instead of Property 3, the whole process of the proof of Theorem 3 is assured to hold good and, consequently, the above mentioned characters are quite similarly preserved in \( \{ i-r_k \} \) of the complementary channel graph (where \( n \) is replaced by \( m-n \)).

An example is seen in Tables 9 and 10. Thus, Table 16 also covers the comparison between \( m \) and \( n \) of \( n=2 \). Table 16 corresponds to the comparison of \( m=6, 3+4 \) and \( 2+2 \) already given in 3.4. Thus, Theorem 4 immediately distinguishes between (0123) and (0124) through the results in 3.1(v), of which superiority can not be discriminated neither by \( g \) nor by C.P.

Property 8 : The C.P. \( \text{c'} \) of a complementary channel graph is given by \( \text{c'} \) of the original channel graph, as \( \text{c'} = \text{c} + m-2n; k=1,...,m-1 \).  

6. CONDITION FOR OPTIMAL CHANNEL GRAPH

6.1 BEST COMMON PATTERN

Since the most uniformly assigned C.P.s seem to cause the optimums in the preceding results, the best Common Pattern \( \text{c'} k=1,...,m-1 \) is such an identical C.P. as consists of:

\[
\begin{align*}
\text{n(n-1)} & \text{ pieces of } \text{c'} = \text{c} + 1 \\
\text{(m-1)(m-2)} & \text{ pieces of } \text{c'} = \text{c} + \text{m-2n} \\
\end{align*}
\]

where \( d = [n(n-1)/(m-1)] \), \( \{ m \} \) means integer part.

Theorem 5 : The best common pattern is a necessary condition for the optimal channel graphs.

Proof: C.P. \( \text{c'} \) is determined when \( m \) and \( m-2 \) i.e. when No.0 1st link (in consideration) and No.k 1st link which has caused the \( \text{c'} \) in question (cf. Fig.7) are simultaneously idle. Thus, the number of 3rd nodes which have \( \text{c} \) becomes \( \text{c'} \). Theorem 3. Consequently, the number of 3rd nodes with \( \text{c} = \text{c'} \) becomes \( 2(n-c) \) and with \( \text{c} = \text{m} \) becomes \( 2(n-m) \). Thus, the number of different elements \( c_x \) and \( c_y \) in two C.P.s \( \text{c'} \) and \( \text{c'} \) respectively, which satisfy \( c_x = c_x' \) or \( c_y = c_y' \), then obviously the latter one cannot be the best. Hence \( \text{c' = c} \) denote Eq.(4) corresponding to \( \text{c} \) on cof2dilation that the 1st link in consideration and the other 1st link which has caused the \( \text{c} \) (or \( \text{c'} \)) in question are simultaneously idle. Similarly, let \( \text{P}_2, \text{P}_3 \) (or \( \text{P}_2, \text{P}_3 \)) denote that to \( \text{c'} \) on condition that the \( \text{2} \) 1st links caused the \( \text{c} \) (or \( \text{c} \)) are simultaneously idle. Then the premise of Theorem 1 holds from the above mentioned relation between \( \text{c} \) and \( \text{c'} \). Consequently, \( \text{P}_2, \text{P}_3 \) (or \( \text{P}_2, \text{P}_3 \)) is superior. Similar conclusion is obtained for \( i=\text{m}-2 \), from the above-mentioned relation to the optimal channel graph which must be superior under every \( i \) (if exists) cannot help having the former C.P. \( \text{c}' \). Which, if not the best, is sure to be replaced by the best C.P. satisfying the above relation.

Corollary : When \( m \neq 2n(n-1)+1 \), the number of subgraphs \( \text{g}' \) is a necessary condition for the optimal channel graphs.

Proof: Under this condition, if \( \text{g}' \geq 2 \), the C.P. can not be the best, thus denies to be optimal from Theorem 5.

