A comparison of Sampling Theory and Bayesian estimates of parameters has been made based on the simulation results of a complex telecommunication system. The analysis identifies merits as well as weaknesses of these approaches. It is concluded that while statistical inference may be made by either method the Bayesian estimate offers a more attractive alternative for deriving measures of error in simulation.

1. INTRODUCTION

Computer simulation is a generally accepted method for describing the behaviour of teletraffic systems whenever complexity precludes the use of analytical means. However, the high cost of simulation runs often limits the complexity of the systems to be simulated and the accuracy of the results. It is essential therefore to establish a means of controlling the simulation process by establishing a measure of the error of the results obtained. This is usually done within the framework of classical sampling theory (1). However, an alternative and somewhat different measure of error can also be achieved by means of the Bayesian approach (2).

In the present paper, these two important statistical approaches are examined. They are applied to analyse the simulation results of a complex telecommunication system (3), which are characterised by fault report reception and terminal transactions. The analysis identifies both the merits and the weaknesses of the two approaches. A concluding discussion summarises the findings.

2. SAMPLING THEORY VERSUS BAYESIAN APPROACH

In the simulation of a stochastic system, it is frequently required to determine the a priori unknown parameters, namely the mean \( \mu \) and the variance \( \sigma^2 \) from \( n \) consecutive observations \( x_1, x_2, \ldots, x_n \). Assuming independent trials, \( \mu \) and \( \sigma^2 \) are given by

\[
\mu = \frac{\sum x_k}{n}, \quad \sigma^2 = \frac{1}{n-1} \sum (x_k - \mu)^2
\]  

(1)

(2)

2.1 Sampling Theory

In the Sampling Theory approach, it is conventional to employ the method of frequency distribution, namely the histogram, for evaluating measured data. This is usually coupled with a confidence interval estimate as a measure of error.

The histogram analysis is characterised by the frequency probability density function, \( f_n(x) \), given by

\[
f_n(x) = \frac{1}{\Delta x} \frac{k}{n}
\]

(3)

where the observed \( x \)-range is divided into \( m \) intervals \( I \) of equal length \( \Delta x \) and \( k \) gives the number of events falling into \( I \). The principal difficulty of this method is in the choice of interval length \( \Delta x \), which is rather arbitrary. In fact, by examining the frequency density function one can conclude that \( \Delta x \) should be made small in order to avoid losing information provided by the measured data.
On the other hand, it is necessary to accumulate as many events as possible in each interval in order to reduce the approximation error and this implies a tendency towards large $\Delta x$. In conclusion, one sees that the choice of $\Delta x$ is subjective and the conflict between the amount of information loss and the accuracy of the data cannot be compromised easily.

It is customary to approach the error measure by a confidence interval estimate. This can be achieved by approximating $\mu$ to a normal distribution. This approximation is guarded by the Central Limit Theorem with a proviso that $n$ ought to be sufficiently large. For a moderate $n$, a Student $t$-distribution is usually employed. Hence, a $100(1-\alpha)$ per cent confidence interval for the mean value of $x$ is given by

$$
(u - t_{k-1}(a) \cdot \left( \frac{1}{\sqrt{k}} \right), u + t_{k-1}(a) \cdot \left( \frac{1}{\sqrt{k}} \right))
$$

where $t_{k-1}(a)$ is the $(1-\alpha/2)$ point of the $t$-distribution with $k-1$ degrees of freedom. This error measure is nevertheless subjective in nature. It depends on the choice of the confidence probability as well as the relevant parameters (5). The subjectivity implies a likely personal prejudice in the decision making.

2.2 Bayesian-statistics

Bayesian-statistics relates the a posteriori knowledge of a statistical measurement to its a priori knowledge. In general, it requires a proper formulation of the a priori information in terms of probability distribution function for the parameter of interest. The choice of the function is, however, subjective and open to much criticism (6).

In a typical teletraffic simulation, the parameter of interest is usually a priori completely unknown. Under the circumstances, the a priori knowledge can be described by a uniform distribution, thus giving rise to the objective Bayesian-statistics (5)

$$
f(u|y) = \frac{f(y|u)}{\int_{\mu} f(y|u)\,du}
$$

where $f(y|u)$ is the probability density function. The validity of this approach has been justified in Ref. (5) to which the reader is referred for more details.

