Superposition of Interrupted Poisson Processes and Its Application to Packetized Voice Multiplexers

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Abstract

This paper analyzes a model of a statistical multiplexer for packetized voice. The packet arrival stream from a single voice source is approximated by an interrupted Poisson process (IPP). The multiplexer is modeled by a single-server queue with general service time distribution, whose input is superposition of independent and identical IPPs. The steady-state probabilities of this queueing system are analyzed by the supplementary variable technique. Three methods for deciding the IPP parameters are discussed and comparisons with simulations show the 2-moments and peakedness match method to be the most accurate.

1. Introduction

Recently, there has been great interest in voice transmission in packet form[1]. In a packet voice network, all speech signals are digitized, coded, packetized, then multiplexed together and sent to a packet switching network through a high speed line. Significant research effort is currently being devoted to the performance of statistical multiplexers[2]–[6]. The voice-packet arrival process to the multiplexer, which is the superposition of the streams from many voice sources, is a fairly complex process. The complexity is primarily due to the bursty nature (high variability) of a packet arrival process from a single voice source.

References [3,5] use an approach based on the two-moment approximations as in the queueing network analyzer (QNA) [7], and the superposition process is approximated by a renewal process. In [6], the superposition arrival process is approximated by a two-state Markov-modulated Poisson process (MMPP)1, and the resulting queueing model is solved using matrix-analytic methods[8].

In this paper, the packet arrival process from a single voice source is first approximated by a simple renewal process called an interrupted Poisson process (IPP) [9]. Then the overall packet arrival process from $N$ voice sources is approximated by the exact superposition of $N$ identical independent IPPs, which will be henceforth referred to as the N-IPP. The N-IPP is an $(N + 1)$-state MMPP. Therefore, the packet voice multiplexer is modeled by feeding the N-IPP into a single-server queue with unlimited waiting room and the first-in-first-out (FIFO) service discipline. It is assumed that the service times are independent of the arrival process and i.i.d. with a general distribution. (The service-time distribution is often deterministic.) The steady-state probabilities of this queueing system are analyzed by means of the supplementary variable technique. All eigenvalues and eigenvectors related to the equilibrium equation are given explicitly. Transforms of the queue length distributions and the delay distributions are obtained. The moments of these performance measures are also obtained.

To determine 3 defining parameters of the IPP from statistical characteristics of the packet arrival process from a single voice source, the following three methods are considered: 1) the mean interval method; 2) the 3-moments method; and 3) the 2-moments and peakedness method. Comparisons with simulations show the 2-moments and peakedness method to be the most accurate.

One advantage of the method presented here is that we need no characterization of the superposition process, but the characterization of a component process. This method is broadly applicable to analyzing superpositions of bursty traffic streams (see[10] and references cited there).

1The MMPP is a doubly stochastic Poisson process where the rate process is determined by the state of a continuous-time Markov chain.
2. Characterization of a Packetized Voice Process

The packet stream from a single voice source is modeled by arrivals at fixed intervals of $T$ ms during talkspurts and no arrivals during silences (Fig. 1). The talkspurt and silence durations can be assumed to be (approximately) exponentially distributed, which implies that the packet arrivals from single voice source form a renewal process with interarrival time distribution given by

$$F(t) = [(1 - \alpha T) + \alpha T(1 - e^{-\beta(t-T)})]U(t-T), \quad (1)$$

where $\alpha^{-1}$, $\beta^{-1}$ are the mean talkspurt and the mean silence period, respectively, and $U(t)$ is the unit step function. The parameters used in this paper are given by $\alpha^{-1} = 352$ ms, $\beta^{-1} = 650$ ms, and $T = 16$ ms, which corresponds to the same single-source model used in [5,6]. The statistical properties of the stream such as moments of interarrival times and the index of dispersion are shown there. An another important characteristics of an arrival stream is the peakedness generalized by Eckberg [11]. The exponential peakedness function $z_{exp}(\mu)$ is defined as the variance/mean ratio of the number of busy servers in a fictitious infinite exponential server system with service rate $\mu$, to which the arrival stream is hypothetically offered. In the single voice source case, $z_{exp}(\mu)$ is given by

$$z_{exp}(\mu) = \left(1 - \left[1 - \alpha T + \frac{\alpha T \beta}{\mu + \beta}\right] e^{-\mu T}\right)^{-1} - \frac{\beta}{\mu(\alpha + \beta)T}. \quad (2)$$

It should be pointed out that the methodology we present is not restricted to the above voice source model. This methodology is applicable to any bursty renewal process.

