CONSIDERATIONS ON LOSS PROBABILITY OF MULTI-SLOT CONNECTIONS

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ABSTRACT

Present telephone connections through PCM networks are set up by switching single-slots with a bit rate of 64 kbit/s. Future ISDN networks demand for switching of multi-slot connections with \( n \times 64 \) kbit/s. This contribution evaluates loss probabilities of multi-slot calls offered to a trunk group and to a switching network.

For the case of a trunk group with full availability an algorithm is presented which greatly simplifies numerical analysis. For the case of trunk group with restricted availability and several PCM links the effect of a practical requirement to place all slots of a multi-slot call on one PCM link is studied. It is shown that the effect of such a restriction is not very significant if an adequate hunting strategy is applied. Four strategies are studied which provide almost equivalent grade-of-service.

Probability of internal loss is evaluated for a 2-stage switching network. Three strategies to hunt for internal slots are considered. They diverge widely in their effect. Internal loss of switching network depends also on the strategy to hunt for outgoing trunks.

1. INTRODUCTION

In telephone networks with PCM switching and PCM transmission connections are switched on the basis of 64 kbit/s-channels ("slots"). Future networks like the wideband ISDN or satellite networks demand for "multi-slot connections" comprising \( n \) times 64 kbit/s-channels where \( n \geq 2 \) is usually assumed. Thus bandwidths of up to 2 Mbit/s are obtained. Examples of potential applications are: high quality audio transmission, facsimile transmission, slow scan video conferences.

Performance evaluation of networks with a mix of multi-slot and single-slot calls is complicated since the various calls differ in many traffic characteristics. Furthermore there is a large variation of possible traffic mixes and performance requirements.

Trunk group performance considering mixes of traffic of different bandwidths (i.e. with \( n \times 64 \) kbit/s-channels or with \( n \times 600 \) bit/s-channels according to CCITT multiplexing scheme X.50) has been investigated by many authors [1]-[5].

Concern of this contribution is directed to:
- a simple algorithm for fully available trunk groups carrying multi-slot calls
- performance of a trunk group comprising several PCM links with 30 channels each and having the restriction that all slots of a multi-slot call have to be placed on one PCM link
- performance of a switching network carrying multi-slot calls

General assumptions are:
- independent traffic classes of single-slot and multi-slot calls
- pure chance traffic with same mean holding times for all traffic classes
- lost calls cleared model.

Performance is expressed in terms of loss probability. Delay problems may be encountered when multi-slot connections are set-up by a series of independent single-slot-connections using arbitrary paths through a network. A special application is discussed in [6].

2. TRUNK GROUP PERFORMANCE

2.1 Fully Available Trunk Group

For the case where all slots are available for any class of traffic analytic solutions derived from state equations are given in [1]-[5]. State probability can be described by two different levels of detail: Probability of microscopic state is

\[
\begin{align*}
\text{Macroscopic state probability is} &
\end{align*}
\]

\[
(1a) \quad P(x_1, x_2, \ldots, x_z) = \frac{T(x_1, x_2, \ldots, x_z)}{V_x} \quad \forall x \in N
\]

\[
(1b) \quad P(x) = \frac{1}{x^{N-1}} P(x)
\]

Equations (1) and (2) comprise terms which may lead to substantial numerical effort. The number of terms to be calculated and to be summed increases rapidly with \( N \) due to increasing number of combinations.

To define loss probability a simple recursive algorithm has been developed.

The following relation holds

\[
U_v = \sum_{x} T(x_1, x_2, \ldots, x_z)
\]

Background and proof is given in [7]. The algorithm is as follows:

\[
(3) \quad B_m = B_0(1 + s_0^{-1} (1 + s_0^{-1} (1 + \ldots (1 + s_{N-2}^{-1} s_{N-2}+ 1) \ldots+ 1) \ldots+ 1))
\]

where

\[
2 \leq m \leq N
\]

\[
B_0 = \text{v}_N
\]
The \( a \)-values are recursively calculated using (2-1) values gained at a step before starting with \( j = 1 \) and ending with \( j = N \). The \( b \)-values are gained recursively, too. Only \( a \) memory cells have to be provided for storage of significant values. To calculate all loss values only about \( z(z/2 + N) \) additions and multiplications and \( 2N \) divisions have to be executed. This algorithm can run on a programmable pocket calculator (e.g., TI-59 for 2 slots).

