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OPTIMAL FEED-BACK CONTROL OF A STOCHASTIC SERVICE SYSTEM: A MARTINGALE APPROACH

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During an overload, the queue is long, and impatient customers drop out. That is a waste of customer time, of service-system resources, and of service-provider goodwill. Before the queue will be too long, it may be better to throttle the input. Then the queue will decrease; but on the other hand, the risk of running empty of customers will increase. That is the dilemma. How to resolve it is outlined in this condensed paper.

1 THE SYSTEM DYNAMICS

The system, assumed to be Markovian, has one server, and the buffer has a limit, N-1. Let Q_t be the number of customers in the system at time t, $Q_t \varepsilon [0,N]$; let A_t be the number of arrivals during $(0,t]$; let I_t be the number of impatients dropping out during $(0,t]$; and let D_t be the number of served departures during $(0,t]$. Then

$$Q_t = Q_0 + A_t - I_t - D_t \qquad t \geq 0 \qquad (1a)$$

or in the form of a stochastic differential equation

$$dQ_t = dA_t - dI_t - dD_t \qquad (1b)$$

Let λ be the arrival intensity, let γ be a customer's drop-out intensity, and let μ be the server's capacity. Also, let $1(\cdot)$ denote the indicator function, which is unity when the argument is true, otherwise zero. Then the driving processes in (1b) have the following semi-martingale representations:

$$dA_t = \lambda u_t 1(Q_{t-} < N)\, dt + dM_t^A \qquad (2a)$$

$$dI_t = \gamma Q_{t-}\, dt + dM_t^I \qquad (2b)$$

$$dD_t = \mu\, 1(Q_{t-} > 0)\, dt + dM_t^D \qquad (2c)$$

where u_t is the feed-back control variable, the "throttle valve", $u_t \varepsilon [0,1]$. Let u_t depend on the present queue length only, $u_t = u(Q_{t-})$, then the Markov property is preserved. M_t^A, M_t^I, and M_t^D are martingales relative to the σ-field generated by Q_s, $0 \leq s \leq t$.

2 THE SYSTEM ECONOMY

Assume a dropped-out customer will cost c_1 units, while a served customer will result in a gain of c_2 units. Then, at the final time, T, the expected total cost for control policy u is $C(u)$:

$$C(u) = E_u(c_1 I_T - c_2 D_T) =$$

$$= E_u \int_0^T (c_1 \gamma Q_{s-} - c_2 \mu 1(Q_{s-} > 0)) ds \qquad (3)$$

The problem is to find a u that minimizes the cost function, $C(\cdot)$. The optimal control policy is denoted u^*.

3 THE OPTIMAL CONTROL POLICY

Guided by Brémaud, ref [1], p 202pp, we now apply a dynamic-programming procedure. Let $V(t,Q_t)$ denote the optimal cost-to-go from time t to the final time point, T, conditioned on the state of the Q-process at time t. Then

$$V(t,Q_t) = E_{u^*}\left\{ \int_t^T (c_1 \gamma Q_{s-} + c_2 \mu 1(Q_{s-} = 0)) ds + V(T,Q_T) \,|\, Q_t \right\} \qquad (4)$$

"Itô-calculus" on the V-process results in

$$V(t,Q_t) = V(0,Q_0) + \int_0^t \left[\frac{\delta V(s,Q_s)}{\delta s} + (V(s,Q_{s-}+1) - V(s,Q_{s-}))\lambda 1(Q_{s-} < N)\, u_s + (V(s,Q_{s-}-1) - V(s,Q_{s-}))(\gamma Q_{s-} + \mu 1(Q_{s-} > 0)) \right] ds + martingale \qquad (5)$$

The Bellman-Hamilton-Jacobi equations for the problem are the following $N+1$ equations, where $q = 0,1,2,\ldots,N$:

$$\frac{\delta V(t,q)}{\delta t} + inf_{0 \leq u \leq 1}\left\{ (V(t,q+1) - V(t,q))\lambda 1(q<N) u_q \right\} + (V(t,q-1) - V(t,q))(\gamma q + \mu 1(q>0)) + c_1 \gamma q + c_2 \mu 1(q=0) = 0 \qquad (6)$$

It is sufficient for optimality, if there exists V and u such that (6) is satisfied. Using a conjecture by Högfeldt, ref [2], we nominate the following candidate:

$$V(t,q) = F(q) + (T-t) const \qquad (7)$$

We substitute (7) into (6), and eureka...., we find an optimal solution:

$$u_q = 1(F(q) > F(q+1)) \qquad (8)$$

The optimal control policy is stationary and has infinite horizon, as it depends neither on t nor on T. Also, it is of type 'bang-bang', the valve turns out to be an on/off switch.

REFERENCES

[1] P. Brémaud, 'Point Processes and Queues: Martingale Dynamics', Springer-Verlag: NewYork, 1981.

[2] P. Högfeldt, personal communication, 1984.