ABSTRACT

Two methods for approximating the blocking probabilities of overflow traffic components are compared for the cases of two and more than two components. A conjecture about the peakednesses of the blocked traffics is stated. A new approximation of the equivalent traffic of Wilkinson's Equivalent Random Theory is given.

1. INTRODUCTION

A number of independent overflow traffic streams is offered to a common overflow route. The determination of the blocking (loss) probabilities of the traffic components on this overflow route is an important and difficult problem. When the blocked traffics are offered afterwards to an other route their peakednesses are also important.

Several approximations for the blocking probabilities of the traffic components are given in the literature. Here we are concerned with a comparison of two methods, namely the Method of Manfield and Downs (MMD) and the Method of Akimaru and Takahashi (MAT). Both use an Interrupted Poisson Process (IPP) as a model for overflow processes. In MMD an IPP is used for each traffic stream whereas in MAT the total overflow traffic is represented by an IPP.

In Section 2 the problem of the blocking probabilities is defined and some notation is given. In Sections 3 and 4 MMD and MAT are treated respectively and in Section 5 they are compared. Section 6 states a conjecture about the peakednesses of the blocked traffics and Section 7 contains the conclusions. Some mathematical details are given in the Appendix.

2. PROBLEM AND NOTATION

We restrict ourselves in the sequel almost always to two traffic streams. Then the loss system drawn in Fig. 1 is considered. For shortness we often use an index i in the sequel where it is tacitly understood that i can be 1 as well as 2. Two mutually independent traffics A1 and A2 are considered. A1 is a Poisson traffic with negative exponential distribution of holding times which is offered to a primary full availability group of Li lines (Li may be zero). The blocked traffic with mean M1 and peakedness (variance/mean) Z1 is offered to an overflow full availability group of N lines. It is not relevant here whether A1 and L1 are explicitly given as characteristics of a real network or whether they are calculated from M1 and Z1 by Wilkinson's Equivalent Random Theory (ERT) as described in [1].

Offered Primary Overflow Overflow Blocking of traffics routes traffics route overflow traffics

\[ A_1 \quad L_1 \begin{pmatrix} \text{M}_1, Z_1 \end{pmatrix} \quad N \quad B_1 \]

\[ A_2 \quad L_2 \begin{pmatrix} \text{M}_2, Z_2 \end{pmatrix} \quad B_2 \]

Fig. 1 Loss system with two overflow traffics offered to a common overflow route

The total overflowing traffic has mean M and peakedness Z given by

\[ M = \frac{M_1 + M_2}{2} \]

\[ Z = \frac{(M_1 Z_1 + M_2 Z_2)}{M} \]

The combined traffic M has a loss probability B on the N lines, while M1 has a loss probability B1. Thus

\[ BM = B_1 M_1 + B_2 M_2 \quad (1) \]

It is well known that B can be determined with sufficient accuracy by ERT or - when M/Z is not too small - by Hayward's approximation (2 and 3). In Section 3 an other approximation is mentioned. The problem here, however, is the approximate calculation of B1 and B2. In the literature several - mostly heuristic - procedures are known. In e.g. [4] some are mentioned. It is not surprising that an exact treatment is virtually impossible because in [5] it is shown that the interarrival time distribution of the traffic (M1, Z1) is a mixture of Li+1 negative exponential distributions.

Here we are concerned with two related methods, namely that of Manfield and Downs [6] and that of Akimaru and Takahashi [7] respectively, which we treat below.

3. METHOD OF MANFIELD AND DOWNS (MMD)

The overflow process (M1, Z1) is modeled as an Interrupted Poisson Process (IPP) described in [6]. Three parameters are chosen as functions of \( M_1, Z_1 \) and A1 such that the first three moments of both processes are equal. Formulas
for the parameters are given in Section A1 of the Appendix. One of the parameters is \( \lambda_1 \) which is the product of the intensity of the IPP when it is on, and the mean holding time (this implies \( \lambda \sim M_1 \)). As shown in Section A2 the formulas of MMD lead to

\[
\begin{align*}
B_1 &= M_1(Z_1-1)f_1 + M \\
B_2 &= M_2(Z_2-1)f_2 + M
\end{align*}
\]

(2)

where

\[
M_1 = \left( \frac{N(N-1)Z_1 + \lambda_1}{\lambda_M} \right)
\]

(3)

In order to use (2) \( \lambda_1 \) and \( \lambda_2 \) have to be determined. When \( \lambda_1 \) and \( \lambda_2 \) are given they can be found with formula (12) of Section A1. If not, \( \lambda_1 \) and \( \lambda_2 \) must be calculated or estimated. This can be done by ERT or e.g. by Rapp's formula or by solving a cubic equation derived in Section A3 which will be discussed in Section 4.