Macroscopic state probability \( P(x) \) can also be calculated by means of a simple algorithm without summing state probabilities according to equation (1b):

\[
P(x) = \left[ b_1 + x a_{x+1} (1 + x a_{x+2} (1 + \ldots a_N) \ldots) \right]^{-1}
\]

For \( 0 \leq x \leq N \)

\[ P(N) = B_1 \]

State probability distribution is calculated using all \( a \)- and \( b \)-values which have to be stored in about \( 2N \) memory cells. Run time is determined by execution of about \( N(N/2 + 2) \) simple additions, multiplications and divisions.

2.2 Trunk Group With Restricted Availability

Cases with various restrictions with regard to slot usage have been investigated in [3,4]. The \( m \) slots of a call had to be allocated equally spaced or consecutively or in predetermined groups. Artificial limitations in order to balance loss performance have been additionally studied in [4].

Restriction considered here results from a requirement that all \( m \) slots of a call have to be allocated to one pcm link but with full availability as regards the channels on the pcm link.

(Remark: If slots of a multi-slot call were spread over several independently synchronized pcm links a mechanism would be required to align byte sequence. Such a mechanism introduces delay (see also [6]). To avoid this the described restriction has to be observed.)

\[
b_j = 1 + a_j b_{j-1}; b_0 = 1
\]

\[
a_j = \frac{1}{j} (a_{j+1} - b_{j+1})
\]

\[1 \leq j \leq N\]

\[A_1, A_2, A_3 \ldots A_{N-1} \geq 0; a_1^2 = 0 \text{ for } s = 0\]

2.2.1 Exact Solution

The impact of such a requirement has been studied by solving state equations. This allows to calculate the expected small performance differences of various hunting strategies.

Since no general solutions could be found numerical results have been calculated for a limited number of models with the following characteristics:

- \( r = 3 \) pcm links
- \( n = 30 \) or \( n = 6 \)
- mix of single slot and one type of \( n \)-slot calls.

Four hunting strategies have been investigated:

a) Sequential hunting where single-slot calls starting from one side and \( n \)-slot calls starting from other side.

b) Sequential hunting from one side for both call types.

c) As strategy a) but single-slot calls hunting for an already partially occupied pcm link by skipping empty pcm links.

d) As strategy a) but single-slot calls hunting for the most heavily loaded pcm link.

These hunting strategies are understood to be superior to others. Reassignment of slots has been considered to be not practical.

Macroscopic state is described by variables \( i,j,k \) corresponding to the number of calls on each of the 3 pcm links. Values \( 0 \) to \( n \) indicate number of single-slot calls, value "\( n+1 \)" seizure by an \( n \)-slot call. There are \( (n+2)^2 \) different states. State probability \( P(i,j,k) \) is calculated. The equations for the state probabilities in equilibrium state are given in Appendix to 2.2.1.

Loss probability of single-slot and \( n \)-slot calls respectively is:

\[
(5) \quad B_1 = P(i,j,k)
\]

\[
(6) \quad B_n = P(i,j,k)
\]
The following can be read from these figures:
- Course of curves and $B_n < B_0$ is typical for multi-slot traffic mixes. Minima and maxima have been discussed in [2] for the case of full availability. Reasoning holds approximately for the restricted case considered here.
- The 4 hunting strategies considered cause no significant loss differences.
- $B_1$ (restricted availability) < $B_1$ (full availability)
- $B_n$ (restricted availability) > $B_n$ (full availability)
- For $p \approx 1$ it holds $B_0 = B_1 = E_{1,1} (A_n)$ where $E_{1,1}(A_n)$ is Erlang's loss formula
- For $p \approx 0$ it holds $B_1 = E_{1,2n} (A_n)$; $B_n \approx E_1$

### 2.2.2 Approximate Solution

An approximate calculation without solving the set of state equations has been presented in [2] considering equally spaced slots and three traffic classes. Requirement followed here is to place all m slots of a n-slot call on one pcm link. The effect of such a requirement is identical to the one considered in the reference cited if only two traffic classes $A_1$ and $A_n$ are assumed where the multi-slot call occupies a pcm link fully. This approximate method has correspondingly been modified.

Sequential hunting from constant start point is assumed for single slot calls. Hunting strategy for channels of n-slot calls is irrelevant for classes $A_1$ and $A_n$ are assumed where the multi-slot call occupies a pcm link fully. This approximate method has correspondingly been modified.

