Reiman [8] showed the precise boundary conditions for a tandem queue which the reflected Brownian queueing networks with the restricted to the nonnegative orthant. Harrison and process [4] that the approximating process must be a one-dimensional elementary return process. The forward equation for the steady-state density functions is derived. Applying this process, an approach for modeling some large queueing systems is shown. Then generating functions for two-station queueing networks and preemptive-resume priority queues are derived by solving a functional equation in two variables. Numerical examples are shown for the preemptive-resume queueing systems. It is also shown that an empty queue probability for these models can be exactly given as well as by one-dimensional case.

1. INTRODUCTION

We consider an n-dimensional diffusion process \( Y_t \) that arises in conjunction with large queueing systems such as networks of queues, preemptive-resume queueing systems, etc. Its state space consists of the interior and boundaries of the parallelepiped of \( \mathbb{R}^n \) (i.e., \( 0 \leq x_i \leq M_i, i=1,2,...,n \)). On the interior of this state space, \( Y_t \) behaves like an ordinary n-dimensional diffusion process (Brownian motion with drift). Whenever \( Y_t \) reaches one of the (n-1)-dimensional hyperplanes (for example, \( x_i = 0 \)) it remains there and behaves as an ordinary (n-1)-dimensional diffusion process until an exponentially distributed finite time lapses or until one of the (n-2)-dimensional hyperplanes is reached. If the exponentially distributed finite sojourn time lapses before the process hitting the (n-2)-dimensional hyperplane, \( Y_t \) jumps instantaneously to a point on the interior of the state space with \( x_i = 1 \). Otherwise an (n-2)-dimensional hyperplane is reached and the process stays there for an exponentially distributed finite time to act as (n-2)-dimensional Brownian motion. In both cases the process then starts from scratch. The behavior of \( Y_t \) on the lower dimensional hyperplane is defined in the same way. The process thus defined is a natural extension of one-dimensional elementary return process [4].

Several works have been devoted to the diffusion approximation for large queueing systems (mainly, for open queueing networks). Kobayashi [10] wrote out the forward equation for open queueing networks with the rough boundary condition that the approximating process must be restricted to the nonnegative orthant. Harrison and Reiman [8] showed the precise boundary conditions for a tandem queue which the reflected Brownian motion should obey. It was also shown that their reflected Brownian motion satisfies the heavy traffic limit theorem [1].

Although the heavy traffic limit theorem was proved for the reflected Brownian motion, it has the disadvantage of failing to approximate the behavior of the light traffic queues. In fact it is known that a serious deviation from exact values arises in a middle or a light traffic condition [10]. With this motivation refinement techniques of the diffusion approximation of queues have been proposed by some researchers [6], [10]. Gelenbe [6] proposed a refinement technique by properly modeling the effect of empty queue probability applying one-dimensional elementary return process defined by Feller. In his approach, the behavior of the light traffic queues can be approximated in a smaller error than by using the reflected diffusion approximation. Our primary goal here is to propose a refinement technique of the multi-dimensional diffusion approximation by extending his work to a multi-dimensional case and to derive the stationary forward equations of the multi-dimensional elementary return process as a model of more general large queueing systems than queueing networks.

In Section 2 and 3 we define the multi-dimensional elementary return process and derive the corresponding stationary forward equations. Using the result derived in Section 3, we then derive in Section 4 the partial differential equations for queueing network with two stations. It is also shown that the empty queue probability for this model can be exactly modeled as well as one-dimensional case of Gelenbe [6]. In Section 5, we discuss a generating function of the two-station queueing networks by analyzing the Laplace transform of the partial differential equations derived in Section 4. In Section 6 the modeling technique is applied to the preemptive-resume priority queues with feedback. In Section 7 calculation methods for mean queue size of preemptive-resume queueing systems are given and some numerical examples are shown to compare with the exact values. A general notation that the infinitesimal volume elements of \( \mathbb{R}^n \) is denoted \( dx = dx_1 dx_2 ... dx_n \) shall be used through the paper.

2. DEFINITION OF THE PROCESS

We shall define a multi-dimensional elementary return process \( Y(X) \) or \( Y(t), t \geq 0 \). General case shall be discussed where the state space \( S^n \) of the process is the parallelepiped of \( \mathbb{R}^n \)

\[ 0 \leq x_i \leq M_i, \quad Y_i > 0 (i=1,2,...,n) \]