B is calculated by modeling the superposition of the two overflow processes as an IPP. The formula for B is given in [6] and [7] and can be written as

\[
B = \sum_{j=0}^{N-1} \left( \frac{N!}{j!(N-j)!} \right) \left( \frac{M}{(r-1)(Z-1) + \lambda_M} \right)
\]

(4)

where \( \lambda \) is calculated in the same way as \( \lambda_1 \) above. In cases like those in [5] where \( \lambda_1 \) and \( \lambda_2 \) are known and hence the first three moments of the total overflow traffic can be calculated, \( \lambda \) can be found by equating these moments to the corresponding moments of the IPP (in general \( \lambda \) is not equal to \( \lambda_1 + \lambda_2 \)). Then \( B_1 \) and \( B_2 \) follow from (1)-(4).

For any two streams \( i \) and \( j \) out of \( k \) (>2) streams (2) is generalized to

\[
\begin{align*}
B_i &= M_i(Z_i-1)f_i + M \\
B_j &= M_j(Z_j-1)f_j + M
\end{align*}
\]

For \( N=1 \) and any value of \( k \) we get

\[
\begin{align*}
B_1 &= M_1Z_1-1 \\
B_2 &= M_2Z_2-1
\end{align*}
\]

(5)

Some results of the method for cases mentioned in [6] are discussed in Section 5.

4. METHOD OF AKIMARU AND TAKAHASHI (MAT)

Here B is calculated for the total overflow process by (4) where especially

\[
\lambda = M_1Z_1 + 3M_2(Z_1-1)
\]

is taken, i.e. a two moment fit is made. This particular choice of \( \lambda \) is not essential for the method. In order to determine \( B_1 \) and \( B_2 \) two steps are taken:

- The special case of an overflow traffic \((M_1,Z_1)\) and a Poisson traffic \((M_2,1)\) is considered. Because \( B_2 \) is then equal to the time congestion on the N lines, \( B_2 \) and hence also \( B_1 \) can be found in this special case.

- By a simple extension \( B_1 \) and \( B_2 \) are found for the case of one overflow traffic \((M_1,Z_1)\) together with an other overflow traffic \((M_2,Z_2)\) which is the addition of two independent traffics, namely \((M_1,Z_1)\) and \((M_2,M_1)\). The formula for \( B_1/B_2 \) in the second case is then generally used. It is as follows

\[
\begin{align*}
B_1 &= 1+(Z_1-1)f \\
B_2 &= 1+(Z_2-1)f
\end{align*}
\]

(6)

where

\[
\frac{df}{M} = \frac{N(\lambda - M)}{M[N(1-Z_1)+\lambda - M]}
\]

(7)

The similarity with the formulas (2) and (3) is evident, but there is only equality in degenerated cases. For any two streams \( i \) and \( j \) out of \( k \) (>2) streams \( B_i/B_j \) is found from (6) with 1 replaced by \( i \) and 2 by \( j \). This method is easier to apply than MMD because no \( \lambda_i \) are needed beside \( \lambda \). If (5) is used the calculation is very simple and only \( B \) requires some more work.

However, two points of criticism can be raised:

- First the decomposition of an overflow traffic according to the description above is in general not possible and it is not easy to see how a similar procedure valid for any two overflow traffics could be found. However, on heuristic grounds (6) and (7) can also be used for the general situation of two or more overflow traffics, and in some - or perhaps even many - cases the results are (very) good. E.g. in Table 1 where MAT is compared with exact results for \( N=1 \). For this value of \( N \) (6) becomes

\[
\frac{B_1}{B_2} = \frac{M_1Z_1-1}{M_2Z_2-1}
\]

Here and in the sequel exact results are either taken from literature or obtained by solving the system of state equations. The exact results for \( L_2=0 \) in Table 1 are given in [8].

Further simulations with 3 and 5 traffic streams (see Section 5) suggest that for more streams the accuracy of MAT may improve.