Application of the objective Bayesian-statistics leads to simple formulae for error measures. For a normal posteriori distribution, the result yields

$$
\text{Mean} \quad \mu = \frac{y}{n}
$$

Absolute error

$$
\sigma_{\mu} = \left( \frac{n}{n(n-5)} \right)^{1/2}
$$

Relative error

$$
d_{\mu} = \frac{1}{\sqrt{n}} \left( \frac{n}{n-5} \right)^{1/2}
$$

Further application enables the evaluation of an empirical density function $F(x)$ from the measured data, which provides the a posteriori approximation to the a priori unknown distribution function $F(x)$. This analysis requires a sorting of the data into an ordered vector $x_r = (x_1, x_2, \ldots, x_n)$, where $x_r \leq x_{r+1}$, $r = 1, 2, \ldots, (n-1)$. The objective distribution function and its complement are given by

$$
F_n(x) = \frac{n+1}{n+2}
$$

$$
G_n(x) = 1 - F_n(x) = \frac{n-r+1}{n+2}
$$

The error is measured by

Absolute error $\sigma_F(x) = \frac{1}{n+2}$

$$
\left[ \frac{(r+1)(n-r+1)}{n+2} \right]^{1/2}
$$

Relative error $d_F(x) = \left[ \frac{n-r+1}{(n+3)(r+1)} \right]^{1/2}$

$$
d_G(x) = \left[ \frac{r+1}{(n+3)(n-r+1)} \right]^{1/2}
$$

These uncomplicated error formulae can be used to control simulation run length.

3. Teletraffic Simulation

A single process-based processor system with complex structure (3) has been simulated. The simulation model is composed of four types of transactions which are considered to dominate actions during the system busy hour. These transactions are represented by Message Sequence Charts as follows:

1. Passive Fault Report Reception
2. Remote Man-Machine Language (MML)
3. Fault Servicing Process (FSP) MML Local

As a data collecting centre servicing a telephony world, the system supports both bulk data transfer from Exchanges (item 1) and Terminal transactions...
A total of 12 user software processes are involved, running under a fixed priority structure, activated either on demand or periodically. Preemptive scheduling is used for periodic processes whereas for on-demand processes the scheduling is non-preemptive.

The fault reports arrived at the centre are stored on the Current Fault File (CFF) of a disc over a 3-week period. The number of faults per location varies. Three cases of interest are considered, namely CFF = 100, 500 and 1000. The number of disc accesses in FSP MML, and consequently the system occupancy, increases considerably with fault reports.

The simulation program is data driven. Hence, the system environment can be constructed by modelling the Message Sequence Charts. This is followed by coding, testing and an eventual system superimposition.

Long simulation runs have been carried out successfully for the three cases of interest. This is summarised in Table 1. Each run takes about 2-4 hours CPU time on the mainframe AMDAHL V7 (equivalent IBM 3031). The simulated system time varies from 1 to 1.5 hours. It deteriorates as occupancy increases. This indicates a very lengthy warm-up period.

<table>
<thead>
<tr>
<th>No. of Fault Reports on CFF</th>
<th>System Occupancy (per cent)</th>
<th>IBM CPU Time (min)</th>
<th>Simulated System Time (min)</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>32.4</td>
<td>124.8</td>
<td>98.2</td>
</tr>
<tr>
<td>500</td>
<td>40.2</td>
<td>211.2</td>
<td>98.2</td>
</tr>
<tr>
<td>1000</td>
<td>49.6</td>
<td>242.3</td>
<td>74.9</td>
</tr>
</tbody>
</table>

Table 1 System Simulation Runs

4. CORRELATION CONSIDERATIONS

The error measure based on either Sampling Theory or Bayesian statistics assumes independent trials. This assumption is not realistic in practice. The output from a simulation frequently contains data which are highly correlated. This makes unbiased estimation difficult and sometimes impossible.

A measure of the correlation, \( \text{Corr}(x_i, x_{i+k}) \), between observations a distance \( k \) apart is given by the autocorrelation function, ACF, which can be estimated as (7)

\[
\text{ACF}_k = \frac{1}{N-k} \sum_{i=1}^{N-k} (x_i - \bar{x})(x_{i+k} - \bar{x})
\]

\[
\text{ACF}_k = \frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x})^2
\]

where \( \bar{x} = \frac{1}{N} \sum_{j=1}^{N} x_j, k = 1, 2, \ldots \)

From the simulation, the correlation between observations has been analysed. A typical correlogram is shown in Fig. 1 where the ACF for Fault Report response is plotted for all three runs. For the case of low CFF, the ACF decends very rapidly, showing little influence of correlation. However, as CFF increases, the ACF takes a long time to die out. It is apparent that for the worst case, observations far apart are also correlated.