Note that \( M_i \) may be infinite, \( S^n \) consists of its interior and the lower dimensional cells (hyper-
planes). The number of $0$-cells (vertexes) of $\mathbb{S}^n$ is $2^n$. Generally the number of $(n-k)$-cells of $\mathbb{S}^n$ is $2^k$, $k=1,2,\ldots,n$ and sum to $2^n$. We number these cells through so that the $(n-k)$-cells are denoted by $C_i$ ($\text{L}^k \text{I} i = 1, 2^{n-k}$ for $k=0$ and by $C_0$ for $k=0$). In particular, that $C_0$ represents the interior of $\mathbb{S}^n$. Suppose $C_i$ is an $m$-cell, ($0 \leq m \leq n$), then we write $C_i = (y) C_j$ if $C_i$ is (not an) $(m-1)$-cell and (or not) contained in the boundary of $C_j$. Let $\pi^k$ be the natural projection map from $\mathbb{R}^n$ to $\mathbb{R}^{n-k}$ which is the sub-space of $\mathbb{R}^n$ parallel to the $(n-k)$-cell $C_i$. The index of the coordinate axis on $\pi^k(\mathbb{R}^n)$ which is orthogonal to $\pi^k(\mathbb{R}^{n-k})$ is denoted $j^i$ if $C_j \subset C_i$.

On the interior of $\mathbb{S}^n$, $Y_t$ behaves as an $n$-dimensional Brownian motion which has the (constant coefficient) forward operator

$$L^* = \sum_{i=1}^n \sum_{j=1}^k \frac{\partial^2}{\partial x_i \partial x_j} - \sum_{i=1}^n \sum_{j=1}^k \frac{\partial}{\partial x_i} \left( \frac{\partial}{\partial x_j} \right)$$

where the covariance matrix $(\pi^k)^*$ is assumed to be non-negative definite. Whenever $Y_t$ reaches an $(n-1)$-cell $C_{ij}$ at $t$, it stays there for a finite sojourn time $T_{ij}$ and behaves as an $(n-1)$-dimensional Brownian motion with the (constant coefficient) forward operator

$$L_{ij}^* = \sum_{i=1}^n \sum_{j=1}^k \frac{\partial^2}{\partial x_i \partial x_j} - \sum_{i=1}^n \sum_{j=1}^k \frac{\partial}{\partial x_i} \left( \frac{\partial}{\partial x_j} \right)$$

where the covariance matrix $(\pi^k)^*$ is assumed to be non-negative definite. If $Y_t$ reaches on an $(n-2)$-cell $C_{ij} < C_{ik}$ at $t$, it stays there for an infinite sojourn time $T_{ij}$ and behaves as an $(n-2)$-dimensional Brownian motion which has the (constant coefficient) forward operator

$$L_{ij}^* = \sum_{i=1}^n \sum_{j=1}^k \frac{\partial^2}{\partial x_i \partial x_j} - \sum_{i=1}^n \sum_{j=1}^k \frac{\partial}{\partial x_i} \left( \frac{\partial}{\partial x_j} \right)$$

with the non-negative definite covariance matrix $(\pi^k)^*$. The behavior of $Y_t$ and the forward operator on the lower dimensional cells are defined in the same way except for the $0$-cells (vertexes). If one of the $0$-cells is reached then $Y_t$ stays there for an exponentially distributed finite sojourn time. The finite sojourn time $T_{ij}$ on $C_{ij} (0 \leq k \leq 2^{n-1})$ is exponentially distributed if any boundary cell $C_i$ of $C_{ij}$ (i.e., $C_i \subset C_{ij}$) is not reached. As soon as the exponential sojourn time lapses before the process hitting the boundary, a jump in the direction of $x_{k^*}$-axis occurs to the interior of $C_{ij}$ from $C_{ij}$ ($C_{ij} > C_i$) according to some probability density function over $0 < x_{k^*} < \pi_k m^*$. Let $\lambda_k$ be the positive constant which represents the rate at which jumps occur to a point on the interior of $C_i$ from $C_{ij}$ ($C_{ij} > C_i$). Then

\[ \text{Prob} \{ T_{ij} > t \} \text{ if any } C_i < C_{ij} \text{ is not reached} = \exp( - \lambda_k t \text{ for } \lambda_k \text{ in } \mathbb{R}^m \]  

where the summation in the exponential function is taken over all $m$ such that $C_i > C_k$. For our purpose, let the density function for the jump be the Dirac delta function $\delta(x_{k^*} - M(k,m))$ in which $M(k,m)$ is defined as follows:\n
\[ M(k,m) = \begin{cases} 0 & \text{if } x_{k^*} = 0 \text{ for } x \in C_k \\ 1 & \text{if } x_{k^*} = M_k \text{ for } x \in C_k \end{cases} \]

After the jump, the process starts from scratch. The process $Y_t$ thus defined is the Markov process because there are no point with memory. The notation (1) shall be used hereafter.