Loss probabilities of n-slot calls and single-slot calls are calculated consecutively:

\[ Q(x) = \frac{1}{x} \sum_{j=0}^{x-1} \left( \frac{E_{1,1}(A_n)}{E_{1,1}(A_n)} \right) r \]

where $Q(x)$ is the probability that $x$ pcm links are partially or fully occupied if there were single-slot traffic only. $Q(x)$ can be found from probabilities $P_i$ that the i-th channel relative to start point is occupied:

\[ P_i = A_i \left(E_{1,1,i-1}(A_n) - E_{1,1}(A_n)\right) \]

for $1 < i \leq x$ and

\[ P_x = A_x \]

Probability $V_j$ that j-th pcm link is occupied with one or more single-slot calls is

\[ V_j = 1 - \prod_{i=j}^{x-1} (1-P_i) \]

for $1 < j < x$.

### 3. SWITCHING NETWORK

A simple one-sided 2-stage network has been investigated. Configuration and graph is shown in Fig. 3 and 4.

#### 3.1 Path Search Strategy

Single-slot calls are set up by conditional path search or by step-by-step assuming exhaustive number of trials. Multi-slot calls are split into single-slot requests set up consecutively like single-slot requests (Note: The restriction to place all single-slot requests on one pcm link as followed depends on strategy chosen. Three strategies have been evaluated:

- a) Random hunting

- b) Sequential hunting from fixed starting point

- c) Hunting controlled by a toggle switch directing each single-slot call or single-slot request alternately to one of the two pcm links.

\[ Q(0) = \begin{cases} \frac{1}{x} & \text{if } x \leq 1 \\ 0 & \text{if } x > 1 \end{cases} \]

Note: $Q(0)$ can also be found directly

\[ Q(0) = \frac{1}{x} \sum_{j=0}^{x-1} \left( \frac{E_{1,1}(A_n)}{E_{1,1}(A_n)} \right) r \]

In actual numerical examples this correct value $Q(0)$ can be used to improve the approximate values of $Q(x)$ particularly when $Q(x) = 0$ for $x \geq 2$.

Numerical results found with this approximate calculation are shown in fig. 1 and 2. Compared to the exact figures of all 4 hunting strategies it can be said that the approximate method gives sufficiently good results. Accuracy increases with increasing proportion $p$. 

#### 2.2.3 Conclusions Concerning Trunk Group With Restricted Availability

From the exercise on a limited number of models the following conclusions are drawn:

- The four hunting strategies considered are practically equivalent particularly for the normal case of $n = 30$. Huntng strategy a) or b) should be selected which are considered to be the simplest to implement.

- Effect of the restriction is generally not very significant related to the fully available group if one of the above hunting strategies is applied. Differences of loss probabilities are of minor importance in applications where $p \approx 0.5$. In case of unknown $p$ the dimensioning of the trunk group has to take into account the worst $B_n$. For a given value of $B_n$ it has been calculated that in case of full availability the traffic offered $(A_1 + A_n)$ could be up to 20% higher than in case of restricted availability.

Single-slot loss profits from the worse performance of the n-slot traffic.

### Fig. 3 Switching network considered

### Fig. 4 Graph of switching network considered
3.2 Internal Loss Probability

Internal loss probability $B_m$ is the conditional probability that an $m$-slot call is blocked in the switching network under the condition that at least $m$ slots are free on the originating and terminating switching element. A mismatch situation of an $m$-slot call is described by the following relation (see fig. 4):

$$ g = \min(n-i,n-u) + \min(n-j,n-v) < m $$

Assuming independence of states between the stages of the switching network and keeping the general prerequisites of section 1 it holds

$$ B_m = \sum_{i,j} Q(i,j)Q(u,v) $$

where $Q(i,j)$, $Q(u,v)$ are probabilities of state describing the number of busy slots on the two pcm links of the two interstage link groups. The state probabilities are calculated for the three hunting strategies defined.

3.2.1 State Probability in Case of Random Hunting (a)

State probability distribution $P(x)$ of a fully available trunk group is given by equation (4) where the variable $x$ describes the total number of busy slots.

