APPROXIMATE METHODS FOR MULTI-USER-CLASS SYSTEM ANALYSIS

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ABSTRACT
In the paper approximate methods for estimating the performances of multi-user-class blocked-calls-cleared and blocked-calls-delayed systems are presented. Closed form formulae for calls congestions and mean waiting times for subscriber classes are derived.

1. INTRODUCTION
The paper proposes approximate methods for multi-user-class blocked-calls-cleared and blocked-calls-delayed system analysis. The need for this analysis arises if circuit-switched computer network are designed. The presented methods were mainly destined for the traffic design on line concentrators with output channels organized according to CCITT Rec. X.50 (or X.51), nevertheless they may be applied for networks internodal traffic analysis. The concentrator is assumed to serve data terminals subscribers of class 3-6 according to CCITT Rec.X.1 (synchronous 0.6,2.4,4.8,9.6 kbit/s rates). To serve, for example, a 2.4 kbit/s terminal four BC are needed simultaneously thus a multi-user-class (multidimensional) problem arises.

The problem of the traffic analysis of a line concentrator with different principles of allocating BC was considered by several authors [1],[2],[4],[5],[7],[9],[10],[12],[15]. The approximate methods presented in this paper assume arbitrary BC allocation. This principle gives the maximum utilization of BC for fixed call congestions.

Approximate formulae of the following parameters was derived:
- call congestions $B_i$ for $i$-th subscriber class (SC) for the multi-user-class blocked-calls-cleared system ($i=1,\ldots,k, k$ - number of classes),
- call congestions $B_i$ and mean waiting times $W_{qi}$ for the SC $i, (i=1,\ldots,k)$, for the multi-user-class blocked-calls-delayed system.

In the appendix an approximate method for the output channel dimensioning is presented.

2. THE MODEL

List of symbols
- $m_o$ - number of basic channels (BC)
- $k$ - number of subscriber classes (SC)
- $\lambda(i)$ - calling rate of an idle subscriber from SC $i, (i=1,\ldots,k)$
- $\mu(i)$ - service rate of the subscriber from SC $i$
- $m_i$ - number of BC simultaneously needed to serve a subscriber from SC $i$
- $\rho(i)=\lambda(i)/\mu(i)$ - traffic rate of an idle subscriber from SC $i$
- $B_i$ - call congestion for subscribers from SC $i$
- $W_{qi}$ - mean waiting time for subscribers from SC $i$
- $M_o$ - number of queueing places
- $S=\sum_i N_i m_i / m_o$ - concentration factor

In the analysis of multi-user-class blocked-calls-cleared system the following assumptions are considered:
- intercalling time distribution for individual subscribers from SC $i$ (when idle) is of general type with mean $1/\lambda(i)$, $i=1,\ldots,k$,
- service time distribution of the individual subscribers from SC $i$ is of general type with mean $1/\mu(i)$, $i=1,\ldots,k$.

In the analysis of multi-user-class blocked-calls-delayed system the following assumptions are made:
- intercalling time distribution for individual subscribers from SC $i$ (when idle) is negative-exponential with $\lambda(i)$ parameter, $i=1,\ldots,k$,
- service time distribution of individual subscribers from SC $i$ is negative-exponential with $\mu(i)$ parameter, $i=1,\ldots,k$
- FIFO queuing discipline
- the number of the queueing places is finite
- a subscriber from SC $i$ occupies one queueing place (when queued) and $m_i$ BC simultaneously (when served).
3. SOLUTION TECHNICS

3.1. Blocked-calls-cleared system \( m_e = 0 \)

3.1.1. Exact method

Exact analytical solutions for this case are well known (generalized Engset formula \([3]\)). The call congestion \( B_i \) of the SC \( i \) is given by

\[
B_i = \sum_{j \in A} \left\{ \begin{array}{l}
\text{P} (j) \left( N_i - j \right) P (j)
\end{array} \right.
\]

where:

- \( j \in \{0, 1, \ldots, m_0\} \) is the state of the system \( j \) is the number of simultaneously served subscribers from the SC \( i \)
- \( A = \{j: j = \sum_{i} j_i m_i \leq m_0, 0 \leq j_i \leq N_i \} \) is the set of all system states
- \( A_g = \{j: m_0 - m_0 \geq \sum_{i} j_i m_i - m_0, 0 \leq j_i \leq N_i \} \) is a set of the system states for which the system is congested for SC \( i \) calls
- \( P(j) \) is the probability of state \( j \),
- \( D(j) = \frac{\left( \sum_{i=1}^{K} \binom{n_i}{j_i} \right)^{g_i} \cdot P(j)}{\sum_{i=1}^{K} D(j)} \)

The following are the disadvantages of calculating \( B_i \) basing on (1):

1. If the value \( m_0 \) and \( k \) are large, the calculation of the \( B_i \) is time consuming \([4]\).
2. Formula (1) is inconvenient for the link dimensioning problem (no straightforward inverse formula available).

