DEPENDENCE EFFECTS IN COMMUNICATIONS NETWORKS WHICH PRODUCE SERVICE DEGRADATIONS FOLLOWING INCREASES IN NETWORK RESOURCES

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The purpose of this paper is to focus attention on some cases where circuit incrementation in a network can lead to a reduction in the total carried traffic rather than an increase. Although fundamental and exploratory in nature such studies may provide some answers to the question of how to avoid resource wastage in telecommunications networks.

1. INTRODUCTION

In an earlier paper [1] it was shown that, for a particular network, calculations based on the equivalent random method give a surprising conclusion: the addition of more circuits to a link of the network results in a decrease in the total traffic carried on the network. In this case the result is entirely due to the approximating assumptions of the equivalence method. That a decrease in total carried traffic can result from an increase in the number of circuits on a link of some networks (exact analysis) is established in [2].

We shall search for conditions under which an increase in circuits in a communications network either will or could result in a reduction in the total carried traffic. Because this effect is somewhat counter intuitive, although not strictly a logical paradox, it is convenient to refer to this problem as the circuit paradox problem.

In section 2 we confine our attention to arbitrary arrival realizations (arbitrary sequences of arrival instants) without requiring information on the underlying arrival or holding time distributions. We introduce the concept call displacement and show how it is related to the circuit paradox problem. Some simple propositions are established for a single channel in a multi-service network and illustrative examples of the circuit paradox problem are given.

In section 3 we assume particular distributions for call inter-arrival and holding times and consider the circuit paradox problem under equilibrium conditions on the network. Further examples are given and some general principles determined. The paper concludes with comments on the practical relevance of the study.

2. REALIZATIONS AND CALL DISPLACEMENT

Consider a single communication link with n circuits (servers) operating on a first come first served lost calls cleared basis. For any call arrival realization and any holding times we establish inequalities relating the number of busy servers at time t to the call service demands. Fig.1 illustrates the changes in the number of busy servers as a function of time for a typical arrival realization.
Let us denote the successful calls in an interval \([0,T]\) by \(c_1,c_2,\ldots,c_p\) their arrival times by \(t_1,t_2,t_p\) and the number of servers required per call by \(m_1,m_2,\ldots,m_p\) respectively. The step function \(N(t)\) gives the number of servers busy on the link at time \(t\). For the same realization, when the number of servers is incremented by \(\Delta n\) it may happen that (i) some calls previously lost are served and (ii) some calls previously served are lost. If both (i) and (ii) occur then we say that call displacement occurs. When a network or system has the property that call displacement cannot occur, and new calls can be accepted under the incrementation, we say that the link is displacement free. (These concepts are defined mathematically later.) For such networks the circuit paradox does not occur. When displacement is possible, then the displacement of a call by another with a shorter holding time can result in a reduction in the total carried traffic.

We obtain a system of inequalities, the solution of which is necessary and sufficient for the occurrence of call displacement.

\[
N'(t) = N(t) + \sum_{i=1}^{r} m_i
\]

At the instant \(t_i\), immediately prior to the arrival of the call \(c_i\),

\[
N'(t_i^-) = N(t_i^-) + \sum_{i=1}^{r} m_i
\]

Suppose that \(c_i\) is the first displaced call and let \(d_1,d_2,\ldots,d_r\) be that subset of the calls lost under the \(n\) server regime which are served when the link has \(n+\Delta n\) servers and survive until \(t_i\). (These are the only calls which can influence the acceptance or rejection of \(c_i\)). The step function \(N'(t)\) gives the number of servers busy on the \(n+\Delta n\) server link at time \(t\). We suppose that calls \(d_1,d_2,\ldots,d_r\) require \(m_1,m_2,\ldots,m_r\) servers and arrive at times \(t_1',\ldots,t_r'\).

We obtain a system of inequalities, the solution of which is necessary and sufficient for the occurrence of call displacement.

\[
m_i > n-N(t_i')
\]

\[
m_i \leq n+\Delta n-N'(t_i')
\]
for \( k = 1,2,\ldots,r \).

To record the success of \( c_i \) (the first displaced call) with \( n \) servers and its failure with \( n+6n \) servers, we have

\[
\begin{align*}
m_i &\leq n-N(t_i^-) \\
m_i &> n+\Delta n-N'(t_i^-)
\end{align*}
\]  

We also note that \( N(t_i^-) = N(t_i^+) \). Also, from (1), (3) and (5) a necessary condition for displacement is

\[
N(t_i) > N(t_i^-). 
\]  

**Proposition:** For any call realization, when each call requires the same number of servers \((m \geq 1)\) call displacement cannot occur.