\section{3. STATIONARY FORWARD EQUATION}

Let the alternative representation of the process be

\[ Y(t) = (x(t), k(t)) \]

where $k(t)$ represents the index of the cell on which the process sojourns and $x(t) = \pi^k(Y(t))$. We assume the process has the transition probability density defined by

\[ p_{xy} = \text{Prob} \{ (k(t), x(t)) = (j, y(t)) \text{ for } t > t_0 \} \]

where $y(t)$ represents the small volume element on $C_j$. We decide $p_{xy}$ represents the transition probability if $C_j$ is a $0$-cell.

Let $x^k$ denote $\pi^k x$. Let $\psi^k = 0, 1, \ldots, 2^n - 1$ be the set of continuous functions $f_0^k : \pi^k(\mathbb{R}^n) \rightarrow \mathbb{R}$ that are twice continuously differentiable in $x^k$ except at $x^k$ whose component $x_k$ of $\mathbb{R}^n$ if contained in 1 or $M_k$ for some 1. Let $f^k(x^k) \in \psi^k$ be the density function on $C_k$ that satisfies

\[ \int_{x_k = 0}^{x_k = M_k} f^k(x^k) \, dx^k = 1 \]

For the density function $f^k$, we define the transition operator by

\[ (T^* f)^k_j(z) = \sum_{k=0}^{2^n-1} f^k_{xy} \int_{x_k = 0}^{x_k = M_k} \psi^k \, dx^k \]

where $z \in \psi^k C_j$. Then the forward operator of the process is defined by

\[ \Lambda^* = \lim_{t \to 0^+} \frac{T^* f}{t} = \text{I} \]

where I represents the unit operator on $\psi^k \times \psi^k \times \cdots \times \psi^k$. The stationary forward equation of the process can be represented in the form

\[ \langle \Lambda^* f \rangle = 0, \quad f = 0, 1, \ldots, 2^n - 1 \]

It will be convenient to define the differential operators

\[ j^i [f_x(j)] = - \frac{1}{2} \sum_{k} \int_{x_k = 0}^{x_k = M_k} \frac{\partial f_{xy}}{\partial y_j} \, dx^k + \int_{x_k = 0}^{x_k = M_k} f_{xy} \, dx^k \]

and

\[ \Lambda^* \Lambda^* \]

\[ j^i [f_x(j)] = \lim_{j \to x} j^i [f_x(j)] \]

where $N(k,i) = \{ 1/2 \} \{ 1 - \text{sgn}(k,i) \} M_{k+1}, i = 1, \ldots, 2^n - 1$, and $\phi^k \text{ represent indexes of the components of } x$ and the summation is taken over all such indexes. The following theorem gives the concrete form of the equation (3). We shall maintain all of the notation established earlier.

\textbf{Theorem.} Assume that $Y_t$ has a stationary density function $f^k(\pi^k x)$, $k=0, 1, \ldots, 2^n - 1$. Then it is the solution of the equations

\[ -\lambda^* M_0 f_0(x) = \sum_{i=1}^{2^n} \lambda^* M_k f_{xy}(x) \delta(x_{k^*} - M_{i+1}) \]

\[ -\lambda^* M_0 f_0(x) = \sum_{i=1}^{2^n} \lambda^* M_k f_{xy}(x) \delta(x_{k^*} - M_{i+1}) \]

\[ -\lambda^* M_k f_{xy}(x) = \sum_{j=1}^{2^n} \lambda^* M_{j+1} f_{xy}(x) \delta(x_{k^*} - M_{j+1}) \]

\[ 0 = \sum_{m=1}^{2^n} \text{sgn}(k,m) \int_{x_k = 0}^{x_k = M_k} f_{xy}(x^k) - \lambda^* M_k f_{xy}(x^k) \]

\[ 0 = \sum_{m=1}^{2^n} \text{sgn}(k,m) \int_{x_k = 0}^{x_k = M_k} f_{xy}(x^k) - \lambda^* M_k f_{xy}(x^k) \]

\[ 0 = \sum_{m=1}^{2^n} \text{sgn}(k,m) \int_{x_k = 0}^{x_k = M_k} f_{xy}(x^k) - \lambda^* M_k f_{xy}(x^k) \]

\[ 0 = \sum_{m=1}^{2^n} \text{sgn}(k,m) \int_{x_k = 0}^{x_k = M_k} f_{xy}(x^k) - \lambda^* M_k f_{xy}(x^k) \]
4. TWO-STATION QUEUEING NETWORKS

4.1. Stationary forward equation

In this section we present the forward equation of the elementary return process which approximates two-station queueing networks. Let the mean rate and the variance of exogenous inter-arrivals of jobs to the queue $Q_i$ be denoted by $\lambda_i$ and $\sigma_i^2$, $i=1,2$, respectively. Service times at each station are i.i.d. with mean $1/\mu_i$ and variance $\mu_i$. Having completed services at each station, jobs proceed to the other station for services with probability $Y_i$, with probability $\beta_i$ jobs rejoin the same station, and with probability $1-Y_i-\beta_i$ jobs leave the system.

