SOME EXACT CLOSED-FORM EXPRESSIONS ON THE PERFORMANCE OF INTEGRATED NETWORKS

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A technique which provides exact closed-form expressions for the equilibrium probabilities of a Markovian network is developed and applied to the First-Come First-Served scheme of voice-data integration problem. This technique has less computational complexity compared to the previous work and can be applied to a large class of Markovian networks.

1. INTRODUCTION

The network of the future will be required to handle a variety of traffic, such as digital voice, video, facsimile, and remote control of information [1,2]. In order to fulfill such requirements, the future networks are expected to combine circuit and packet switching technologies. In Fig. 1, a model is given for the integration of circuit-switched and packet-switched traffic.

The performance of an integrated network depends upon how intelligently the output channels are distributed among different classes of users. Hence, the controller which controls the use of output channels has utmost importance. Especially, at high traffic intensities, when all the channels are occupied, the channel allocation should be made intelligently according to rules defined by a control scheme. The networks with different performance objectives use different control-schemes.

The problem of analytically determining the traffic performance of integrated networks has been studied in the literature. Most of the researchers have accepted that the exact analysis of the system is a very difficult problem and they proposed approximation techniques [5]-[11]. The exact analyses for small systems have been found to be very difficult to generalize [7]-[9]. Simplified analyses have been found to be too naive to capture the essence of the system [6].

Most of the models proposed for integrated networks shared by multiple classes of users lead to Markovian queueing networks. The primary interest in many problems involving Markovian networks is the determination of the equilibrium probabilities. In this study, we will use the key-state approach [3]. This approach makes it possible to find exact closed-form expressions for the equilibrium probabilities and eliminates the computational problems that have arisen previously. Moreover, it can be applied to a large class of integrated network models with different control-schemes.

2. ANALYSIS

Assuming that each arrival process is Poisson and that the holding times are exponentially distributed, it is possible to give a Markovian model for the network under consideration. In this study, we consider a special class of Markovian queueing networks

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which is very suitable for many integrated networks, especially for those where circuit and packet-switched traffics are multiplexed. A state of this model is defined as:

\[ S = [d, s_1, s_2, \ldots, s_L], \quad 0 \leq d, \quad 0 \leq s_i \leq K_i < \infty, \quad K_i \in I^+ \]  

(1)

Here, \( d \) is the number of packet-switched users in the system and \( s_i \) are the specifiers of the control scheme. As an example, if we consider the multiplexing of circuit-switched voice and packet-switched data under First-Come First-Served (FCFS) control scheme, \( d \) is the number of packets, \( L = 1 \) and \( s_1 \) corresponds to the number of voice calls in the system. The number of specifiers must be enough to describe the control scheme completely. Many other control schemes such as Movable Boundary (MB) where voice channels and data channels are separated by a boundary, require more than one specifier. In the MB scheme, data users are allowed to use the voice channels if there is any available. Accordingly, \( s_1 \) is defined as the number of voice calls in the system, and \( s_2 \) is the number of data users occupying the voice channels.

A subset of states with the same packet size \( d \), is called the set of \( d \)-states. The equilibrium probabilities of the \( d \)-states form the probability vector \( \bar{P}_d \).

\[ \bar{P}_d = [P_{d0} \ P_{d1} \ldots P_{d,M-1}]^T. \]  

(2)

Here, \( M \) is the total number of states in the set of \( d \)-states.

The arrivals are assumed to be coming from a memoryless source of infinite population, therefore arrival rates are always independent of the packet-size, \( d \). Departure rate from the system, however, is proportional to the number of packets in the system. Nevertheless, the departure rate can be expressed in terms of only the specifiers when the data queue is not empty, i.e., when \( d \) is larger than the maximum number of data packets that can be served simultaneously. Therefore, when the packet-size, \( d \), is greater than a certain threshold, \( m \), and assuming that there cannot be more than one departure and arrival simultaneously, the steady-state (equilibrium) probability vectors satisfy the following constant coefficient difference equation:

\[ A_1 \bar{P}_{d+1} + A_0 \bar{P}_d + A_{-1} \bar{P}_{d-1} = \vec{0}, \quad d > m. \]  

(3)

Here, \( A_1, \ A_0, \ A_{-1} \) are \( M \times M \) matrices and \( m \) is the maximum number of data terminals that can be served simultaneously. The entries of \( A_1 \) are the transitions from \( (d+1) \)-states to \( d \)-states. Note that these are the departure rates. \( A_{-1} \) contains the arrival rates from \( (d-1) \)-states to \( d \)-states. The transitions within \( d \)-states are defined by the matrix \( A_0 \).

