ITC 11, Minoru Akiyama (Editor)
Elsevier Science Publishers B. V. (North-Holland)

TIME CONGESTION AND THE CONTINUED FRACTION EXPANSION OF THE ERLANG FORMULA

Per LINDBERG

Swedish Telecommunications Administration
Farsta, Sweden

This contribution gives formulas for the computation of the time congestion B_T of the secondary group in the simple overflow system shown in fig.1. By use of them B_T is computed with an arbitrary precision for all positive real j and all real i. This has a large practical application due to Wilkinson's Equivalent Random Theory.

$$A \to o \text{---} i \text{---} o \to m, v \to o \text{---} j \text{---} o \to \tilde{m}$$

Fig.1

Define λ_{ij} by $\lambda_{ij} = \tilde{m}/B_T$, where $\tilde{m} = A\ E_{i+j}(A)$.

Thus $\lambda_{0j} = A$ and $\lambda_{i0} = A\ E_i(A)$.

One recurrence formula (see [2]) for λ_{ij} is

$$\lambda_{ij} = A - i - j - 1 + A(j+1)/\lambda_{i\ j+1} \qquad (1)$$

From the state equations it is simple to prove the following generalization of the Erlang recurrence.

$$\lambda_{ij} = A/(1+i/(j+\lambda_{i-1\ j})) \qquad (2)$$

Starting with the initial value for i = 0, we can by (2) compute λ_{ij} for all positive integer i. That instead start from j = 0 and use (1) backwards is not as suitable since for $i+j+1 < A$ each step would increase the relative error. If (1) and (2) are combined we obtain the recursive step of the wellknown continued fraction expansion of Erlang's formula.

$$\lambda_{ij} = A - i/(1+(j+1)/\lambda_{i-1\ j+1}) \qquad (3)$$

When $i < A$ each step of (3) will decrease the relative error. Therefore, by using an approximate initial value and a number of steps large enough, we can compute λ_{ij} with an arbitrary precision for all $i \leqslant A$ and all j. By afterwards applying (2), the result is extended to all i. As initial value we use Frederick's approximation [2] with a slight modification.

$$\lambda_{i-K\ j+K} \approx \tfrac{1}{2}\left(A-i-j-\tfrac{1}{2} + \sqrt{(A-i-j-\tfrac{1}{2})^2 + 4A(j+K+\tfrac{1}{2})}\right)$$

In order to have a simple algorithm we must know in advance the required number of steps K. The following formulas for this have been proved analytically to give a relative error less than exp(-2r), i.e. 0.87r significant digits [3].

1. for $r \leqslant i \leqslant A$

$$K = ((A-i)^{1.5} + 1.5r\sqrt{A})^{2/3} - A + i \qquad (4)$$

2. for $0 \leqslant i \leqslant r$

$$K = r + (r-i)^2/(4A) \qquad (5)$$

3. for $i \leqslant 0$

$$K = r\sqrt{(A-i)/A} + r^2/(4A) \qquad (6)$$

With these formulas the continued fraction provides an efficient way of calculating the time congestion except when A is close to 0. For $A < 0.1$ the continued fraction gives only a limited precision in a reasonable computing time.

For the calculation of the Erlang formula. i.e. the special case j = 0, the formulas can be sharpened. Thus (4) can be substituted by (4') and (6) by (6'). These are based on the previous formulas and tested numerically.

$$K = r/2 + \frac{r^2 A}{(A-i)^2 + r\sqrt[3]{4rA^2/9}} \qquad (4')$$

$$K = r + \max(0, r+i/3)^2/(4A) \qquad (6')$$

A heuristic formula for the number of steps for a fixed precision was published by Farmer and Kaufman [1]. One for negative i was given by J Oppelstrup. An important feature of (4') is that it is uniformly bounded for $i \leqslant A - c\sqrt{A}$ if c is a positive constant. In the complementary area the Erlang formula can for a large A be calculated with a fixed precision by the normal distribution approximation recommended in [1]. Combining these two methods the Erlang formula is computed with at least six significant digits within a bounded CPU-time for arbitrarily large A and i.

REFERENCES

[1] R.F. Farmer, I. Kaufman, "On the numerical evaluation of some basic traffic formulæ," Networks 8:2 (1978), pp 153-186.

[2] A.A. Fredericks, "Approximating parcel blocking via state dependent birth rates," Proc. ITC 10, Montreal 1983.

[3] P. Lindberg, "Simple computation methods for Erlang's formula and Wilkinson's ERT," Report Nst85 008, Televerket Sweden 1985.