In the link group considered here the busy slots are partitioned into $i$ slots on the upper and $j$ on the lower pcm link. Since all patterns of $i$, $j$ busy slots are equiprobable the following relation holds:

$$ Q(i,j) = \frac{\binom{n}{i,j}}{\binom{2n}{i+j}} P(i+j); \ 0 \leq i,j \leq n $$

Equation (10) holds for each half of the graph.

3.2.2 State Probability in Case of Sequential Hunting (b)

State probability $P(i,j,k)$ is calculated where $i$ and $j$ defines the number of busy slots in the upper and lower pcm link and $k$ indicates the number and distribution of $n$-slot calls.

State equations in equilibrium state and transition rates are given in appendix to 3.2.2. It follows:

$$ (11) \ Q(i,j) = \sum_{k} P(i,j,k) $$

3.2.3 State Probability in Case of Toggle Switch (c)

State probability $P(i,j,k,s)$ is introduced. Meaning of $i$, $j$, $s$ as in sections 3.2.2. Variable $s = 0$ or $1$ corresponds to position of toggle switch. The state equations are given in appendix to 3.2.3. It follows:

$$ (12) \ Q(i,j) = \sum_{k,s} P(i,j,k,s) $$

3.2.4 Numerical Results

Loss probabilities have been calculated assuming equal hunting strategies on both halves of the graph. Figures 5 and 6 show loss curves for a constant traffic offered versus $n$-slot traffic proportion $p$. From the loss curves one can read:

- An oscillating course as for trunk groups but more pronounced and with totally different evolution. There are ranges of $p$ where $B_{30}$ becomes smaller than $B_1$.

- Diverging loss curves for the three hunting strategies. With regard to $B_1$ figure 5 shows that strategy b is worst while strategies a and c are about equivalent.

With regard to $B_{30}$ figure 6 reveals that in the range of $p > 0.5$ strategy c is much better. Approaching the point $p = 1$ strategy b is worst, followed by strategy a. For exactly $p = 1$ it is contrary.

In case of strategy c loss $B_1$ as well as $B_{30}$ are approaching the value zero.

Fig. 5 Internal loss probability $B_1$ versus $n$-slot traffic proportion $p$

Fig. 6 Internal loss probability $B_{30}$ versus $n$-slot traffic proportion $p$
Single-slot calls are blocked if opposite link of the two halves of the graph are fully busy. This condition may be more frequent in case of strategy b than in case of strategy a and c due to "filling" of links. Thus a worse ranking of strategy b with regard to b, could be expected applying strategy a.

Multi-slot calls are blocked under many mismatch conditions (see 3.1). From the cases considered a generally valid explanation cannot be given.

Results of strategy c approaching zero loss are explained by the fact that only states (i,j) = (0,0) or (15,15) can exist where no mismatch is possible.

3.2.5 Simulation Results

Internal loss probability calculated is checked by roulette-type simulation. An n-slot call selects randomly a first stage switching element out of all which have at least n free slots. The terminating element is selected in the same manner. Thus external loss contributions where single-slot calls and n-slot calls are hunted with a 95% confidence interval or vice versa.

In general the simulation results confirm the analytic calculations with deviations to lower and higher loss. Deviation is explained by the fact that the dependencies between the two halves of the graph are neglected in case of analytic calculations and that a single loss single-slot calls will be compensated by a higher loss of multi-slot-calls or vice versa.

3.2.6 Impact of External Trunk Hunting Strategy on Internal Loss

So far, internal loss probability has been considered assuming random selection of the two switching elements involved in setting up a call. By random selection of a terminating element the traffic is distributed equally over outgoing trunks. This is a valid approach for fully available trunks. In case of restricted availability a well adapted hunting strategy is usually applied which may result in traffic imbalances.

The impact of such a trunk hunting strategy on internal loss probability is studied by simulation. A single outgoing trunk group is assumed where single-slot calls and n-slot calls are hunting for slots from different sides (according to strategy a of section 2.2.1). Simulation results are shown in fig. 7. A drastic increase of loss can be seen.

3.3 Conclusion on Internal Loss Probability

From the exercise on a simple switching network the following conclusions are drawn:

- There is no general optimal strategy to hunt for internal slots. Loss develops very differently, and drastically with proportion p of n-slot traffic. Selection of a strategy depends on loss objectives to meet, whether B2 or B3 should be minimum or whether equalized losses are required.

- Internal loss is highly sensitive to traffic imbalance which is produced when trunk groups with restricted availability are applied. Loss calculations or simulations should therefore take into account the actual trunk hunting strategy.