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Table 1. Comparison of blocking values from MAT with exact values for \( L_1=M=1 \)
- The second point of criticism concerns the choice of $\lambda$ according to (5) which is more suitable for A than for $\lambda$. It can therefore be argued that A be calculated or estimated like $A_1$ and $A_2$ in MMD. Here this is done in the following way.

It is shown in Section A3 that for $N=1$ or 2

$$B = E_L(A)/E_L(A)$$

where $E_L(A)$ is the Erlang loss formula for A erlangs on L lines and L and A are the fictitious number of primary lines and the fictitious traffic respectively of ERT. This means that for $N=1$ and $N=2$ B agrees with ERT. The equality is proved for $N=1$ and $N=2$ by showing that both sides of the equation are the same function of $M, Z$ and $A$. For $N=3$ there is only equality if $A$ is a root of a cubic equation (18) which lies between $Z(M+Z-1)$ and $+\infty$.

It appears numerically in Section A4 that this root lies closer to the equivalent random traffic of ERT than $Z^2+3Z(Z-1)$. It is in fact a very good approximation of the ERT traffic value and it is easier to obtain than by manipulation of the Erlang formula with non-integer numbers of lines.

Substitution of the root in (12) gives the corresponding value of $\lambda$. Table 2 gives a comparison between exact values of $B_1$ and $B_2$, calculations with $\lambda$ according to (5), and $\lambda$ according to (18) and (12), in some cases mentioned in [7] (the exact values in Table 2 of [7] are mostly likely misquoted). The second kind of calculation is called MMAT (Modified Method of Akimaru and Takahashi).

It appears that MMAT is better than MAT if $L_2=0$, but there is hardly any difference between the two methods if $L_2=0$. The performance of both methods is not very good. Errors of 5% in $B_1$ or $B_2$ are common.

One can think of another modification of MAT, namely computing $\lambda$ from (12) with $A=Z+3Z(Z-1)$. This modification has not been considered here, however, because a numerical investigation showed that MMAT performs better. It appeared also that $\lambda$, and therefore also $\lambda-M$, is highly sensitive to small changes in $A$ which in turn has a great influence on $B_1/B_2$ but much less on $B$.

5. COMPARISON OF MMD AND MMAT

For some cases mentioned in [6] and [7] MMD and MMAT are applied and compared with each other and with an exact calculation. In all cases $\lambda$ is calculated via (18) and (12). This leads for MMD to blocking values which sometimes deviate slightly from those in [6]. If necessary, $\lambda$ is found with (12).

For both methods it should be noted that it may sometimes be easier - and for not too small $M/Z$ equally accurate - to calculate $B$ by Hayward’s approximation

$$B = E_{M/2}(M/Z)$$

than by (4).

The results of the comparison are given in Table 3.

The following conclusions can be drawn from this table.

- The approximation of $B$ by MMD and MMAT is good. It seems to be an exception when the error is 5%.
- When there is one Poisson traffic ($L_2=0$) MMAT is better than MMD which may show errors of 30% in $B_1$ or $B_2$.
- In cases with $L_2=0$ the errors of MMD in $B_1$ or $B_2$ are mostly in the order of some percents and exceptionally 10%. The errors of MMAT in $B_1$ or $B_2$ are generally larger and often in the range of 5-10%.
- In cases with $L_2=0$ where the ratio between $M_1/Z_1$ and $M_2/Z_2$ differs considerably from 1, MMD is better than MMAT (e.g. where $\lambda_1=22$).
- When the ratio between $M_1/Z_1$ and $M_2/Z_2$ is in the order of 1, MMD and MMAT perform equally well. This happens e.g. for $A_1=8$, $A_2=12$ and $A_1=14$, $A_2=20$.

Some simulations with three and five traffic streams suggest that the above mentioned trends may still be present but less pronounced and that the relative errors in $B_1$ and $B_2$ are smaller than for two streams. If this is true, MMAT may be preferable for more streams because of its greater simplicity.

