TECHNOLOGY REPLACEMENT MODELS BASED ON POPULATION DYNAMICS

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This paper presents a model of multilevel technology replacement inspired by the classical Lotka-Volterra predator-prey equations. An implementation of the model is described and an application is made to an analysis of a subset of the North American switching market.

1. INTRODUCTION

Techniques for modelling the interactions among competing technologies have been studied for many years. In the early 1970s, a model for technological substitution was developed by R. H. Pry and J. C. Fisher [3]. It can be used to predict the future market shares of two competing technologies, once the newer of the two has captured a few percent of the market. In telecommunications, as in many other diverse industries such as energy and transportation, however, many substitutions are occurring simultaneously [6]. The numerous interrelationships among the competing technologies make this situation particularly difficult to model.

A selected subset of the North American switching market, shown in Figure 1, will be used as an illustration in the paper. If we look at the later years, we see that there are actually three technologies in competition: step-by-step (SXS), crossbar (XBar), and electronic (i.e. analog and digital ESS combined); manual switches had been retired by the mid-1960s. In general, at any time, a given technology is replacing

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older ones and is being replaced by the newer. This situation may be modeled as a collection of interacting populations, with the older technologies serving as prey for the predating newer technologies. This paper presents such a model, describes its implementation and studies its ability to:

- reconstruct the history of the switching market, and
- forecast future market evolution.

2. MODEL FORMULATION

An important aspect of population dynamics is the investigation of the interaction between a predator species with population $y$ and its prey with population $x$. The most well-known formulation of this interaction is due to Lotka and Volterra [1], [4]; this formulation employs a pair of non-linear differential equations of the form:

$$\frac{dx'}{dx} = A - By \quad , \quad \frac{dy'}{dy} = Cx - D \quad (1)$$

The terms on the left sides of the equations are the (logarithmic) rates of change of the prey and predator populations, respectively. The terms on the right have the following interpretations:

$A$: Reflects a constant per capita food supply, allowing population growth in the absence of predators.

$By$: Reflects the predation rate due to $y$ predators.

$Cx$: Reflects the per capita food supply available to predators, as represented by a prey population of $x$ individuals.

$D$: Reflects the per capita food requirements to just sustain the predator population.

If, in the above, we interpret $y$ as the market share held by a new switching technology challenging the market share $x$ held by an older, defending technology, we obtain a potential model of the resulting replacement dynamics. In particular, if these two technologies constitute the whole market, we must have:

$$x + y = 1$$

Additionally, if there is no challenging technology (i.e. $y = 0$), the market share of the defender should be fixed, so that there is no further change in the defender's growth rate (i.e. $x' = 0$), implying that $A = 0$. Similarly, if all defenders have been eliminated by the challenger, there can be no further change in the challenger's growth rate (i.e. $y' = 0$), implying that $B = 0$. The system (1) becomes, under these restrictions:

$$x' = -Bxy \quad , \quad y' = Cxy$$

Adding, and using the fact that $x + y = 1$, we obtain $0 = x' + y' = (C - B) xy$, so that $B = C$. Using $x = 1 - y$, the model for the predator population is, then:

$$\frac{y'}{y} = B(1-y)$$

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which can be readily solved to produce:

$$\log \left( \frac{y(t)}{(1-y(t))} \right) = K + Bt$$

where $K$ and $B$ are constants. This has the form of the technology replacement model first studied by Fisher and Pry [3], and subsequently by many other researchers. This model has proven applicability to simple (i.e., one challenger, one defender) replacement situations in the telecommunications arena [5],[7]. In this section, we generalize the above to a multi-species predator-prey model of technological replacement.

We postulate $n$ species, the $k^{th}$ of which has a market fraction at time $t$ given by a function $f_k(t), 1 \leq k \leq n$. We must have, for all times $t$ ($t$ suppressed for simplicity in what follows):

$$\sum_{k=1}^{n} f_k = 1$$

For each $k$, $1 \leq k \leq n$, we postulate that:

1. **Technology $k$ replaces older technologies** (i.e., $i < k$) at the rate $C_{ik}$ in the absence of other competitors, that is:

$$\frac{f'_k}{f_k} = C_{ik} f_i$$

2. **Technology $k$ is replaced by newer technologies** (i.e., $i > k$) at the rate $C_{ki}$ in the absence of other competitors, that is:

$$\frac{f'_k}{f_k} = -C_{ki} f_i$$

Assuming that the effects of all competitors can be superposed, the overall dynamics of the $k^{th}$ technology are given by:

$$\frac{f'_k}{f_k} = \sum_{i < k} C_{ik} f_i - \sum_{k < i} C_{ki} f_i$$  \hspace{1cm} (2)

It will be advantageous to allow the coefficients $C_{ik}$ to be functions of time $t$; the exact nature of the time dependence will be considered later.