3. Superposition of the IPPs

The IPP is a Poisson process which is alternately turned on for an exponentially distributed time and then turned off for another independent exponentially distributed time. Let $\gamma^{-1}$ be the mean on-time, $\omega^{-1}$ be the mean off-time, and $\lambda$ be the arrival rate during on-time. The arrival process from a single voice source is approximated by the IPP, and the aggregated arrival process is approximated by the superposition of the IPPs (N-IPP). Let the state (phase) of the N-IPP at time $t$ be described by $J(t)$ where $J(t)$ is the number of IPPs in their 'on' state. $J(t)$ is an $(N+1)$-state continuous-time Markov chain — a birth and death process — and the N-IPP is an MMPP. When the chain is in state $j$, ($j = 0, 1, \ldots, N$), the birth rate is $(N - j)\omega$, the death rate is $j\gamma$, and the arrival process of the N-IPP is a Poisson process with rate $j\lambda$.

4. Analysis of the N-IPP/G/1 queue

This section discusses an analysis of a single-server queue whose input is the N-IPP (Fig. 2). The service times of customers (packets) are i.i.d. with the mean $h$ and the Laplace-Stieltjes transform (LST) $H^*(s)$. This queueing model is a special case of MMPP/G/1 [12], N/G/1 [8], and semi-Markov/G/1 [13] queues. Transforms and moments of the steady-state probability distributions are obtained using the supplementary variable technique.

The state of this queueing system at time $t$ is given by

$I(t) = \text{number of customers in the system (waiting and being served)},$

$J(t) = \text{phase of the N-IPP, and}$

$X(t) = \text{remaining service time for the customer being served}.$

The traffic intensity $\rho$ is given by $\rho = N \lambda h \gamma / (\gamma + \omega)$. We discuss the stationary joint probability densities at an arbitrary epoch under the assumption $\rho < 1$. We define

$$p_{0j} \overset{\text{def}}{=} \lim_{t \to \infty} \Pr\{I(t) = 0, J(t) = j\}, \quad j = 0, \ldots, N; \quad (3)$$

$$p_{ij}(x) \overset{\text{def}}{=} \lim_{t \to \infty} \Pr\{I(t) = i, J(t) = j, x \leq X(t) < x + dx\}, \quad \begin{cases} i = 1, 2, \ldots; & j = 0, \ldots, N; \quad x \geq 0. \end{cases} \quad (4)$$
We introduce the generating function and the Laplace transform:

\[ p_j(z, x) \overset{\text{def}}{=} \sum_{i=1}^{\infty} p_{ij}(x) z^i, \quad \Psi_j(z, s) \overset{\text{def}}{=} \int_0^\infty e^{-sx} p_j(z, x) \, dx. \]

Let \( p_0, \Psi_j(z, s), \phi(z) \) be the \((N+1)\)-row vectors whose components are \( p_{0j}, \Psi_j(z, s), p_j(z, 0) \) for \( j = 0, \ldots, N \), respectively. From the equilibrium equation, the following equation is obtained \(^2\):

\[
\Psi_j(z, s) (sI + R(z)) + H^*(s)p_0 R(z) - \left( 1 - \frac{H'(s)}{z} \right) \phi(z) = 0, \quad |z| \leq 1, \quad \text{Re} s \geq 0, \tag{5}
\]

where \( I \) is an \((N+1) \times (N+1)\) unit matrix and \( R(z) \) is an \((N+1) \times (N+1)\) tridiagonal matrix whose components are

\[
R_{j,j+1} = j \gamma, \quad R_{j,j} = j \lambda(z) - [(N-j) \omega + j \gamma], \quad R_{j,j+1} = (N-j) \omega, \tag{6}
\]

Let \( \xi_m(z) \) for \( m = 0, \ldots, N \), denote eigenvalues of \( R(z) \) and let \( \xi_0(z) \) be the maximum eigenvalue for \( 0 < |z| \leq 1 \) related to the Perron-Frobenius eigenvalue (see Theorem 5.3.1 in [8]). Let \( u_m(z) \), \( v_m(z) \) be the right and left eigenvectors of \( R(z) \) respectively corresponding to \( \xi_m(z) \), and satisfy \( v_m(z) u_m(z) = 1 \).

Let \( U(z) \) be an \((N+1) \times (N+1)\) matrix whose columns are \( u_m(z) \), and let \( V(z) \) be an \((N+1) \times (N+1)\) matrix whose rows are \( v_m(z) \).