A useful guide in studying the sample ACF is the estimation of warm-up period which is correlation dependent. This can be done by regarding a certain number of the initial observations as "warm-up" numbers and discarding them. From Fig. 1 the warm-up numbers are estimated to be 10, 40 and 180 for CFF = 100, 500 and 1000 respectively. This approach eliminates the initial transient fluctuation which tends to enhance the unwanted correlation.

One way to overcome the correlation is to use the method of repeated simulations. This is, however, rather costly as much simulation time is wasted in the initial warm-up period which has to be eliminated. The alternative is to carry out a long simulation run. The collated data can be either processed while simulating or stored in memory for later analysis. The separation of analysis from simulation is feasible only if the size of the dataset is manageable. The advantage is that repeated analysis can be made on the stored dataset. The latter strategy is adopted for our simulation as different analyses are required on the same dataset for the purpose of comparison.

A common method of acquiring an uncorrelated set from a sequence of measured data is by means of Batch Means analysis (8). In this approach a large sample of measured data is divided into non-

![Fig. 1 Autocorrelation function, ACF, for fault report response](attachment:image.png)
overlapping sub-samples with averages given by
\[ Y_i = \frac{1}{m} \sum_{j=1}^{m} x_{m(i-1)+j} \]  
(15)
i = 1, 2, \ldots, K, K = \text{integer} \left\lfloor \frac{N}{m} \right\rfloor

where K is the number of batches and m the number of observations in a batch. If \( Y_1, Y_2, \ldots, Y_K \) are independent, an unbiased error estimate can be obtained either by confidence interval or objective Bayesian function. The problem therefore is to determine an m such that \( Y_1, Y_2, \ldots, Y_K \) are independent. This can be achieved by the following test statistics
\[ C_K = 1 - \frac{K-1}{\sum_{i=1}^{K} (Y_{i,m} - Y_{i+1,m})^2} \]  
(16)

The hypothesis of independence is acceptable if
\[ |C_K| < c(\alpha) \left( \frac{(K-2)}{(K^2-1)} \right)^{1/2} \]
where \( c(\alpha) \) is given by
\[ c(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = 1 - \frac{\alpha}{2}. \]

5. ERROR MEASURES

The simulation produces quantitative information about system throughputs and response time performance. The former includes system task queue length, buffer pool size of messages, disc access times, and buffer pool size of disc controller unit. The latter yields response times of Passive Fault Report Reception, Remote MML, FSP MML Local and FSP MML Remote. As mentioned earlier, the parameters of interest are sampled and stored in memory during simulation. The results are then analysed, correlated and compared using Sampling and Bayesian approaches. As the methods of analyses are similar for various parameters, we present here only the relevant results pertinent to this article.

5.1 Confidence Interval versus Objective Bayes-function

In simulation, the distribution function of a parameter is usually a priori unknown. The Central Limit Theorem predicts that the sampling distribution of the mean can be approximated closely with a normal distribution. This is true provided that the population has the shape of a normal distribution or if the number of independent trials, n, is sufficiently large. If neither of the conditions hold, the approximation is doubtful.

From the simulation, a typical distribution of probability function is given in Fig. 2. It is obvious that the sample population does not follow a normal distribution.

Thus, one requires a large n in order to satisfy the normal criterion. However, this can not be obtained easily as one is constrained by the high simulation cost. This situation, which is common to all simulations, makes error estimates difficult.

Of the large number of observations, N, the Batch Means analysis produces an uncorrelated set with the effective number of independent trials being \( n < N \). By applying the Sampling Theory, the 95 per cent confidence interval with limits \( \mu_1 \) and \( \mu_2 \) is estimated using Student's t-distribution.

Under the same conditions, a parallel estimate is made using objective Bayes-function (eqns 6-8). The results are compared in Fig. 3 for a number of independent trials, n. The curves indicate that for all values of n, \( \sigma_\mu < (\mu_2-\mu_1) \).

A pictorial representation on the accuracy of the measured data can be obtained by drawing the confidence interval on the figure. A similar presentation can also be made for the objective measure for error. This is shown in Fig. 4 where both the confidence interval, \( \mu_2-\mu_1 \), and the absolute error, \( \sigma_\mu \) are marked for comparison. The curve is derived from three simulation runs, yielding the response time performance for Passive Fault Report Reception under various CFT conditions. The error increases with the CFT parameters, showing a growth of uncertainties at high system occupancies. The absolute error is consistently less than the confidence interval, thus providing a more realistic estimate of accuracy.