In order to construct a usable model of the switching market, we need a way to "tune" the free parameters in the model to the historical values for the market fractions $f_k$. Let $h_k(t)$ denote the observed fraction of technology $k$ at time $t$, and $[t_0, T]$ represent the interval over which a solution is desired. Since technology $i$ is a replacement for technology $j$, $i > j$, it is likely that not all of the $n$ technologies were in existence at time $t_0$. Suppose $m < n$ technologies existed initially, and had measured market shares:

$$h_k(t_0) \quad 1 \leq k \leq m < n$$

If the coefficients $C_{ik}$ were known, we could set $f_k(t_0) = h_k(t_0)$, and then integrate the system of $m$ evolution equations of the form (2) up to the time $t_f$ of introduction of the
at time \( t_1 \) to solve the system of \( m+1 \) evolution equations until the time of introduction of the \((m+2)\)nd technology. Proceeding in this way, we can solve the evolution equations (2) over the whole interval \([t_0, T]\), incorporating technologies as they are introduced. The coefficients \( C_{ik} \) can be chosen to minimize a suitable measure of modelling error. One possible such measure is the root mean square error:

\[
\sum_{k=1}^{n} \int_{t_0}^{T} (f_k(C(t);t) - h_k(t))^2 \, dt \quad (3)
\]

This measures how well the trajectories \( f_k(t) \) track the historical data \( h_k(t) \). In the above expression, the \( f_k \) are shown to be explicit functions of the \( \frac{n(n-1)}{2} \) parameters:

\[ C(t) = (C_{ik}(t)) \]

If the functions \( C_{ik}(t) \) are sufficiently simple (e.g. linear functions of \( t \)), the optimization is straightforward.

3. MODEL IMPLEMENTATION

Considerable simplification is achieved by considering only discrete time versions of the model (2). Since market share data [2] are generally available only yearly, this results in no essential loss of generality.

The discrete time analogue of (2) is:

\[
f_k(t+1) - f_k(t) = f_k(t) \left\{ \sum_{i \neq k} C_{ik} f_i(t) - \sum_{k \neq i} C_{ki} f_i(t) \right\} \quad (4)
\]

If, at each time \( t \), we set the model market fractions \( f_k(t) \) to the measured fractions \( h_k(t) \), we can view \( f_k(t+1) \) as calculated from (4) as a prediction of the market share at time \( t+1 \). Let \( h(t) = (h_k(t)) \), and define \( f_k(t;h(t-1)) \) to be the estimate for the \( k \)th market fraction at time \( t \), using \( h(t-1) \) as the initial value for the estimate. Then, the values of the single period prediction errors can be represented by:

\[ N_k(t) = h_k(t) - f_k(t;h(t-1)), \quad 1 \leq k \leq n \]

If we square these errors, and sum over all technologies \( k \), and all times \( t \) which satisfy \( t_0 < t \leq T \), we obtain a measure of the total single period prediction error:

\[
N(C) = \sum_{t, k} N_k^2(t) \quad (5)
\]

In this paper, we consider only \( C_{ik} \)'s that are either constant:

\[ C_{ik}(t) = A_{ik} \quad (6) \]
or are linear functions of time:

\[ C_{ik}(t) = A_{ik} + B_{ik}(t-t_0) \]  

(7)

In these cases, (5) can be minimized by differentiating with respect to the \( A_{ik} \)s (and the \( B_{ik} \)s, if applicable), setting the results equal to zero, and solving the system so obtained.

It is also instructive to consider other measures of modelling error. The measure \( N(C) \) represented by (5) will not lead to trajectories \( f_k(t) \) that fit well globally. Whereas \( N(C) \) only considers single period forecast deviations, we can define a new measure, \( M(C) \) (which is the discrete analogue of (3)):

\[ M(C) = \sum_{t,k} \left( f_k(t;h(t_0)) - h_k(t) \right)^2 \]  

(8)

which measures the global deviation of trajectories developed from time \( t_0 \). These trajectories will more closely track the measured values, \( h_k(t) \). Again, for the simple forms of \( C_{ik} \) used in this paper, estimates for \( C_{ik} \) can be readily found.

Regardless of which error measure is used, some potential sources of estimation difficulty can arise from differences in the technology mix from time to time. Consider, for example, the replacement of manual equipment (technology 1) by electronic (technology 4). In the data set used for Figure 1, \( h_1(t) \) is negligible from 1972 on, and \( h_4(t) \) is negligible prior to 1968, so that only four years of data are available to estimate \( C_{14} \). By contrast, 31 years (1952-1982) of data are available for estimating \( C_{23} \), which represents the replacement of SXS (technology 2) by XBar (technology 3). Not surprisingly, this leads to relatively unreliable estimates of \( C_{14} \), which can result in difficulties when using the model predictively. For this reason, all data with market fractions less than 0.005 were deleted, thereby eliminating all interactions between technologies 1 and 4, and leading to no essential loss of generality. As the next section shows, however, not all problems arising from small amounts of available estimation data can be solved so simply.

