This contribution gives formulas for the computation of the time congestion $B_T$ of the secondary group in the simple overflow system shown in Fig. 1. By use of these $B_T$ is computed with an arbitrary precision for all positive real $j$ and all real $i$. This has a large practical application due to Wilkinson’s Equivalent Random Theory.

Define $A_{ij}$ by $A_{ij} = \frac{m}{B_T}$, where $m = A E_i+j(A)$.

Thus $A_{ij} = A$ and $A_{i0} = A E_i(A)$.

One recurrence formula (see [2]) for $A_{ij}$ is

$$A_{ij} = A - i - j - 1 + A(j+1)/A_{i+1,j+1} \quad (1)$$

From the state equations it is simple to prove the following generalization of the Erlang recurrence.

$$A_{ij} = A/(1+i/(j\lambda_{i-1,j})) \quad (2)$$

Starting with the initial value for $i = 0$, we can by (2) compute $A_{ij}$ for all positive integer $i$. That instead start from $j = 0$ and use (1) backwards is not as suitable since for $i+j+1 < A$ each step would increase the relative error. If (1) and (2) are combined we obtain the recursive step of the wellknown continued fraction expansion of Erlang’s formula.

$$A_{ij} = A - i/(1+(j+1)/\lambda_{i-1,j+1}) \quad (3)$$

When $i < A$ each step of (3) will decrease the relative error. Therefore, by using an approximate initial value and a number of steps large enough, we can compute $A_{ij}$ with an arbitrary precision for all $i < A$ and all $j$. By afterwards applying (2), the result is extended to all $i$. As initial value we use Frederick’s approximation [2] with a slight modification.

$$\lambda_{i-K,j+K} = \frac{1}{2} \left( A - i - j + \sqrt{(A - i - j)^2 + 4A(j+K+i)} \right) \quad (4)$$

In order to have a simple algorithm we must know in advance the required number of steps $K$. The following formulas for this have been proved analytically to give a relative error less than $e^{-2\pi}$, i.e. 0.077 significant digits [3].

1. For $r < i < A$

$$K = \frac{(A-i)^{1.5} + 1.5r \sqrt{A}}{2} \cdot A + i \quad (4)$$

2. For $0 < i < r$

$$K = r + (r-i)^2/(4A) \quad (5)$$

3. For $i < 0$

$$K = r \sqrt{(A-i)^2 + r^2/(4A)} \quad (6)$$

With these formulas the continued fraction provides an efficient way of calculating the time congestion except when $A$ is close to 0. For $A < 0.1$ the continued fraction gives only a limited precision in a reasonable computing time.

For the calculation of the Erlang formula, i.e. the special case $j = 0$, the formulas can be sharpened. Thus (4) can be substituted by (4′) and (6) by (6′). These are based on the previous formulas and tested numerically.

$$K = r/2 + \frac{r^2}{(A-i)^2 + r^2/(4A)} \quad (4')$$

A heuristic formula for the number of steps for a fixed precision was published by Farmer and Kaufman [1]. One for negative $i$ was given by J. Oppelstrup. An important feature of (4′) is that it is uniformly bounded for $i < A - c/A$ if $c$ is a positive constant. In the complementary area the Erlang formula can for a large $A$ be calculated with a fixed precision by the normal distribution approximation recommended in [1]. Combining these two methods the Erlang formula is computed with at least six significant digits within a bounded CPU-time for arbitrarily large $A$ and $i$.

REFERENCES