6.2 CONSIDERATION ON OPTIMIZING CONDITION

All the slip connections with \( m \geq 8 \) have been compared, besides, the comparisons with every non­ slip one have been made for \( m \leq 5 \). They have been extended by the supplement of the results in 3.1(v) and Theorem 4 (for \( m \leq 6 \)). The optimal channel graphs have been materialized in these ranges. As a result, the following characters have always been found in common among the optimums.

(1) The number of subgraphs \( \text{g}' = 1 \)
(2) The best common pattern
(3) Slip connections

The method in this paper is generally applicable to \( m \geq 9 \), however, it will be troublesome for non­ slip cases, especially, the identification may become prohibitive. So if the above conditions are generally approved, the selection of optimums will effectively be carried out. Thus, each item is briefly considered below.

(1) When \( n=2 \), the isomorphism is uniquely determined by \( g \) and the separation. The optimality of \( \text{g}' = 1 \) have been proved by Ref.12), (cf. 3.1) For general \( n \), the necessity of \( \text{g}' = 1 \) is confirmed for a practical range, including all the examples in this paper, by Corollary of Theorem 5. The comparisons in 3.4, \( \text{g}' = 1 \) seems to dominate the superiority in \( 1=m/2 \) or so of \( n=2 \), contrary to Theorem 5. It may be desirable to ascertain this effect generally for \( m \geq 3 \).

(2) The necessity of the best C.P. is proved by Theorem 5. However, it presupposes the existence of the best C.P. and the optimal channel graph, which has not always been assured.

(3) Non-slip connections, which do not yield identical C.P.s in general (cf. 5.2), hardly have the best C.P.. This reasoning may account for the necessity of slip connections, which on the contrary easily yield the best C.P. (verified for \( m \leq 8 \); e.g. for \( m=9 \), (013) and (0146) are always the best of \( n=3 \) and \( n=4 \), respectively).

7. CONCLUSION

For a class of 4-stage link systems without so­ called "parallel" limitation, a notation based on the slip numbers has been introduced, which has made the comparison possible between any slip channel graphs without omission. The comparisons have been fairly reduced by use of two necessary conditions to classify isomorphic channel graphs and three slip connections to identify them, utilizing the slip notation.

General blocking probability calculation methods
have been obtained under the assumption of so-called random hunting (regardless of probability distribution) in the 1st links and the mutual independence of every 2nd and 3rd link. For the case of two 2nd links per 2nd or 3rd node, the calculation becomes analytical, especially under Lee's model, another very simple recursion. For more than two 2nd links per node, however, some procedures are involved.

The channel graph comparisons have been made possible by some Theorems utilizing only the structural values from the general calculation procedures, without obtaining blocking probabilities quantitatively. The generality of the calculations (i.e., the comparisons) has been shown as much that the 2nd and 3rd links can be substituted by any multi-stage graphs.

As examples of this method, all the slip channel graphs have been compared and the optimums have been concretely determined for the case of not more than eight 1st links. These generalized optimums have been proved superior to the former "parallel" optimums. Numerical results have shown, however, that the differences are so small that the link system design by "parallel" optimums may be practically sufficient. Also, the comparisons have been made with every non-slip channel graph of not more than six 1st links. Which have had complementary channel graphs introduced with a related theorem and have extended the comparison ranges.

The common characters of the optimal channel graphs have been found to be such that they consist of: non-separable 2nd-3rd node subgraphs, the best common patterns, and slip connections; without exception from the obtained results. The applicability of these characters have been studied for general (more than eight 1st links) cases, and the former two have been proved to be necessary conditions for generalized optimal channel graphs, though including a certain incompleteness. Thus, the method and the obtained results in this paper may help obtaining generalized optimums.

However, the following items are to be confirmed: existence of optimal channel graphs, existence of the best common patterns, their realizability by slip connections, and the unconditional necessity of non-separable subgraphs. Moreover, it may be desirable to establish a more direct and efficient method to reach optimums, also desirable to extend to the cases of unequal numbers of the 2nd and the 3rd nodes, to more than 4-stage channel graphs, and to more general blocking probability calculation assumptions. Thus, many problems have been left unsolved for the theoretical generalization of the optimal channel graphs, if the "parallel" limitation is to be removed.

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