4. HISTORICAL DATA RECONSTRUCTIONS AND FORECASTING

Figure 2 shows a historical reconstruction based on the constant model, and Figure 3 one based on the linear model. In both cases, all data over the period 1952-1982 were used for parameter estimation, and the error measure \( M(C) \) (9) was employed. In Figures 2 through 8, the dashed lines represent the actual data, while the solid lines show the modeled data.

It is clear from Figure 3 that the linear reconstruction is remarkably accurate, strongly suggesting a time dependency in the coefficients \( C_{ik} \). Unfortunately, the linear model tended to produce poor forecasts. Typically, one of the \( C_{ik} \)s would become negative a few years outside the region of model fitting (1952-1982), producing meaningless forecasts. Fits over smaller regions such as 1952-1972 produced similar results: great accuracy in the fit region, followed by unstable forecasts. This behavior arises in part from the instability of certain estimates of \( A_{ik} \) and \( B_{ik} \) which are made in the face of small market fractions and small amounts of data. In fact, this latter situation is a central attribute of this type of forecasting problem. Typically, the forecaster has to deal with a situation in which the older, defending technologies all have relatively large market fractions and lots of history, but the challenger has, at best, a few years of history, for some of which its market share is miniscule.
The above behavior suggested abandoning attempts to represent time dependencies in favor of using a simple, constant coefficient model \( \text{\( \text{\( C_{ik}(t) = A_{ik} \)} \)\) .} 
Forecasts based on fits over various regions are shown in Figures 4 to 6. The following guidelines were used to produce these forecasts:

- at least five years of data following the introduction of a new technology would be collected prior to forecasting, and
- forecasts would not be for more than five years.

These guidelines are arbitrary, but show that good results are possible if sufficient data are available. In all cases, data from 1952 up to the beginning of the forecast period were used to estimate the model parameters. As the amount of data for the new technology increases, the forecast quality improves.
5. IMPROVING FORECASTING ACCURACY

It was found that estimates of $A_{ik}$ and $B_{ik}$ following the introduction of a new technology $k$ tended to be overstated, especially for the case $i = k-1$. Constraints can be introduced in a natural way to alleviate this problem. These constraints are based on an interpretation of $C_{ik}$ as the efficiency with which technology $k$ replaces technology $i$. In fact, rewrite (2) as

$$(\log f_k)' = C_{ik}(f_i - h) + C_{jk}(f_j - h) + C_{ik}h + C_{jk}h + \text{other terms}$$

where $0 < i < j < k$ and $h > 0$ is small enough to ensure that $0 < f_i - h < 1$, $0 < f_j - h < 1$. The term $C_{jk}h$ represents the net contribution that the technology $j$ market fraction $h$ makes to the technology $k$ relative growth rate $(\log f_k)'$. Since technology $i$ is older than technology $j$, an equivalent technology $i$ market fraction $h$ should make a greater contribution to $(\log f_k)'$ than that made by technology $j$, or:

$$C_{ik} \geq C_{jk} \quad \text{for } i < j$$

A similar line of reasoning suggests the constraints:

$$C_{ik} \leq C_{ij} \quad \text{for } k < l.$$

When sufficient (i.e. more than a few years) of data are available to estimate the $A_{ik}$ (in the constant model), the above constraints are invariably satisfied. In fact, stronger constraints: $C_{ik} \geq \alpha_{ijk}C_{jk}$, where $\alpha_{ijk} > 1$ will, in general, hold. The impact of imposing the constraint: $C_{24} \geq \alpha C_{34}$ for $\alpha = 6$ is shown in Figure 7. Values of $\alpha$ were varied between 1 and 9. The optimum results were obtained for $\alpha = 6$. Figure 8 shows the same result for the unconstrained case. It is clear that selecting an appropriate value of $\alpha$ is a matter of judgement. However, this is unavoidable, as it constitutes adding information to the basic model represented by (2).
6. SUMMARY

This paper has presented a model of multilevel technology replacement based on the classical Lotka-Volterra predator-prey equations. An implementation of the model, together with an application to a subset of the North American switching market was discussed. The best forecasting results were achieved using a simple constant coefficient model, with auxiliary constraints on the coefficients.

A number of techniques were applied to the linear coefficient model presented in an attempt to improve its predictive performance. The consistent lack of success strongly suggests that there are too many parameters in the linear model, providing too many degrees of freedom to produce good fits under the exacting conditions described. Further investigation into these time dependencies needs to be carried out.

7. REFERENCES


Figure 7 Forecast with $\alpha = 6$

Figure 8 Unconstrained Forecast