The process behaves as an ordinary two-dimensional diffusion process approximating the busy period queue sizes on the interior of $R^2$. Whenever it reaches $x_2=0$, i.e., the origin, it jumps instantaneously to a point with the same $x_1$-coordinate and with $x_3-1=1$, i.e., $1$-i.d. Otherwise the origin would be reached and the process stays there for an exponentially distributed finite time, after which jumps occur to a point $(1,0)$ or $(0,1)$ with the rate $\lambda_1$ or $\lambda_2$, respectively. In both cases the process then starts from scratch.

The exponential sojourn times of the process on the $x_2$-axis, $i=1,2$, or at the origin approximates the periods of one empty queue or of both queues empty by supposing the arrival processes to be the Poissonian with the rate $\lambda_1, \lambda_2$ when either queue is empty.

The stationary forward equations of the two-dimensional elementary return process are given in section 2. These are the natural extension of one-dimensional elementary (or instantaneous) return process which is proposed in [4] and applied to approximate computer system models in [6]. The partial differential equations for the queue size equilibrium density read as follows:

$$\lim_{t \to 0^+} E[Q_i(t^+\Delta t) - Q_i(t) | Q_j=0, Q_k>0] = C_i$$

$$\lim_{t \to 0^+} \text{var}[Q_1(t^+\Delta t) - Q_1(t) | Q_j>0, Q_k>0] = A_i$$

where $f_0$: density function on the interior of $R^2$, $f_1$: density function on the interior of $R^2$, $f_2$: probability mass at the origin, $f_3$: probability density on the boundary of $R^2$.

$$f_0(x_1,0)=f_0(0,x_2)=0$$

$$f_1(x_1,0)=f_1(0,x_2)=0$$

with boundary conditions

$$\lim_{x \to 0} f_0(x^2) = \lim_{x \to 0} f_2(x^2) = 0$$

$H_j$: is the component of $x^k$, $0 < k \leq 3^n-1$.

Remark. If all $L_k$ are finite and there are no absorbing states, $Y_i$ is positive recurrent and has a stationary density.


4.2. Functional equation

In this section we show the stationary functional equation for the generating function, and

$$\frac{\partial f_0}{\partial x_1} + B \frac{\partial f_0}{\partial x_2} + \frac{1}{2} A_1 \frac{\partial^2 f_0}{\partial x_1^2} + C_1 \frac{\partial f_0}{\partial x_1} - C_2 \frac{\partial f_0}{\partial x_2} = 0$$

$$\frac{\partial^2 f_1}{\partial x_1^2} + E \frac{\partial f_1}{\partial x_1} + \frac{1}{2} B \frac{\partial^2 f_1}{\partial x_2^2} + A_1 \frac{\partial f_1}{\partial x_1} = 0$$

$$\frac{\partial^2 f_2}{\partial x_1^2} + E \frac{\partial f_2}{\partial x_1} + \frac{1}{2} A_2 \frac{\partial^2 f_2}{\partial x_2^2} + C_2 \frac{\partial f_2}{\partial x_1} = 0$$

$$\frac{\partial f_3}{\partial x_1} - \frac{1}{2} A_3 \frac{\partial^2 f_3}{\partial x_1^2} + \frac{1}{2} B \frac{\partial^2 f_3}{\partial x_2^2} + f_0(0,x_2)=0$$

$$f_3(x_1,0)=0$$

and the parameters $C_1, A_1, B, A_2$ are determined directly through the central-limit-theorem type argument [8]. Similarly we have

$$\lim_{t \to 0^+} \text{cov}[Q_1(t^+\Delta t) - Q_1(t) | Q_j=0, i \neq j, f_i=0] = E_i$$

$$\lim_{t \to 0^+} \text{var}[Q_1(t^+\Delta t) - Q_1(t) | Q_j=0, i \neq j, f_i=0] = D_i$$

The parameters $C_1, A_1, B, A_2$ can be determined directly through the central-limit-theorem type argument under the condition that $Q_i=0$, if $f_j=1$ and $j=1,2$. We assume therefore that the infinitesimal covariance matrix defined by

$$\begin{pmatrix}
A_1 & B \\
B & A_2
\end{pmatrix}$$

is positive definite.
discuss the traffic intensity (or the utilization factor). Let define the Laplace transformations

\[ L_0(s_1, s_2) = \int_{0}^{\infty} \exp(-s_1 x_1 - s_2 x_2) f_0(x_1, x_2) \, dx_1 \, dx_2 \]

\[ L_i(s_i) = \int_{0}^{\infty} f_i(x_i) \exp(-s_i x_i) \, dx_i \text{, } (i=1,2) \]

and

\[ F_i(s_{3-i}) = \int_{0}^{\infty} H_i(x_{3-i}) \exp(-s_{3-i} x_{3-i}) \, dx_{3-i} \text{, } (i=1,2) \]