6. CONJECTURE

So far we were only concerned with the blocking probabilities of the k different traffic streams. The mean $M_i$ of the traffic of stream i that flows over from the $N$ lines can be found from

$$M_{i1}=B_{i1}M_i$$

(8)
Table 3. Comparison of blocking values from MH (1), MMAT (2) and exact calculation (3) for two traffic streams

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<td>3</td>
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The conjecture is that the peakedness $Z_{oi}$ of this traffic can be approximated by using the formula

$$M^{oi}Z_{oi}^{-1} = B_i \cdot M^{o}Z_{o}^{-1}$$

where $M$ and $Z_o$ are the mean and peakedness of the total overflow from the $N$ lines. Now

$$M^o = BM$$

and $Z_o$ can be found from Hayward's approximation. By assuming that the traffic $M/Z$ offered to $N/Z$ lines is Poissonian, the peakedness of the overflow can be calculated by

$$BM \left( 1 - \frac{M}{N} \right) \frac{1}{1 + \frac{M}{Z} + \frac{M}{Z} \frac{M}{Z} \frac{M}{Z}}$$

and $Z_o$ can be found as $Z$ times this peakedness. Hence

$$Z_{o} = Z_{o} - \frac{M}{N}$$

Thus by applying (10), (11), (8) and (9) $Z_{oi}$ can be calculated. Simulations as well as results in (6) suggested this conjecture.

7. CONCLUSIONS

Many parameters play a role in the two methods considered for approximating the blocking probabilities of overflow traffic components. Therefore, the conclusions here cannot be definitive, but are necessarily provisional. Under this restriction the following can be said.

The method of Manfield and Downs is preferable in the case of two traffic streams, neither of which is Poissonian. The method of Akimaru and Takahashi is preferable in the case of two streams one of which is Poissonian, and in the case of more than two streams.

Errors in the order of 5% in the approximations occur.

Two remarks can be made here. First, the extension in the method of Akimaru and Takahashi to the general case of two overflow traffics should perhaps be further studied.

Secondly, an approximation of the equivalent traffic of Wilkinson's ERT has been found which is easier to calculate than the exact value and is more accurate than Rapp's approximation.

8. ACKNOWLEDGEMENT

I am grateful to Ir. B. Sanders of the Dr. Neher Laboratorium of the Dutch PTT who provided me with some important calculation results.

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**APPENDIX**

**A1. THE PARAMETERS OF AN IPP**

An IPP has three parameters denoted in [6] and [7] by λ, ω and γ respectively, where

- λ = intensity during on-time of the Poisson process
- 1/γ = mean holding time
- 1/ω = mean off-time of the Poisson process

In formula (12) of [6], λ, ω and γ are expressed in A and the first three factorial moments M(j) (j=1,2,3) of the number of occupied overflow lines. These factorial moments can be written as

\[ M(0) = M \]
\[ M(1) = M(M+Z-1) \]
\[ M(2) = \frac{2AM(M+Z-1)^2}{A+Z(M+Z-1)} \]

Substituting this in formula (12) of [6], after some algebraic manipulation we get

\[ \lambda = \frac{(M+Z-1)[A(2Z-M-2)+MZ(M+Z-1)]}{2(M+Z-1)(M+2Z-2)-AM} \]  
\[ \omega = \frac{M}{\lambda (Z-1)} \]
\[ \gamma = \omega (\frac{\lambda}{M}) \]

When M, Z and A are known, λ can be calculated by (12) and hence also ω and γ.

**A2. B1/B2 ACCORDING TO MMD**

Equations (4.9) and (4.10) of [6] give the mean numbers O1 and O2 of calls of stream 1 and 2 respectively, overflowing from the N lines in a time interval wherein all N lines are occupied. Because

\[ B_1 = \frac{O_1}{M_1} \]
\[ B_2 = \frac{O_2}{M_2} \]  

it is necessary to evaluate O1 and O2. It follows from the above equations that

\[ O_1 = \frac{M_1}{1 - \lambda_1(N)} + \frac{M_2}{N} \]
\[ O_2 = \frac{M_2}{1 - \lambda_2(N)} + \frac{M_1}{N} \]

where \[ \lambda_i(\cdot) \] is the Laplace transform of the interarrival time distribution of the IPP concerned. Formula (8) of [7] gives

\[ \lambda_1(N) = \lambda_1 \frac{N}{N+M_1+\gamma_1(N)} \quad (i=1,2) \]

Using (13) and (14) we find that the right hand side of this equation is equal to

\[ \frac{M_1}{N} \left( \frac{\lambda_1-\lambda_1 M_1}{N+M_1+\lambda_1(N)} \right) \]