4. CONCLUSION

Loss in case of trunk groups with full and restricted availability has been studied by many authors. The investigations in this paper extend the results with regard to trunk group and consider internal loss of switching networks.

For the case of a trunk group with full availability an algorithm is presented which greatly simplifies numerical analysis.

For the case of a trunk group with restricted availability and several pcm links the effect of a practical requirement to place all slots of a multi-slot call on one pcm link is studied. It is shown that the effect of such a restriction is not very significant if an adequate hunting strategy is applied. The four strategies compared provide equivalent grade-of-service.

Probability of internal loss is evaluated for a 2-stage switching network. Three strategies to hunt for internal slots are considered. They differ widely in their effect. Internal loss is also very sensitive to traffic imbalance produced by strategy to hunt for outgoing trunks.

This contribution is based on investigations of simple models derived from practical applications. Investigations of more complex models and with other traffic characteristics are ongoing.

APPENDIX to 2.2.1

State probabilities of trunk group with restricted availability considering hunting strategies a, b, c, d are

\[
\begin{align*}
&f_0 = P(I,J,k) + f_1 P(I-1,J,k) + f_2 P(I,J,k) + f_3 P(I,J-1,k) + f_4 P(I,J,k-1) + f_5 P(I,J-k) + f_6 P(I+1,J,k) + f_7 P(I,J+1,k) + f_8 P(I,J,k+1) + f_9 P(I,J-k) + f_{10} P(I,J-1,k) + f_{11} P(I,J-k-1) + f_{12} P(I,J-k+1)
\end{align*}
\]

with 0 ≤ j, k ≤ n + 1

Transition rates and validity range are (out of range f_0 = 0 for 0 ≤ j ≤ k):

\[
\begin{align*}
f_{01} &= A_1; j < n or j < n or k < n \\
f_{02} &= B_1; i = 0 or j = 0 or k = 0 \\
f_{03} &= I + j; k, i, j, k ≠ n + 1 \\
f_{04} &= 0 + j; 1, 1, j, k ≠ n + 1; k + n + 1 \\
f_{05} &= 0 + j; k, i, j, k ≠ n + 1; j + n + 1 \\
f_{06} &= 1 + j; k, i, j, k ≠ n + 1; j + n + 1 \\
f_{07} &= 0 + j; 1, 1, j, k ≠ n + 1; k + n + 1 \\
f_{08} &= 0 + j; k, i, j, k ≠ n + 1; j + n + 1 \\
f_{09} &= 0 + j; 1, 1, j, k ≠ n + 1; k + n + 1 \\
f_{10} &= f_9 = 1 + j; 0 ≤ i ≤ n + 1; f_8 = 1 + j; 0 ≤ i ≤ n + 1 \\
f_{11} &= f_1; i + 1; 1, 1, j, k ≠ n + 1 \\
\end{align*}
\]

The other rates are given according to hunting strategy (a), (b), (c), (d)

\[
f_1 = A_1 \text{ for } 1 ≤ i ≤ n + 1; 0 ≤ j ≤ n + 1; 0 ≤ k ≤ n + 1
\]
The state equations read for the four areas of validity defined by the k-value:

**area 1:**

\[ k = 0, \text{ or } n \leq i, j \leq n; \]

\[ (A_1 (i-1,j-1) + A_n (i,j-1) + i) \quad \text{or} \quad A_1 (i,j-1) \quad \text{or} \quad (i,j,n+1) \]

\[ + \quad (i+1,j) \quad \text{or} \quad (i+1,j) \quad \text{or} \quad (i,j,n+1) \]

\[ + \quad A_n (i,j-1) \quad \text{or} \quad A_n (i,j+1) \quad \text{or} \quad A_n (i+1,j-1) \]

\[ + \quad 2n \quad \text{or} \quad 2n \quad \text{or} \quad 2n \]

\[ \text{with} \quad x = \max(1,i) \]

\[ \text{The state equations read for the four areas of validity defined by the k-value:} \]

**area 2:**

\[ k = 2n + 1, \quad 0 \leq i, j \leq n; \]

\[ (A_1 (i+1,j) + A_n (i,j-1) + 1) \quad \text{or} \quad A_1 (i,j-1) \quad \text{or} \quad (i-1,j,n+1) \]

\[ + \quad (i,j,n) \quad \text{or} \quad (i,j,n) \quad \text{or} \quad (i-1,j,n+1) \]

\[ + \quad 2n \quad \text{or} \quad 2n \quad \text{or} \quad 2n \]

\[ \text{with} \quad x = \max(1,i) \]