Taking the two-dimensional Laplace transformation of (4), we have

\[ \frac{1}{2} A_1 s_1^2 + B_1 s_2 + \frac{1}{2} A_2 s_2^2 - C_1 s_1 - C_2 s_2 \]

\[-d_1 L_2(s_2) \exp(-s_2) - d_2 L_1(s_1) \exp(-s_1) = 0 \]

Further, Laplace transformation of (5) and use series expansion for \( s_i \), we have

\[ \sum_{k=0}^{\infty} \frac{(-s_i x_i)^k}{k!} f_k(x_i) \, dx_i \]

Notice that \( E[X_1] \) is given by

\[ E[X_1] = \int_{0}^{\infty} \int_{0}^{\infty} x_1 f_0(x_1, x_2) \, dx_2 \, dx_1 \]

\[ E[X_1] = \int_{0}^{\infty} \int_{0}^{\infty} f_1(x_1) \, dx_1 \]

Thus we have obtained the empty queue probability exactly. Notice that Gelenbe [6] introduced the elementary returning boundary to model the empty queue probability (or the utilization factor) in one-dimensional case. Thus we can see that our process is a natural extension of one-dimensional case of Gelenbe in this respect too.

5. GENERATING FUNCTION

5.1. Solution for the special case of \( B=0 \).

In this section, we seek the generating function of the diffusion model of two station queueing networks. The main mathematical technique used in this section is so-called Riemann-Hilbert boundary value problems. The technique developed can be applied to another diffusion model of queueing systems directly as will be done in the next section.

Rewriting (7) and (8), we have

\[ R(s_1, s_2) = G_1(s_1, s_2) L_1(s_1) + G_2(s_1, s_2) L_2(s_2) - g(s_1, s_2) f_3 \]

where

\[ L_0(s_1, s_2) = \frac{1}{2} A_1 s_1^2 + B_1 s_2 + \frac{1}{2} A_2 s_2^2 - C_1 s_1 - C_2 s_2 \]

\[ G_1(s_1, s_2) = d_2 \exp(-s_2) + \frac{1}{2} s_2 + E_1 s_1 - d_2 \]

\[ G_2(s_1, s_2) = d_1 \exp(-s_1) + \frac{1}{2} s_1 + E_2 s_2 - d_1 \]

Notice that from (14) to find \( L_0(s_1, s_2) \), it is enough to find \( L_1, L_2 \). We assume through this subsection that \( B=0 \).

The following observation will enable us to determine the generating functions \( L_1, L_2 \). The right-hand side of (14) vanishes whenever \( R(x, y) = 0 \), provided that \( L_0(x, y) \) is finite. For a complex variable \( w \) define

\[ R_1(x, y) = \frac{1}{2} A_1 x^2 - C_1 x + w = 0, \quad Re x \geq 0 \]

\[ R_2(y, w) = \frac{1}{2} A_2 y^2 - C_2 y - w = 0, \quad Re y \geq 0 \]

where \( R(x, y) = \frac{1}{2} A_1 x^2 - C_1 x + w = 0 \). As a consequence of the complex function \( \frac{1}{2} A_1 x^2 - C_1 x + w = 0 \) with multiplicity one for \( Re w \), we have

\[ R_1(x, y) \] with multiplicity one for \( Re w \), and similarly for \( Re w \). Thus we have obtained the generating function exactly. Notice that Gelenbe [6] introduced the elementary returning boundary to model the empty queue probability (or the utilization factor) in one-dimensional case. Thus we can see that our process is a natural extension of one-dimensional case of Gelenbe in this respect too.

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Rewriting (7) and (8), we have

\[ R(s_1, s_2) = G_1(s_1, s_2) L_1(s_1) + G_2(s_1, s_2) L_2(s_2) - g(s_1, s_2) f_3 \]

where

\[ L_0(s_1, s_2) = \frac{1}{2} A_1 s_1^2 + B_1 s_2 + \frac{1}{2} A_2 s_2^2 - C_1 s_1 - C_2 s_2 \]

\[ G_1(s_1, s_2) = d_2 \exp(-s_2) + \frac{1}{2} s_2 + E_1 s_1 - d_2 \]

\[ G_2(s_1, s_2) = d_1 \exp(-s_1) + \frac{1}{2} s_1 + E_2 s_2 - d_1 \]

Notice that from (14) to find \( L_0(s_1, s_2) \), it is enough to find \( L_1, L_2 \). We assume through this subsection that \( B=0 \).