**Proof:** Note from (1) that \( N'(t_i^-) = N(t_i^-)+rm \). Substitution in (5) gives

\[
(r+1)m > n+\Delta n-N(t_i^-)
\]  

Considering \( t_i \) and noting that some previously unserved calls may be served with \( n+6n \) servers and not survive until \( t_i \)

\[
N'(t_i^-) \geq N(t_i^-)+(r-1)m. 
\]  

Substitution in (3) gives

\[
rm \leq n+\Delta n-N(t_i^-) 
\]  

Since \( N(t_i^-) \equiv N(t_i) < N(t_i^-) \) contradicts the previous failure of call \( d_r \) and \( N(t_i^-) \geq N(t_i^-) \) implies \( N(t_i^-) \geq N(t_i^-)+m \) it follows that

\[
(r+1)m \leq n+\Delta n-N(t_i^-)
\]  

contradicting (7). We conclude that a first displacement is impossible.

We also note in passing that previously served calls are still served by the same servers.

**Corollary:** For the case when each call requires the same number of servers, incrementing the number of servers on a single link during an interval \([T_1, T_2]\) cannot reduce the total carried traffic during that interval.

It should be noted that circuit incrementation for part of an interval can lead to a reduction in the total carried traffic over the whole interval. This is illustrated in Fig. 2. Perhaps this has implications for dynamic circuit allocation?

![Figure 2](image-url)
Returning to the general case where calls may demand variable numbers of servers we see that displacement can occur with circuit incrementation. Thus, in this case, by displacing a long holding time call by a short holding time call we can construct realizations for which the total traffic carried would decrease with circuit increase.

We define the circuit requirement set \( S = \{a_1, a_2, \ldots, a_q\} \) where \( a_1 < a_2 < \ldots < a_q \) are positive integers. This is the set of \( q \) possible (and distinct) server requirements for \( q \) classes of calls.

**Proposition:** For any \( S \), if \( q > 1 \) displacement is always possible (but dependent on \( n \) and \( \Delta n \)).

**Proof:** It suffices to construct the following example.

Let \( n = a_2, \Delta n = a_1, m_1 = a_1, m'_1 = a_2, m_2 = a_2. \)

Whether or not, for a given \( S \) (with \( q > 1 \)) a link is displacement free for certain values of \( n \) and \( \Delta n \) is an interesting question. To further analyse and simplify equations (1)-(6) we introduce the following notation

\[
S' = \{\text{all non-negative integer combinations of elements of } S \text{ that are } \leq n\}.
\]

\[
S'' = S' - \{0\}.
\]

Let \( x_1, x_2, \ldots, x_r, x_{r+1} \in S \) and \( y_1, y_2, \ldots, y_r, y_{r+1} \in S' \). \( x_\ell = m_\ell, y_\ell = N(t_\ell') \); \( \ell = 1, 2, \ldots, r \) and \( x_{r+1} = m_1, y_{r+1} = N(t_{r+1}) \) (\( c_i \) is the first displaced call). It is also convenient to denote \( x_{r+1} + y_{r+1} \) by \( z \).

**Displacement Conditions:** Single link call displacement occurs for given \( S, n \) and \( \Delta n \) iff

\[
\exists x \in S, y \in S' \text{ and } z \in S' - \{0\}
\]

satisfying

\[
\begin{align*}
\sum_{\ell=1}^{r} x_\ell + y_\ell & \leq n + \Delta n \\
x_\ell + y_\ell & > n \\
\end{align*}
\]

\( \ell = 1, 2, \ldots, r. \) (9)

and

\[
\sum_{j=1}^{r} x_j + z > n + \Delta n \]

(10)

for some positive integer \( r \). Physically \( r \) corresponds to the number of displacing calls. It is easy to show that, from physical constraints, it is only necessary to consider \( r \leq (n + \Delta n)/a_1 \).

**Definition:** If \( \not\exists \) a solution to (9) we say that we have circuit wastage. That is, the circuit increase \( \Delta n \) cannot result in the acceptance of new calls. Of course, it is recognized that such circuits may be useful in the case of circuit outages, but under normal conditions they play no part.

**Definition:** If \( \exists \) a solution to (9) which does not satisfy (10) then the link is said to be displacement free.

Significantly, the circuit paradox cannot occur on a displacement free link. When the displacement conditions (9) and (10) are both satisfied the circuit paradox is possible (but dependent on underlying distributions).