3.1.2. Approximate method

Assuming that

\[ B_i \ll 1 \] (2)

an approximate method for the analysis of the blocked-calls-cleared system is evaluated (for circuit-switched computer network, \( B_i = 2.5 \times 10^{-3} \) is usually assumed \([3]\)).

The idea of the method

The idea of the method is to substitute the multi-user-class system by a usual one-user-class GI/G/m/m/N \( \lambda / \mu / N \) system. The subscriber of this system, so called "average subscriber" AS, is characterized by the following parameters:

- \( N^* \) - the number of individual AS
- \( g^* \) - traffic rate for individual AS, when idle
- \( m^* \) - the number of simultaneous BC needed

1/ the notation from \([4]\)

to serve one AS \( m^* \) may be a real number.

The parameters \( N^*, g^*, m^* \) are evaluated from the parameterized Engset formula \([3]\) with the parameters

\[
m^* = \sum_{i=1}^{K} m_i p_i,
\]

where:

- \( p_i = L_i \sum_{i=1}^{K} L_i m_i \)

- \( L_i \) indicates the mean number of subscribers from SC \( i \), simultaneously served

Assuming that \( B_i \ll 1 \) and using Little's formula to approximate the following expressions for the \( L_i \) are obtained

\[
L_i = N_i \left( 1 - B_i \right)^{\frac{1}{1+B_i}}\frac{1}{(1+B_i)^{1+B_i}} \approx \frac{N_i}{1+B_i} \]

(5)

1. \( N_i \) - number of subscribers from SC \( i \). The parameters \( p_i \) and \( cL_i \) (\( L_i \) is given by [5]) are evaluated as follows.

The exact formulae for \( B_i \) are given by

\[
B_i = N_i - \frac{cL_i}{N_i - L_i} \cdot P_i
\]

(6)

where:

- \( cL_i \) is the mean number of SC \( i \) subscribers served simultaneously when the system is in the congestion state for SC \( i \).
- \( P_i \) indicates the time congestion for the SC \( i \). The parameters \( P_i \) and \( cL_i \) (\( L_i \) is given by [5]) are evaluated as follows.

\[
P_i = P(0)\left( \frac{N_i}{m^*} \right)^{g^*} \left( \frac{1}{m^*} \right)^{g^*}
\]

1. \( [x] \) is an integer, so that \( [x] \leq x < [x+1] \)
An approximate formula for \( P_i \) is given by

\[
P_i \approx P(m^*_i) \cdot m_i / m^*
\]

(11)

**3.2. Blocked-calls-delayed system (\( M_0 > 0 \))**

**3.2.1. Exact method**

In this case, exact calculation of the call congestions and the mean waiting times for SC \( i \) can only be done by solving the set of linear state equations. This is practical only for systems with small number of BC and SC \( i \) call congestions. So an approximate calculation method is needed.

**3.2.2. Approximate method**

Assuming that

\[
B_i \ll 1
\]

(14)

\[
W_{qi} \ll \mu(1)
\]

(15)

an approximate method for the analysis of the multi-user-class blocked-calls-delayed system is evaluated. The inequalities (14) and (15) are usually assumed for circuit-switched computer networks [13].

**The idea of the method**

The approximate method presented in this case is in fact an extension of the former (point 3.2.1). The major difference is that now the "average subscriber" served by BC is different than the "average subscriber" queued. This is due to the fact that a subscriber from SC \( i \) occupies

\( m_j \) BC simultaneously (when served) and one queueing place (when queued).

The idea of the method is to substitute the multi-user-class system by the one-user-class M/M/m/m/N system with calling rates dependent on the systems state. For this system the "average subscriber" is denoted by ASS. The following is assumed for the ASS group:

\( N^* \): the number of the ASS

\( \lambda^* \): service rate of the ASS

\( \lambda^*_q \): calling rate for ASS calls stepping into service without queueing

\( \lambda^*_q \): calling rate for ASS calls which one queued before being served

\( m^*_q \): the number of simultaneous BC needed to serve one ASS (\( m^*_q \) may be real).

The parameters \( N^*_i, \lambda^*_i, \lambda^*_q, m^*_q \) on the basis of \( \{m_0, M_0, k, (N_i, \lambda(1), \mu(1)), m_i \} \) are evaluated as follows.