The following observation will enable us to determine the generating functions \( L_1, L_2 \). The right-hand side of (14) vanishes whenever \( R(x, y) = 0 \), provided that \( L_0(x, y) \) is finite. For a complex variable \( w \) define

\[ R_1(x, y) = \frac{1}{2} A_1 x^2 - C_1 x + w = 0, \quad Re x \geq 0 \]

\[ R_2(y, w) = \frac{1}{2} A_2 y^2 - C_2 y - w = 0, \quad Re y \geq 0 \]

where \( R(x, y) = \frac{1}{2} A_1 x^2 - C_1 x + w = 0 \) with multiplicity one for \( Re w \), we have

\[ R_1(x, y) \] with multiplicity one for \( Re w \), and similarly for \( Re w \). Thus we have obtained the generating function exactly. Notice that Gelenbe [6] introduced the elementary returning boundary to model the empty queue probability (or the utilization factor) in one-dimensional case. Thus we can see that our process is a natural extension of one-dimensional case of Gelenbe in this respect too.
\[ f^+(t) = \lim_{z \to \infty} f(z), \quad z \to +t \]

\[ f^-(t) = \lim_{z \to \infty} f(z), \quad z \to -t \]

where \( G(z) \) and \( e(z) \) are functions defined on \( \Phi \), satisfy the Hölder condition \([1]\) on \( \Phi \), and do not vanish everywhere on \( \Phi \). Furthermore the following conditions are incurred:

i) \( f(z) \) is regular for \( z \in \Phi \) \( (\Phi^-) \) and is continuous for \( z \in \Phi \) \( (\Phi \cup \Phi^-) \).

ii) \( f(z) + c = z \pm \infty \), where \( c \) is a constant.

Notice that in our problem \( c = 0 \).

The form of solution of the Riemann boundary value problem depends on what is known as the "index" of the problem. This is denoted by and is defined by:

\[ \chi = \frac{1}{2\pi} \int \frac{d \log G(t)}{t} \]

\[ \{ \text{increment of the argument of } G(t) \text{ when } t \text{ traverses in the positive direction} \} \]

where \( i = \sqrt{-1} \).

To reduce our problem to the Riemann boundary value problem, rewrite (16) as:

\[ \begin{align*}
L_1(K_1(w)) &= G(w)L_2(K_2(w)) + e(w)f_3, \\
G(w) &= -G_2(K_1(w),K_2(w))/G_1(K_1(w),K_2(w)) \\
e(w) &= g(K_1(w),K_2(w))/G_1(K_1(w),K_2(w)).
\end{align*} \]

Furthermore set \( \phi = \{ w; Re w = 0 \} \), and set \( \Phi = \{ w; Re w < 0 \} \).

It is easily seen that \( G(.) \) and \( e(.) \) have finite derivatives everywhere on \( \Phi \).

Since

\[ \lim_{w \to +\infty} G(w) = (D_2 / A_1) (A_2 / A_2) \]

and

\[ \lim_{w \to -\infty} e(w) = 0, \]

\( G(w) \) and \( e(w) \) satisfy the Hölder condition on \( \Phi \) and there exists a solution for the boundary value problem. For the analysis of the problem we have to investigate the index. From the definition the index of \( G(w) \) is given by the increment of the argument of \( G_1(K_1(w),K_2(w)) \) subtracted by the increment of the argument of \( G_2(K_1(w),K_2(w)) \) and devided by \( 2\pi \) when \( w \) traverses \( \Phi \) from \(-\infty \) to \(+\infty \).

First consider the asymptotic solution for \( w \to +\infty \).

Let

\[ \gamma_{1}(w) = \frac{1}{2\pi} \int \frac{\log G(is)}{s} ds, \]

and where

\[ \gamma_{2}(it) = \frac{1}{2} \log \{ G(it) \} + \gamma_{1}(it). \]

Notice that \( \gamma_{2} \) can be determined from \( 10 \) and \( 11 \). Thus the generating function has been found:

\[ L_1(s_1) = f(\frac{-s_1^2}{2}A_1^2 + C_1s_1), \quad Re s_1 > 0, \]

\[ L_2(s_2) = f(\frac{s_2^2}{2}A_2^2 - C_2s_2), \quad Re s_2 > 0, \]

with \( f(.) \) given in \( 17 \). All the equilibrium moments of \((x_1,x_2)\) can now be computed through numerical computation.

