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AN ANALYSIS OF THE SENSITIVITY OF KRUITHOF BIPROPORTIONAL MATRIX SOLUTIONS TO TRAFFIC DATA PERTURBATIONS

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The well-known Kruithof biproportional model is frequently used by traffic network planners to derive forecasts of point-to-point traffic demands (flows) between exchanges in a closed network. The model assumes that the $m \times n$ forecast matrix $X \equiv [x_{ij}]$ is related to the given $m \times n$ non-negative (base) matrix $A \equiv [a_{ij}]$ according to the functional form

$$x_{ij} = P_i a_{ij} Q_j \quad . \qquad (1)$$

The elements of X are required to satisfy the constraints

$$\sum_i x_{ij} = C_j \ (j=1,\ldots,n) \qquad (2)$$

$$\sum_j x_{ij} = L_i \ (i=1,\ldots,m) \qquad (3)$$

$$x_{ij} \geq 0 \ (\forall\ i,j) \qquad (4)$$

$$a_{ij} = 0 \text{ implies } x_{ij} = 0 \ (\forall\ i,j). \qquad (5)$$

The strictly positive quantities L_i and C_j are, respectively, (known) estimates of futur total originating and terminating traffic demands for each exchange. Condition (5) requires that zero elements in the base matrix A remain zero in the forecast matrix X. The L_i and C_j are also assumed to satisfy $\sum_i L_i = \sum_j C_j$.

A forecast matrix X can also be derived by minimizing with respect to x_{ij} the strictly convex functional

$$S = \sum_{i,j} x_{ij} \log(x_{ij}/a_{ij}) \qquad (6)$$

subject to the constraints (2)-(4). The (unique) solution to this (convex) nonlinear program gives a "maximum entropy" solution [1]. Such a solution is said to give the "most probable" distribution X of point-to-point traffic demands for the given planning scenario, i.e., for given L_i, C_j and A.

A close relationship exists between the maximum entropy and Kruithof biproportional matrix adjustment problems: their optimal solution sets are identical and the iterative adjustment procedure [4]

$$P_i^{(k+1)} = L_i \left(\sum_j Q_j^{(k)} a_{ij}\right)^{-1} \qquad (7)$$

$$Q_j^{(k+1)} = C_j \left(\sum_i P_i^{(k+1)} a_{ij}\right)^{-1} \qquad (8)$$

($k \geq 0$ and $Q_j^{(0)} = 1$) used to obtain P_i and Q_j in (1) can be easily derived from the maximum entropy model by affixing the constraints (2)-(3) to the objective function (6) by means of Lagrange multipliers. The necessary conditions for a stationary point of the resulting Lagrangian function lead, after a simple change of variable, to (7)-(8).

This paper exploits the close relationship between the Kruithof and maximum entropy models to quantify the effect on X of changes in the problem data L_i, C_j and A. This quantitative information is obtained by regarding the given data as "problem parameters" and carrying out a sensitivity analysis on the optimal solution set of the strictly convex separable nonlinear program which results when (6) is minimized subject to (2)-(4). By "sensitivity analysis" is meant an analysis of the effect on the optimal solution and optimal objective function value of small perturbations in the problem parameters [3]. Since we are interested in studying the effects on the elements of X of perturbations in some or all of the elements of L_i, C_j and A, our results pertain only to the prediction of changes in the optimal solution, given corresponding changes in the problem parameters.

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