**3.4. Session**

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i.e. the "average subscriber" served by the BC is different from the "average subscriber" queued.

\[ N^* = \left[ \sum_{i=1}^{k} \frac{N_i m_i}{m_S} \right] \]

(the formula (6) with \( m_S = m^* \)).

\[ \mu^* = \sum_{i=1}^{k} \mu_i \frac{N_i m_i}{\sum N_i m_i (1 + g)} \]

(23)

(24)

(25)

(26)

\( \chi^* \): The \( \chi^* \) parameter approximated in the following cases:
- if \( l = m_S / m_q \) is an integer, then
  \[ \frac{1}{\chi^*} = \frac{1}{l} + \frac{1}{m_S} + \cdots + \frac{1}{m_q} \]
- if \( l = m_S / m_q \) is a non-integer, then
  \[ \frac{1}{\chi^*} = \frac{1}{m_S} + \frac{1}{m_q} + \cdots + \frac{1}{m_q} \]

The Fig. 1 is obtained from formulae (27) and suitable interpolation for non-integer \( m_S / m_q \) and \( m_S / m_q \). The curve from Fig. 1 may be used to evaluate \( \chi^* \).

\[ W_q = \left( \frac{N}{L} \right) \left( \frac{m_S}{m_q} \right) \left( \frac{m_q}{m_S} \right) \]

(34)

Session 3.4

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paper #6
2. \( m^* \) from (24)
3. \( N^* \) from (21)
4. \( m_0 \) from (26)
5. \( \lambda^* \) from (24)
6. \( \lambda^* \) from (26)
7. \( \lambda^* \) from (27) or Fig. 1
8. \( P(m^*, M_0^*) \) from (30)
9. \( cl_t \) from (39)
10. \( B_t \) from (33)
11. \( W_{qi} \) from (34)

4. NUMERICAL EXAMPLES, CONCLUSIONS

The comparison of results obtained from the approximate, exact (the blocked-calls-cleared system case) and computer-simulation (the blocked-calls-delayed system case) methods are now presented.

Table 1. shows the values of parameters for blocked-calls-cleared system used in the verification of the approximate calculation method from point 3.1.2.

Table 1.

<table>
<thead>
<tr>
<th>( x_0 )</th>
<th>80</th>
<th>160</th>
<th>240</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k )</td>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( { s_i } )</td>
<td>( { s_{i1}, s_{i2}, s_{i3}, s_{i4}} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( { y_i } )</td>
<td>3, 4, 5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( { z_i } )</td>
<td>( { z_{i1}, z_{i2}, z_{i3}, z_{i4}} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( { g_i } )</td>
<td>( { g_{i1}, g_{i2}, g_{i3}, g_{i4}} )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Fig. 2a-m show the call congestions \( B_t \) \( (i=1, 2, 3, 4) \) as a function of the concentration factor \( S \) for several sets of parameters \( (m_0, \{ s_i \}, \{ c_{ik} \}) \). For example, the set \( (80, A, C) \) indicates that \( m_0 = 80 \), the \( \{ s_i \} \) parameters are given by \( A \) and \( \{ c_{ik} \} \) parameters are given by \( A \). The sets \( (m_0, C, A) \) are the traffic prognostic for the NPDN network [13].

Table 2. shows the values of parameters of the blocked-calls-delayed system used in the verification of the approximate method from point 3.2. The traffic volumes are taken from [13].

Table 2.

<table>
<thead>
<tr>
<th>Class number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_{1i} )</td>
<td>20</td>
<td>12</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>( x_{2i} )</td>
<td>23</td>
<td>45</td>
<td>8</td>
<td>4</td>
</tr>
<tr>
<td>( x_{3i} )</td>
<td>29</td>
<td>54</td>
<td>10</td>
<td>5</td>
</tr>
<tr>
<td>( x_{4i} )</td>
<td>51</td>
<td>3</td>
<td>54</td>
<td>1</td>
</tr>
</tbody>
</table>

Fig. 3a-h show that the difference between the approximate and simulation results for call congestions \( B_t \) and mean waiting times \( W_{qi} \) \( (i=1, 2, 3, 4) \) is insignificant, in most cases the approximate results are contained by the 95% confidence intervals.

APPENDIX

An approximate method for calculating the number of BC for fixed \( (N^i, \xi^i, B^i, i=1, \ldots, k) \) parameters and bounded call congestions \( B^i \leq B_{max}, i=1, \ldots, k \).