5.2 Approximate solution for general case.

To seek the solution for general case, i.e., for \( B \neq 0 \), we have to determine the functions \( K_1(.) \) and \( K_2(.) \) on complex \( w \)-plane with the following property:

\[ \begin{align*}
(i) \quad & R(K_1(w),K_2(w)) = 0, \text{ for } w \in \{ w; Re w = 0 \} \\
(ii) \quad & K_1(w) \text{ is analytic for } Re w < 0 \text{ and continuous for } Re w > 0, \text{ and } K_2(w) \text{ is analytic for } w > 0 \text{ and continuous for } Re w > 0. \\
(iii) \quad & K_1(w) \text{ is multiplicity one for } Re w \leq 0 \text{ in } Re K_1(w) \geq 0, K_2(w) \text{ is multiplicity one for } Re w \geq 0 \text{ in } Re K_2(w) \leq 0, \text{ and } \lim_{w \to \infty} Re K_1(w) = + \infty, \text{ i.e., } i=1,2.
\end{align*} \]

Now consider a sequence \( w_i \), \( i=1,2 \), which uniformly converges to \( \Phi \).

Then we obtain for \( Re w \to +\infty \):

\[ \begin{align*}
\gamma_{2}(w) &= b_1, \quad \gamma_{2}(w) = w^\gamma, \quad (w \to +\infty), \\
\gamma_{2}(w) &= b_2, \quad \gamma_{2}(w) = w^\gamma, \quad (w \to -\infty).
\end{align*} \]

Since the infinitesimal covariance matrix is positive definite, we obtain from \( 20 \):

\[ \begin{align*}
a_1 &= \sqrt{(A_2 / A_1)} b_1, \\
0 < \gamma &= (1/ \pi) \cos^{-1} \left[ \sqrt{\frac{A_1}{A_2}} \right] \leq 1/2.
\end{align*} \]

Now let determine the solution for (i), (ii), (iii) of the form

\[ \begin{align*}
K_1(w) &= a_1(z_0 - w) \gamma + Z_1 w \to +\infty \quad a_1(z_0 - w) \gamma \\
K_2(w) &= b_1(z_1 + w) \gamma + Z_2 w \to +\infty \quad b_1(z_1 + w),
\end{align*} \]

assuming that \( K_1(0)=K_2(0)=0 \). To show the existence of the solution \( 21 \), we employ the well known "theorem for the coincidence of limiting function in analytic function theory: suppose a sequence \( f_n(z) \), \( n=0,1, \ldots \) of analytic function on a simply connected domain \( D \) be uniformly bounded on any bounded region in \( D \). If \( f_n(z) \) converges to \( f(z) \) at \( z \) of \( D \) with \( z \) converging to \( z \) of \( D \) as \( i \to +\infty \), then \( f_n(z) \) weakly uniformly converges to \( f(z) \) analytic in \( D \).

Now consider a sequence \( w_i \), \( i=1,2 \), which converges to zero with \( Re w_i = 0 \). Define function
sequences

\[ K_1^0(w) = a_1(z_0 - w) + \sum_{i=-n}^{0} a_i(z_0 - w)^i, \quad K_2^0(w) = b_1(z_1 + w) + \sum_{i=-n}^{0} b_i(z_1 + w)^i, \]

where \( z_0, z_1, a_i \) are arbitrary positive real numbers (and thus \( b_1, b_i \) is a positive real number), and where the complex numbers \( a_i, b_i \) are determined by solving a system of simultaneous quadratic equations:

\[ R(K_1(w), K_2(w)) = 0, \quad j = 1, 2, \ldots, 2n, \]

\[ K_1^0(0) = K_2^0(0) = 0. \]

Then \( K_1^0(w) \) is analytic for \( c < 0 \) and \( K_2^0(w) \) is analytic for \( c > 0 \) for arbitrary \( n \). Furthermore \( K_1^0(w) \) for any \( n \) is bounded on any bounded region in \( \mathbb{R} \times \mathbb{R} \) and \( K_2^0(w) \), the infinite series in \( (21) \) converges in \( \mathbb{R} \times \mathbb{R} \) for \( c > 0 \) for \( i=1,2 \), respectively. Thus \( K_1^0(w) \), \( i=1,2 \), is uniformly bounded on any bounded region in \( \mathbb{R} \times \mathbb{R} \) and vanishes at infinitely many points \( w', i=1,2 \).