The solution to (3) is found by using standard techniques:

\[ \bar{P}_d = \sum_{i=1}^{2M} k_i \vec{b}_i \ z_i^d, \quad d > m, \]  

(4)

where \( \vec{b}_i = [b_0 \ b_1 \ldots b_{M-1}]^T \quad i = 1, 2, \ldots 2M \) are from the nullspace of \( R(z_i) \) which is defined as:

\[ R(z) = A_1 + A_0 z^{-1} + A_{-1} z^{-2}, \]  

(5)

and \( z_i \) are the roots of \( \det\{R(z)\} \).

The techniques used to locate the zeros of a polynomial and to find the nullspace of a matrix can be employed to obtain the roots \( z_i \) and the vectors \( \vec{b}_i \). The coefficients \( k_i \), however, depend on the state probabilities \( \bar{P}_d \) for \( d \leq m \), which are the initial conditions of equation (3). From the balance equations of these states, each of these coefficients can be expressed in terms of the initial state probabilities and solved, but this is a tedious task.
and it becomes untractable when the system size gets larger. We will show how to obtain $k_i$ by using the key-state approach.

2.1 Key-state approach

It can be shown that one particular smaller subset of the state space captures the essence of whole system. In other words, all other states can be rather easily found by employing the knowledge of the equilibrium probabilities in that subset only.

For the class of Markovian networks under consideration, the coefficient matrices with the exception of $A_1$ are not constants for $d \leq m$. Hence, we have the following matrix difference equation for $d \leq m$:

$$A_{1d} \tilde{P}_{d+1} + A_{0d} \tilde{P}_d + A_{-1} \tilde{P}_{d-1} = 0, \quad d \leq m. \quad (6)$$

At $d = 0$, we have:

$$A_{10} \tilde{P}_1 + A_{00} \tilde{P}_0 = 0. \quad (7)$$

For almost all control schemes, $A_{10}$ is invertible. Therefore, (7) can be written as:

$$\tilde{P}_1 = -A_{10}^{-1} A_{00} \tilde{P}_0. \quad (8)$$

The entries of the vector $\tilde{P}_0$ are called key-state probabilities [3]. If the key-state probabilities are known, any state probability can be obtained from these starting from (8) by simple iterations without solving any set of linear equations. We will take advantage of the key-state approach to find the initial conditions of (3). Key-state approach allows us to write:

$$M-1 \sum_{j=0}^{M-1} c_{dtj} P_{0j} = \ell = 0,1,\ldots M-1. \quad (9)$$

Here, $P_{00}, P_{01}, \ldots, P_{0,M-1}$ are the key-states and $c_{dtj}$ are called key-state coefficients. The solution found for the equilibrium state probabilities in (4) suggests that:

$$c_{dtj} = \sum_{i=0}^{2M-1} k_{ij} b_{\ell i} z_i^d, \quad d > m, \quad j = 0,1,\ldots, M-1, \quad \ell = 0,1,\ldots, M-1. \quad (10)$$

Here, $b_{\ell i}, z_i$ are as in (4). The coefficients $k_{ij}$ are yet to be found. Substituting (10) into (9), the following expression is obtained for the equilibrium probabilities:

$$P_{d} = \sum_{i=0}^{2M-1} b_{\ell i} z_i^d \sum_{j=0}^{M-1} k_{ij} P_{0j}, \quad d > m, \quad \ell = 0,\ldots, M-1. \quad (11)$$

Now, using (11) and (4) we can express $k_i$'s in terms of $k_{ij}$'s and the key-state probabilities:

$$k_i = \sum_{j=0}^{M-1} k_{ij} P_{0j}, \quad i = 0,1,\ldots 2M-1, \quad (12)$$

which allows us to rewrite (4) in the following matrix form:

$$\tilde{P}_d = B_d K \tilde{P}_0, \quad (13)$$

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where
\[
\begin{pmatrix}
{b_{00}z_0^d} & {b_{01}z_1^d} & \ldots & {b_{0,W}z_W^d} \\
b_{10}z_0^d & {b_{11}z_1^d} & \ldots & {b_{1,W}z_W^d} \\
\vdots & \vdots & \ddots & \vdots \\
b_{M-1,0}z_0^d & b_{M-1,1}z_1^d & \ldots & {b_{M-1,W}z_W^d}
\end{pmatrix}
\]

\[
K = \begin{pmatrix}
k_{00} & k_{01} & \ldots & k_{0,M-1} \\
k_{10} & k_{11} & \ldots & k_{1,M-1} \\
\vdots & \vdots & \ddots & \vdots \\
k_{W,0} & k_{W,1} & \ldots & k_{W,M-1}
\end{pmatrix}
\]