Fig. 3 illustrates the set relations between the three link conditions.
D \equiv \text{displacement}. \quad \text{DF} \equiv \text{displacement free.} \quad \text{CW} \equiv \text{circuit wastage}

We note that unused circuits will also result if \( n \notin S' = \{ \text{all non-negative integer combinations of elements in } S \} \). An efficient algorithm for solving the Diophantine equation to test whether \( \exists \) such a solution \( w \) to \( \sum_{j=1}^{\infty} a_j w_j = n \) is given in [3].

It is of interest to determine the relative frequencies of occurrence of the above three regions (CW, DF and D) for circuit requirement sets \( S \) as \( n \) and \( \Delta n \) vary.

In Table 1 for \( S = \{6,8\} \) and \( n=10 \) we note that as \( \Delta n \) increases all three possibilities are exhibited.

<table>
<thead>
<tr>
<th>( \Delta n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>LINK STATE</td>
<td>CW</td>
<td>D</td>
<td>D</td>
<td>D</td>
<td>D</td>
<td>DF</td>
<td>DF</td>
</tr>
</tbody>
</table>

In Table 2 the ranges of \( a_1 \) are shown. The other variables were related as follows: \( a_2 \) ranged from \( \min a_1 +1 \) to \( \max a_1 +1 \); \( n \) ranged from \( a_2 \) to \( 5a_2 \); and \( \Delta n \) from 1 to \( a_2 -1 \). The percentage occurrence of CW, D and DF are given.

<table>
<thead>
<tr>
<th>( a_1 )</th>
<th>CW</th>
<th>D</th>
<th>DF</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 to 10</td>
<td>8.12%</td>
<td>89.34%</td>
<td>2.54%</td>
</tr>
<tr>
<td>11 to 50</td>
<td>13.98%</td>
<td>79.64%</td>
<td>6.38%</td>
</tr>
</tbody>
</table>

Clearly link displacement occurs with very high probability for randomly chosen \( S, n \) and \( \Delta n \). Circuit wastage has significant occurrence and it is relatively difficult to obtain displacement free cases.

We complete this section by considering the simple network shown in Fig.4. An arrival realization is shown for the two streams on the time axis. The routes are A-B-C and A-B-D for streams 1 and 2 respectively. With \( n=1 \), the total carried traffic is 1.1e (0.3+0.8) but with \( n=2 \) it is 0.6e (0.3+0.3). The additional circuit results in a 36% reduction in total carried traffic.

\[ \begin{align*}
\alpha_1 & \quad \alpha_2 \\
0 & \quad .2 & \quad .4 & \quad .6 & \quad .8 & \quad 1.0 \\
\beta_1 & \quad \beta_2 \\
\text{Stream 1} & \text{Stream 2} \\
\end{align*} \]

FIGURE 4

\( \alpha_i \) arrival, \( \beta_i \) departure
This effect has also been demonstrated by simulation (see Table 3). An increase in \( n \) from 9 to 10 resulted in a decrease from 10.537e to 10.403e. The same arrival realization (1,600 arrivals) was used in each case.

**TABLE 3:** Real Time Simulations 1,600 arrivals (loading from an empty state)

<table>
<thead>
<tr>
<th></th>
<th>CARRIED TRAFFIC</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>STREAM 1</td>
</tr>
<tr>
<td>8</td>
<td>4.440</td>
</tr>
<tr>
<td>9</td>
<td>4.326</td>
</tr>
<tr>
<td>10</td>
<td>4.207</td>
</tr>
<tr>
<td>11</td>
<td>4.170</td>
</tr>
</tbody>
</table>

3. DISTRIBUTIONS AND CALL DISPLACEMENT

Offering some generally distributed stream of traffic, otherwise lost, (for example superimposed overflow traffics in an alternative routing network) to a long route can result in a decrease in total carried traffic. To illustrate this we consider Fig.5.

FIGURE 5
A chain offered \( n+1 \) Poisson streams

The overflow stream with mean \( M_1 \) is offered to a route of \( n \) links. Each link in the route consists of 1 circuit, and is also offered a Poisson stream with mean \( M_2, M_3, \ldots, M_{n+1} \) respectively. The stream with mean \( M_1 \) can be viewed as carried on a high capacity link offered to node 1. It has been shown in [2] that removal of these circuits (an immediate loss of \( M_1 \)) leads to an increase in total carried traffic on the network whenever

\[
\sum_{i=2}^{n+1} \frac{M_i}{(1+M_i)H_i} > \frac{1}{H_1}
\]

where \( H_i \) is the mean holding time for stream \( i \). This example illustrates the circuit paradox and, since the left hand side of (11) is monotonic increasing in \( n \), suggests an important practical consequence. As traffic offered to a long route in a network can result in an increase in the total lost traffic, caution is needed when contemplating long route access to origin-destination pairs in non-hierarchically routed networks. The example also gives support to the network management principle: "give priority to calls requiring a minimum number of links to form a connection when all available circuits are in use".