For the sake of numerical evaluation, we may make an approximation of \((21)\) by using the truncated series (22), e.g., for \( n=10 \). The equation (23) for the coefficients is rewritten in the form (24) numerically computed by so-called Picard iteration: i.e.,

\[ K^1 = x^{-1}g(x^0), \quad x^0 = (a_0, b_0, \ldots, a_0, b_0, \ldots, a_0, b_0)^T, \]

assuming \( x \) is nonsingular. It is convenient to set

\[ a_1 = (2A_1)^{-1} A_1, \quad z_0 = z_1 = z_1^{(2)}, z_2 = z_1^{(2)}, \]

\[ \infty = (a_0, b_0, 0, 0, 0, \ldots, 0) \text{ with } a_0 = C_1 / A_1 \text{ and with } b_0 = C_2 / A_2. \]

Repeating the same discussion as subsection 5.1, we obtain

\[ L_1(s_1) = f(K_1^1(s_1)) \]

\[ L_2(s_2) = f(K_2^1(s_2)) \]

where \( f(.) \) is given by (17) and \( K_1, K_2 \) is replaced by (21).

6. APPROXIMATION OF GI_1 GI_2/G_1 G_2/1 PRIORIY QUEUES

Consider preemptive-resume queueing systems (which may have inter-queue feedback structure) with two types of customers (see Figure 1). Type 1 customers are given preemptive-resume priority over type 2 customers and both types of customers are served by a single server. Externally customers arrive at each queue according to a renewal process whose interarrival times have mean \( 1 / \lambda \) and variance \( \sigma^2 \). The generating function is given by

\[ P_i = P_{i1} + P_{i2}, \quad i = 1, 2. \]

The generating function is given by (17) and (21) with replacing the diffusion parameters by (27).

Figure 1. Preemptive-resume priority queues with self-feedback.

The generating function is given by (17) and (21) with replacing the diffusion parameters by (27).

Now consider the important special case with \( P_{i1} = 0 \) and \( \lambda_1 = \lambda_2 \). This model is an ordinary preemptive-resume priority queue. In this case we have two expressions: for \( w > 0 \) and the same analytical method as in section 5.1 is applicable with a little additional discussion. Notice that in this model \( C_2 > 0 \) and thus \( K_1(0) = 2C_2 / \lambda_1 > 0 \). Therefore (17) determines \( L_2(s_1) \) for \( \Re s_1 \geq 0 \) uniquely but \( L_1(s_1) \) only for \( \Re s_1 \geq K_1(0) \). To determine \( L_1(s_1) \), we need the other branch of \( K_1(w) \), i.e.,

\[ T_1(w) = [ C_1 - \sqrt{C_1^2 + 2A_1w} / A_1 \].

Since \( (T_1(w), K_1(w)) \) and \( (K_1(w), K_2(w)) \) for \( w \in [0, C_1^2 / (2A_1)] \) are both zeros of the kernel \( R(s_1, s_2) \), we have two expressions: for \( w \in (0, C_1^2 / (2A_1)] \)

\[ L_1(K_1(w)) = G(w)L_2(K_2(w)) + e(w)f_3 \]

and

\[ L_1(K_1(w)) = G(w)L_2(K_2(w)) + e(w)f_3 \]
L_1(T_1(w)) = G'(w)L_2(K_2(w)) + e'(w)f_2(w) \tag{29}
\]
where G(.) and e(.) are defined in section 5, G'(w) = -G_2(T_1(w),K_2(w))/G_1(T_1(w),K_2(w))
e'(w) = G(T_1(w),K_2(w))/G_1(T_1(w),K_2(w)) .

From (17) for Re w > 0, the righthand sides of (28), (29) are known, therefore the lefthand sides are given. Hence for s_1 \in [0, C_1 \infty, s_i(r_i), s_2 \in [0, C_2 \infty, s_i(r_i) .

Hence we have again the relation (9),(10),(11) and (12), (13) the mean queue length of both high and low priority queues can be obtained. From (17), (18)

\[b_0 = f(0)\]

\[b_1 = \int \Psi'(0) \exp \Gamma(0) \Psi(0) \Gamma'(0) \exp \Gamma(0)
\]

where ' represents the derivative w.r.t. w.

The method of singular integration [2],[7] is applied to evaluate the integrals appearing in (31). The 16-point Gaussian numerical integration method is applied dividing the integration interval [0,M, iM] with M being sufficiently large into nine sub-intervals. Since the function Log G(w) in these integrals has to be evaluated for many points and used over and over, tables of the function can be constructed. The numerical solution was rapid in computer time. The solution took over about 6 seconds of a middle class computer (MELCOM 800 III, 1.7 MIPS).

In order to evaluate the accuracy of our approximation method we have compared it numerically in Table 1 with the exact results of M/G/1 priority queue. It is remarkable that our approximation method gives exact value in the Markovian case.

| Table 1. Mean queue length of M/G/1 preemptive-resume priority queues. |
\[
S_1 = \infty \quad \rho_1 = \bar{o}\%
\]

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<thead>
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<th>Exact</th>
<th>Diff</th>
<th>Exact</th>
<th>Diff</th>
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