5.2 Histogram versus Objective Empirical Distribution Function

A comparison is made between histogram analysis and the objective empirical distribution function approach. This is shown in Fig. 5 for the Fault Report response. The deficiency of histogram analysis is illustrated in two plots, one using a large interval length and the other a smaller length. The former gives a display of wide step function which does not form an unambiguous distribution curve. The latter takes the shape of a curve, but as remarked in Section 2.1, the
accuracy is in doubt. On the other hand, the objective empirical distribution function preserves all the information provided by the data. For $n = 64$, the function $F(x)$ does approach a smooth curve which represents the posteriori knowledge of the true distribution of the response.

For many applications, it is more useful to express the result in terms of the complementary distribution function, $G(x) = 1 - F(x)$, which gives the probability that the response will exceed a certain threshold value. The relative error is given by $d_G$, which if desired can be graphically represented by vertical bars attached to $G$, indicating the tolerance region. However, it is more informative to provide a separate curve for error. This is shown in Fig. 6 where plots of $G_n$ and $d_G$ are made for the Fault Report response. The $d_G$ curve gives the percentage error for the particular $G_n$ value in the specific $x$ range. The quality of the approximation is good in the high $G_n$ range but deteriorates as $G_n$ decreases.

Additional information can be derived from $d_G$, which is related to the number of independent trials, $n$, and the measured data, $x$. A plot of $n$ versus $x$ is displayed in Fig. 7 for a fixed value of $d_G = 10$ per cent.

The relationship is linear. This knowledge is useful as one can deduce from it the minimum number of trials required to achieve a prescribed error requirement. For instance, a 10 per cent error requires $n = 100$ for the range $x \leq 1.565$ and $n = 200$ for $x \leq 1.627$. To ensure a 10 per cent accuracy for the mean response time of 1.615 seconds, a minimum of 180 trials is needed. Application of this feature enables a dynamical control of the simulation runlength, thus making simulation economical. This is particularly suited for simulation of single parameter systems. The attraction may be less for complex systems with multi-parameters if the cost of the control software is high, which may outweigh the saving from economic simulation.

$G_n(x)$, the function complementary to the objective distribution function $F_n(x)$, is rather sensitive to $n$, the number of independent trials. For large $n$ the quality of approximation improves significantly. This is illustrated in Fig. 8 for Remote MML response, where the smooth curve extends to a wider range. On the other hand, when $n$ is small, the approximation is poor, as shown in Fig. 9. The step function suffers a similar degree of uncertainty to that in the histogram approach.
Fig. 5 Comparison of Histogram analysis with objective approximation function $F_n(x)$ for fault report response Parameters for curves: (a) Histogram with $\Delta x = 0.1$
(b) Histogram with $\Delta x = 0.01$
(c) $F_n(x)$ with $n = 64$

Fig. 6 Objective empirical complementary distribution function $G_n(x)$ and relative error $d_G$ for fault report response with $n=64$

Fig. 7 Variation of the No. of independent observations, $n$, with the range of measured data, $x$, for a fixed relative error $d_G = 10\%$

Fig. 8 Objective empirical complementary distribution function $G_n(x)$ and relative error $d_G$ for remote MML response with $n=256$
the application to simulation is only a recent trend of thought (5). The present analysis takes a step forward to perform the actual implementation of the methodology on a practical simulation.

The problem of correlation is important in simulation. Statistical inference demands a measure of dispersion. The number of observations is meaningless if they are highly correlated. The Batch Means analysis is based on a Sampling Theory approach which has been shown to be unsatisfactory. Alternatively, to treat the correlation self-consistently within the Bayes-statistics requires a proper formulation of prior information in terms of probability distribution function. The choice of a utility function is subjective and arbitrary. Moreover, the full analysis involves a complex integral, which makes computation costly and unattractive. A rigorous treatment of the subject is beyond the scope of this article.

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REFERENCES

(5) Schreiber, F, 'Improved simulation by application of the Objective Bayes-statistics', 9th ITC, Torremolinos, Spain, October 16-26, 1979
Q.1 F. Schreiber

In your paper you apply Bayesian methods to correlated random variables. Could you please explain what methods you recommend in order to "decorrelate" these variables so you can apply Bayes' law?

A.1 G.S. Poo.

The application of Bayes' Theorem is not confined to your 'Objective Bayes' approach. Indeed D.V. Lindley would probably advocate a subjective Bayes approach, based on the notion of exchangeability, and this was what I advised Dr. Poo to do as well as use your method. Correlation among observations appears to be a worthwhile avenue for further research, but only for applications where it can be used in a utility function. The paper 2.4.3 by Manfield and Foers makes this point and also shows how your objective Bayes may be applied to a two parameter problem.