It may, at first, seem surprising that inequality (11) does not depend on \( M_1 \). But, observe that when all calls have the same mean holding time (eliminating \( H_i \)) the right hand side gives the mean carried traffic over the set of time intervals during which stream 1 occupies the route (1e) and the left hand side is the mean traffic that would have been carried for these same time intervals if stream 1 had been switched off. (The reader can note the reason for assuming that streams 1,...,n+1 are Poisson viz. to employ the "memoriless property").

For the case when all calls have the same holding time distribution we are able to state a general principle which has intuitive appeal. Suppose that Poisson streams 2,3,...,n occupy links in a subgraph \( G \) of the network and stream 1 is overflow traffic from \( G^c \), a disjoint subgraph.
**Principle:** The addition of circuits which allow stream 1 to use links in G will result in a reduction in total network carried traffic whenever the total marginal loss for traffic in G due to the reduction by m circuits on each link of any path in G which a call from stream 1 could occupy is less than m.

The proof follows from the observation that during the occupancy of one route in G by a call from stream 1, m erlangs of traffic are added to the total and the decrease in traffic for streams 2, 3, ..., n during these times, being independent of when the occupations by stream 1 occur due to the Poisson assumption, is the marginal loss for traffic in G. We note that for the case of nonhomogeneous calls with different holding times it is necessary to introduce holding time weightings as in (11).

Some examples to illustrate the above principle are now considered for the case that the n+1 streams are Poisson and calls have a common holding time distribution. We confine our attention to networks in which each call is offered to a single route, the offered traffic to route r being $M_r$. A call requesting route r is lost if on any link of the route there is not a free circuit (assume $m=1$).

It is well known [4] that the stationary distribution of the number of calls in progress is given by

$$p(n) = p(0) \prod_{r=1}^{n} \frac{M_r}{n_r}$$

(12)

where $n = (n_r, r \in R)$ is a feasible state.

We first note from (11) that when $n=1$ no solution is possible ($M/1+M<1$). Thus the circuit paradox cannot be exhibited on a single circuit offered two streams one of which is Poisson. (Note that it is easy to construct examples on a single link, e.g. with two deterministic streams, which do show the circuit paradox.)

Returning to Fig.5 with $n=2$ and $n_1, n_2$ circuits on links (1,2) and (2,3) resp., application of the principle gives the following: the circuit paradox occurs if

$$M_2[E_1^{-1}(M_2)-E_1^{-1}(M_2)] + M_3[E_2^{-1}(M_3)-E_2^{-1}(M_3)] > 1$$

(13)

where $E_0(\cdot)$ is the Erlang loss function. The left-hand side represents the total marginal loss when a stream 1 call occupies a circuit on the route. For the case $n_1=2, n_2=1, n_3=1$ it is straightforward to show that the circuit paradox occurs if $M_3 > 2M_1$.

The circuit paradox does not occur in this network

In Fig.6 $M_1, M_2, M_3$ are offered in two hops to destinations 3, 1 and 2 resp., with losses $m_1, m_2, m_3$. Suppose that each link has 1 circuit. The marginal decrease in total carried traffic when stream 1 occupies its route is $(M_3 + M_2)/(1 + M_2 + M_3)$ and since this is less than 1 the circuit paradox cannot occur.
With the slight modification in Fig.7 we note that the circuit paradox will occur whenever $M_2/(1+M_2) + M_3/(1+M_3) > 1$. (Compare the "long chain" example.)

4. CONCLUSIONS

The concepts call displacement, displacement free link and circuit wastage have been introduced. For arbitrary traffic arrival realizations in multi-service multi-server networks ad hoc circuit incrementation is likely to result in call displacement or circuit wastage. It is possible to dimension and increment links using displacement free circuit values. In such cases the circuit paradox cannot occur. Although call displacement occurs with high probability with arbitrary circuit incrementation this does not necessarily imply a service degradation (this is distribution dependent). Some simple network examples have been given to illustrate actual occurrence of the circuit paradox.

The concepts discussed in this paper are relevant to economic dimensioning and network management.

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REFERENCES