Here \( W = 2M - 1 \). We aim to find the coefficient matrix \( K \) explicitly. Starting from (8) and employing (6), the state probability vector at \( d = m \) is expressed in terms of key-state probabilities as:
\[
\vec{P}_m = C_m \vec{P}_0.
\]

Employing (13) and (14),
\[
\begin{pmatrix}
\vec{P}_m \\
\vec{P}_{m+1}
\end{pmatrix} = \begin{pmatrix}
C_m \\
C_{m+1}
\end{pmatrix} \vec{P}_0 = \begin{pmatrix}
B_m \\
B_{m+1}
\end{pmatrix} K \vec{P}_0.
\]

The coefficient matrix \( K \) is selected as follows to satisfy (15):
\[
K = \begin{pmatrix}
B_m \\
B_{m+1}
\end{pmatrix}^{-1} \begin{pmatrix}
C_m \\
C_{m+1}
\end{pmatrix}.
\]

Once the matrix \( K \) is obtained, the coefficients \( k_i \), are found by employing (12). Note that, we still do not know the key-state probabilities. They can be obtained by employing the following argument.

If the system is stable, then the equilibrium probabilities exist and are unique. But equation (11) represents a probability distribution if only the terms with \( z_i \geq 1 \) are exactly cancelled, i.e., \( k_i = 0 \) if \( z_i \geq 1 \). We thus obtain a set of equations for key-state probabilities:
\[
k_i = \sum_{j=0}^{M-1} k_{ij} P_{0j} = 0, \quad z_i \geq 1.
\]

Assuming that \( z_i \geq 1 \) for \( i \in \{i_1, i_2, \ldots, i_j\} \), (17) can be put into matrix form as:
\[
\begin{pmatrix}
k_{i_1,0} & k_{i_1,1} & \ldots & k_{i_1,M-1} \\
k_{i_2,0} & k_{i_2,1} & \ldots & k_{i_2,M-1} \\
\vdots & \vdots & \ddots & \vdots \\
k_{i_0,0} & k_{i_0,1} & \ldots & k_{i_0,M-1}
\end{pmatrix}
\begin{pmatrix}
P_{00} \\
P_{01} \\
\vdots \\
P_{0,M-1}
\end{pmatrix} = \vec{0}.
\]

If the system is stable, then this system of linear equations has a nontrivial solution. This means that the nullity of above matrix is nonzero. Without loss of generality, we can assume that the normalized solution of (18) is expressed as \( [1 \sigma_1 \sigma_2 \ldots \sigma_M]^T \). The key-state probabilities can be related to the empty state probability by employing the normalized solution given in (20). In fact, \( \sigma_i \) is the ratio of \( P_{0j} \) to the empty state probability \( P_{00} \), i.e., \( P_{0j} = \sigma_j P_{00} \). Finally, the empty state probability \( P_{00} \) is found explicitly by using the normalization condition. The final form of the equilibrium probabilities are:
\[
P_{dt} = P_{00} \sum_{i=0}^{2M-1} b_{ti} z_i^d \sum_{j=0}^{M-1} k_{ij} \sigma_j, \quad d > m.
\]

3. **EXAMPLE:**
As an example, we will consider the multiplexing problem of circuit-switched voice and packet-switched data. There are $N$ output channels and voice traffic can only use $N_1$ of these channels. The voice traffic is blocked if transmission facilities are not available. On the other hand, packet-switched traffic is allowed to queue. The channel assignments are made according to First-Come First-Served priority.

Assuming that the arrival processes are Poisson (memoryless) and the service time distributions are exponential (memoryless), let the voice arrival rate be $\lambda_2$ and the data arrival rate be $\lambda_1$. The departure rates are $\mu_2$ and $\mu_1$ for voice and data respectively. The state transition rate diagram of this system is given in Fig. 2 for $N = N_1 = 2$.