Assumptions:
- The traffic offered by SC \( i \) equals
  \[ A_{of}(i) = N_i \xi_i / [1 + \xi_i (1-B_i)] \]
  \( (1) \)

We assume that \( A_{of}(i) \) is a Poisson traffic (the Engset formula is approximated by the Erlang formula).

- The call congestion \( B_t \) \( (i=1, \ldots, k) \) is approximated as follows
  \[ B_t = B_{max} \frac{m_t}{m_{max}} \]
  \( (2) \)

where
  \[ m_{max} = \max m_i \quad (i=1, \ldots, k) \]

The \( m^* \) parameter (cf. (3)) equals
  \[ m^* = \sum_{i=1}^{k} \frac{m_i N_i \xi_i (1-B_i)}{[1+\xi_i (1-B_i)]} \]
  \( (3) \)

where \( B_t \) is given by (2).

The traffic offered \( A^* \) and the call congestion \( B^* \) for the "average subscriber" are given by:
  \[ A^* = \sum_{i=1}^{k} A_{of}(i) m_i / m^* \]
  \( (4) \)

and
  \[ B^* = B_{max} \cdot m^* / m_{max} \]

Let
  \[ m_{0}^* = m_0 / m^* \]
  \( (5) \)

The \( m_0^* \) parameter is evaluated from the inverse Erlang formula (for example, the Farmer-Kauffman approach may be used [5]).
In Table 3, the $m_0$ values for parameters from Table 1 are presented. The results show that:

- for $B_{\text{max}} = 10^{-7} - 10^{-5} - \Delta \leq 15\%$
- for $B_{\text{max}} = 10^{-4} - 10^{-1} - \Delta \leq 8.5\%$

where $\Delta$ is the relative error.

Table 3.

<table>
<thead>
<tr>
<th>Parameters' Set</th>
<th>$S_{\text{a}}$</th>
<th>$S_{\text{b}}$</th>
<th>$S_{\text{c}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_0$ obs.</td>
<td>$B_{\text{max}}$</td>
<td>$m_0$ obs.</td>
<td>$B_{\text{max}}$</td>
</tr>
<tr>
<td>(80,1,1)</td>
<td>85.25</td>
<td>6.81 x 10^{-5}</td>
<td>70.12</td>
</tr>
<tr>
<td>(80,1,2)</td>
<td>22.3</td>
<td>5.97 x 10^{-4}</td>
<td>77.75</td>
</tr>
<tr>
<td>(80,1,3)</td>
<td>81.52</td>
<td>8.11 x 10^{-7}</td>
<td>75.9</td>
</tr>
<tr>
<td>(80,1,4)</td>
<td>76.55</td>
<td>6.7 x 10^{-11}</td>
<td>75.9</td>
</tr>
<tr>
<td>(80,2,0)</td>
<td>22.6</td>
<td>6.9 x 10^{-15}</td>
<td>75.25</td>
</tr>
<tr>
<td>(80,2,1)</td>
<td>81.75</td>
<td>7.77 x 10^{-3}</td>
<td>74.1</td>
</tr>
<tr>
<td>(80,2,2)</td>
<td>79.3</td>
<td>4.5 x 10^{-12}</td>
<td>74.9</td>
</tr>
<tr>
<td>(80,2,3)</td>
<td>63.1</td>
<td>6.7 x 10^{-13}</td>
<td>77.47</td>
</tr>
<tr>
<td>(80,2,4)</td>
<td>78.98</td>
<td>5.5 x 10^{-15}</td>
<td>74.27</td>
</tr>
<tr>
<td>(160,1,1)</td>
<td>137.63</td>
<td>6.31 x 10^{-10}</td>
<td>166.9</td>
</tr>
<tr>
<td>(160,1,2)</td>
<td>137.63</td>
<td>7.6 x 10^{-10}</td>
<td>165.06</td>
</tr>
<tr>
<td>(160,1,3)</td>
<td>246.92</td>
<td>9.7 x 10^{-17}</td>
<td>255.73</td>
</tr>
<tr>
<td>(160,1,4)</td>
<td>269.65</td>
<td>8.6 x 10^{-17}</td>
<td>255.4</td>
</tr>
</tbody>
</table>

References

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Fig. 2.
Call congestions $B_i$ ($i=1, 2, 3, 4$) as a function of the Concentration factor $S$ (for sets of parameters from Table 4): the broken line = approximation results, the continuous line = exact results.

Fig. 3.
Call congestions $B_i$ and average waiting times $W_{q,i}$ ($i=1, 2, 3, 4$), as a function of the number of queueing places $M_0$ (for sets of parameters from Table 4): the broken line = approximation results, the continuous line = simulation results for 0.05 confidence coefficient.