Substitution in (16) and use of (15) yields

\[ B_1 = \frac{M_1}{N} \left( \frac{1+(Z_1-1)\gamma_1(N)}{N+M_1+\lambda_1(N)} \right) \]
\[ B_2 = \frac{M_2}{N} \left( \frac{1+(Z_2-1)\gamma_2(N)}{N+M_2+\lambda_2(N)} \right) \]

**A3. B AND E_{L+NM}(A)/E_{L}(A) FOR N=1,2,3**

A and L are the equivalent Poisson traffic and the equivalent number of lines of ERT when the overflow is characterized by M and Z.
a) $N=1$

The recurrence relation for the Erlang loss formula

$$\frac{1}{E_{L+1}(A)} = 1 + \frac{L+1}{A} \cdot \frac{1}{E_L(A)}$$

leads to

$$\frac{E_L(A)}{E_{L+1}(A)} = \frac{AE_L(A)+L+1}{A} = \frac{M+L+1}{M+Z-1}$$  \hspace{1cm} (17)

Now

$$Z = 1 - M + \frac{A}{M+L+1-A}$$

so that

$$\frac{M+L+1}{A} = \frac{M+Z}{M+Z-1}$$

Substitution in (17) gives

$$\frac{E_L(A)}{E_{L+1}(A)} = \frac{M+Z}{M+Z-1}$$

According to (4), and taking the term with $j=0$ apart, for $N=1$

$$B^{-1} = 1 + \frac{1}{M+Z}$$

$$= \frac{M+Z}{M+Z-1}$$

Hence for $N=1$

$$B^{-1} = \frac{E_{L+1}(A)}{E_L(A)}$$

b) $N=2$

Two times application of the recurrence formula gives

$$\frac{E_L(A)}{E_{L+2}(A)} = \left(1 + \frac{Z}{A(M+Z-1)}\right)^2$$  \hspace{1cm} (18)

On the other hand (4) gives

$$B^{-1} = \frac{M+Z-1}{M+Z-1} + \frac{2(M+Z-1)(\lambda-M)}{M+Z-1}$$

and substitution of $\lambda$ from (12) yields after some algebra

$$B^{-1} = \left(\frac{M+Z}{M+Z-1}\right)^2 + \frac{Z}{A(M+Z-1)}$$

The complicated expression for $B^{-1}$ is not of much use here. Putting it equal to $E_L(A)/E_{L+3}(A)$ we get a cubic equation for $A$:

$$DA^3 + GA^2 + HA + P = 0$$  \hspace{1cm} (19)

with

$$D = 2M$$
$$G = M(M+1)(4Z-M-6)$$
$$H = 2Z^2(M+2-Z-6(Z-1)^2)$$
$$P = 4Z^2(M+1)^3 - (2-M)$$

Putting

$$A = aZ(M+Z-1)$$

(18) can be simplified to

$$da^3 + ga^2 + ha + p = 0$$  \hspace{1cm} (19)

with

$$d = 2MZ$$
$$g = M(4Z-M-6)$$
$$h = 2Z^2(M+2-Z-6(Z-1)^2)$$
$$p = M(2-M)$$

For $Z=1$ the three roots of this equation are $a_1=1$, $a_2=1$, $a_3=(M-2)/2$.

In this case $a=1$ or $A=M$ is clearly the desired solution, i.e. the smallest root in the interval $[1, \infty)$. It is difficult to see for $Z>1$ whether (19) has one or three real roots. The left hand side is then negative for $a=1$, namely $-12(Z-1)^2$, and it is $+\infty$ for $a=\infty$. Therefore, we follow the same rule for $Z>1$, i.e. the smallest root in $[1, \infty)$ should be taken.

c) $N=3$

Proceeding in the same way, but with much more algebra, we find

$$E_L(A) = \left(\frac{M+Z}{M+Z-1}\right)^3 + \frac{Z(M+2-Z-6(Z-1)^2)}{A(M+Z-1)^2}$$

$$+ \frac{Z^2(2-M)}{A^2(M+Z-1)}$$

The solution of (18) has been compared with Rapp's approximation

$$A = MZ + 3Z(Z-1)$$  \hspace{1cm} (20)

and the exact value of the equivalent Poisson traffic for given $M$ and $Z$. The comparison has been made for the cases of Table 3 and the results are given in Table 4.

The table shows that (18) gives a uniformly - though sometimes only slightly - better approximation than (20).
<table>
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Table 4. Comparison of different methods of calculation of the equivalent random traffic