Here, $\mathbf{P}_0 = [P_{00} \ P_{01} \ \ldots \ P_{0N_1}]^T$ are the key-state probabilities and $M - 1 = N_1$. The coefficient matrices are found as:

$$ [A_{id}]_{ij} = \begin{cases} \min(d, N_1 - i) \mu_1, & \text{if } i = j; \\ 0, & \text{otherwise}. \end{cases} $$

$$ [A_{-1}]_{ij} = \begin{cases} \lambda_1, & \text{if } i = j; \\ 0, & \text{otherwise}. \end{cases} $$

$$ [A_{0d}]_{ij} = \begin{cases} j\mu_2 & i = j - 1; \\ -\alpha_{id} & i = j; \\ f(d, j) & i = j + 1; \\ 0 & \text{otherwise}. \end{cases} $$

where

$$ f(d, j) = \begin{cases} \lambda_2 & \text{if } d + j < N \text{ and } j < N_1; \\ 0 & \text{otherwise}. \end{cases} $$

$$ -\alpha_{id} = \min(d, N_1 - j) \mu_1 + \lambda_1 + j\mu_2 + \lambda_2 f(d, j) $$

The probability generating functions are found as:

$$ \mathbf{R}(z) = \begin{pmatrix} r_{00}(z) & r_{01}(z) & 0 & \ldots & 0 \\ 0 & r_{11}(z) & r_{12}(z) & \ldots & 0 \\ 0 & 0 & r_{22}(z) & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & r_{N_1 N_1}(z) \end{pmatrix} $$

$$ (20) $
The entries of the vector $\tilde{b}_i$ defined in (6) are found as:

$$b_{\ell} = (-1)^{\ell} \prod_{j=0}^{\ell-1} r_{j,j+1}(z_i), \quad \ell \leq i; \quad b_{\ell} = 0 \quad \ell > i \quad (21)$$

where $r_{jj}(z) = 0$ for $z = z_{2j}$ and $z = z_{2j+1}$.

The equilibrium probabilities are, therefore, given as:

$$P_{dt} = \sum_{j=0}^{2N_1-1} k_j z_j^d (-1)^{\ell} \prod_{i=0}^{\ell-1} r_{i,i+1}(z_j) u(\ell - j), \quad d > m. \quad (22)$$

Here $u(\ell - j) = 1$ for $\ell \geq j$ and $u(\ell - j) = 0$ otherwise. The coefficients $k_j$ are obtained by employing (16) and (17). The key-state coefficients matrix $C_k$ are found iteratively as:

$$C_{k+1} = -A_{1k}^{-1}(A_{0k} C_k + A_{-1k} C_{k-1}), \quad 0 < k < m \quad (23)$$

$$C_1 = -A_{10}^{-1} A_{00} C_{00}, \quad C_{00} = I$$

The data queueing delay and the voice blocking probability are easily obtained from the expressions given in (22). Fig. 3 and Fig. 4 show the normalized queueing time and voice blocking probability respectively as a function of voice traffic. Here parameter $\alpha$ is the ratio of voice holding time to average packet duration, $N = N_1 = 10$, $a_d = \lambda_1/\mu_1 = 5 E r$. The approximate results given in [9] for the same problem are also compared here with the exact results.

**FIGURE 3**

Comparison of the queueing delay obtained by using exact analysis with the approximate result given in [9]

**FIGURE 4**

Comparison of the blocking probability obtained by using exact analysis with the approximate result given in [9]

### 4. CONCLUSIONS AND COMPARISONS

The analysis of the same problem using moment generating functions has been carried out in [9]. By cancelling the unstable roots inside the unit circle, an expression is obtained
for the $z$-transform of the equilibrium probabilities in terms of the unknown steady-state probabilities for $d \leq m$. These unknown probabilities are found by solving the corresponding local balance equations. But the number of unknowns to be solved for is in the order of $N^2$ which makes the solution very difficult for large systems. In order to avoid such numerical difficulties, it is therefore assumed in [9] that $\mu_2 \approx \mu_1$ and approximate closed-form expressions for the moment generating functions of mean packet delay and voice blocking probability are obtained.

The key-state approach, however, allows us to find exact closed form expressions for the equilibrium probabilities. This is physically more meaningful than to have closed-form expressions for the moment generating functions of the equilibrium probabilities. Furthermore, it can be shown that the key-state approach can be applied to large systems without encountering the numerical difficulties that have arisen previously.

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