**area 3:**

\[ k = 2n + 2, \quad 0 \leq i, j \leq n; \]

\[ (A_1 (i,j) + A_n (i,j-1) + 1) \quad \text{or} \quad A_1 (i,j) \quad \text{or} \quad (i,j) \quad \text{or} \quad (i,j) \]

\[ + \quad 2n \quad \text{or} \quad 2n \quad \text{or} \quad 2n \]

\[ \text{with} \quad x = \max(1,i) \]

**area 4:**

\[ k = 2n + 3, \quad 0 \leq i, j \leq n; \]

\[ (A_1 (i,j) + A_n (i,j-1) + 1) \quad \text{or} \quad A_1 (i,j) \quad \text{or} \quad (i,j) \quad \text{or} \quad (i,j) \]

\[ + \quad 2n \quad \text{or} \quad 2n \quad \text{or} \quad 2n \]

\[ \text{with} \quad x = \max(1,i) \]

**APPENDIX to 3.2.2**

State probability \( P(i,j,k) \), strategy b

The k-value indicates number of n-slot calls and distribution of slots to two pcm links. The k-value defines four areas of validity:

**area 1:** \( k = 0; \) no n-slot call

**area 2:** \( 1 \leq k \leq n; \) one n-slot call seizing \( n \) slots in the 1st and \( k \) slots in the 2nd pcm link

**area 3:** \( k = n + 1; \) one n-slot call in the 1st link

**area 4:** \( n + 1 \leq k \leq 2n; \) two n-slot calls, 1st n-slot call seizing \( k \) slots on the 1st link and \( 2n \) slots on the 2nd link, 2nd n-slot call seizing \( 2n-k \) slots on the 1st link and \( n \) slots on the 2nd link.
area 2: \( 1 \leq k \leq n; \ n - k \leq i \leq n; \ k \leq j \leq n \)
\( s = 0,1 \)

\[
(A_1 (1- \delta_{i,j}^n) + A_0 \delta_{i,j}^n) + i + j - n + 1) P(i,j,k,s)
\]

\[
A_1 (1- \delta_{i,j}^n) P(i,j-1,k,s+1)
\]

\[
A_0 \delta_{i,j}^n (1- \delta_{i-k,j}^n) P(i+1,k,s+1)
\]

\[
A_n (\delta_{i,j+k}^n \delta_{i+k,j}^n + \delta_{i,k}^n (1- \delta_{i,j}^n) \delta_{i+k,j}^n x P(i-n+k,j-k,0,s+1)
\]

\[
(1+1-n-k) (1- \delta_{i,j}^n) P(i+1,j,k,s)
\]

\[
(\delta_{i,j}^n (1- \delta_{i,j}^n) P(i+1,j,k,s)
\]

\[
x \leq \sum (\delta_{i-k,j}^n \delta_{i,j+k}^n + \delta_{i,j}^n (1- \delta_{i,j}^n) x P(n,n,s+1)
\]

\[
x P(n,n,s+1)
\]

area 3: \( k = 2n + 1; \ 0 \leq j \leq n; \ s = 0,1 \)

\[
(A_1 (1- \delta_{i,j}^n) + A_0 \delta_{i,j}^n) + i + j + 1) P(n,j,2n+1,s)
\]

\[
A_1 (1- \delta_{i,j}^n) P(n,j-1,2n+1,s+1)
\]

\[
A_0 \delta_{i,j}^n P(0,j,0,s+1)
\]

\[
(\delta_{i,j}^n + i+1) (1- \delta_{i,j}^n) P(n,j+1,2n+1,s)
\]

\[
\delta_{i,j}^n P(0,n,2n,s)
\]

area 4: \( n + 1 \leq k \leq 2n \)

\[
2 P(n,n,k,s)
\]

\[
A_0 (1- \delta_{i,j}^n) P(k-n,2n-k,2n-k,s+1)
\]

\[
A_0 \delta_{i,j}^n P(0,n,2n+1,s+1)
\]

\[
A_0 \delta_{i,j}^n P(0,n,2n,s+1)
\]

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references