**Theorem 1** If all eigenvalues \( \xi_m(z) \) for \( m = 0, \ldots, N \) are distinct, then the solution of (5) is obtained as follows:

\[
\Psi(z, s) = p_0 U(z) D(z, s) V(z), \tag{7}
\]

where

\[
D(z, s) \overset{\text{def}}{=} \text{diag} \left( \ldots, \frac{-z \xi_m(z) [H^*(s) - H^*(-\xi_m(z))]}{[z - H^*(-\xi_m(z))] [s + \xi_m(z)]}, \ldots \right). \tag{8}
\]

Vector \( p_0 \) is obtained from the following linear equation:

\[
p_0 (e, u_1(z_1), \ldots, u_N(z_N)) = (1 - \rho, 0, \ldots, 0), \tag{9}
\]

where \( e \) denotes a column vector whose components are all 1, and \( z_m, m = 1, \ldots, N \) are solutions of equations such as

\[
z - H^*(-\xi_m(z)) = 0, \quad |z| < 1, \quad m = 1, \ldots, N. \tag{10}
\]

**Proof.** Multiplying (5) by \( u_m(z) \) from right and letting \( s = -\xi_m(z) \) we get

\[
\xi_m(z) H^*(-\xi_m(z)) p_0 u_m(z) - \left( 1 - \frac{H^*(-\xi_m(z))}{z} \right) \phi(z) u_m(z) = 0. \tag{11}
\]

From the above equation we obtain a formula for \( \phi(z) \) and putting this formula into (5) yields (7).

Let \( z = z_m \), then (11) reduces to

\[
p_0 u_m(z_m) = 0. \tag{12}
\]

Equation (9) is obtained from the normalized condition and (12).

We obtain all eigenvalues and eigenvectors explicitly by the generating function method \([14]\).

**Theorem 2**

\[
\xi_m(z) = \omega [(r_2 - r_1)m + N(r_1 - 1)], \tag{13}
\]

\[
= \frac{N[\lambda(z-1) + \omega + \gamma]}{2} + (N-2m) \sqrt{[\lambda(z-1) + \omega - \gamma]^2 + 4\omega \gamma}, \tag{14}
\]

\[
u_m(z) = \left( A_m \sum_{j=0}^{m} \binom{m}{j} (N-m) \frac{r_1^{j-i} r_2^i}{j-i+1}; \quad 0 \leq j \leq N \right)^T, \tag{15}
\]

\[
u_m(z) = \left( B_m \sum_{j=0}^{m} \binom{m}{j} (N-m) \frac{r_2^{m-i} (-r_2^{N-m})(j-i)}{j-i+1}; \quad 0 \leq j \leq N \right). \tag{16}
\]

\(^2\)If we use (1.3.4) in [8] as the definition of \( R(z) \), then equation(5), Theorem 1 and Theorem 4 hold true for the general \( N/G/1 \) queue also \([15]\)
where \( r_1 = r_1(z), r_2 = r_2(z) \) are defined by

\[
\begin{align*}
    r_1 & \overset{\text{def}}{=} (\lambda(z - 1) + \omega - \gamma + d(z))/2\omega, \\
    r_2 & \overset{\text{def}}{=} (\lambda(z - 1) + \omega - \gamma - d(z))/2\omega,
\end{align*}
\]

and \( A_m, B_m \) are arbitrary constants satisfying \( \nu_m(z)u_m(z) = 1 \).

**Proof.** Let \( \xi = \xi(z) \) be some eigenvalue of \( R(z) \) and let \( v(z) \) be the associated left eigenvector. That is

\[
v(z)R(z) = \xi(z)v(z).
\]

Equation (19) is also

\[
(N\omega - (j - 1)\omega)v_{j-1} + [j(\lambda(z - 1) + \omega - \gamma) - (N\omega + \xi)]v_j + (j + 1)\gamma v_{j+1} = 0,
\]

for \( j = 0, \ldots, N \), where \( v_j \) is a component of \( v(z) \).

Let \( V(z, x) \) denote the generating function of \( v(z) \), i.e., \( V(z, x) \overset{\text{def}}{=} \sum_{j=0}^{N} v_j(z)x^j \). By multiplying (20) by \( x^j \) and summing over \( j \) we obtain the following equation:

\[
\frac{\partial V(z, x)}{\partial x} \bigg|_{V(z, x)} = \frac{N\omega x - (N\omega + \xi)}{\omega x^2 - (\lambda(z - 1) + \omega - \gamma)x - \gamma}. \]  

We define \( r_1 \) and \( r_2 \) to be the distinct real roots, \( r_1 > r_2 \) for \( 0 \leq z \leq 1 \), of the quadratic in the denominator of the right-hand side. Equation (21) may now be written as

\[
\frac{\partial V(z, x)}{\partial x} \bigg|_{V(z, x)} = \frac{c_1}{x - r_1} + \frac{c_2}{x - r_2},
\]

where the residues are computed to be

\[
c_2 = N - c_1, \quad c_1 = \frac{N(1 - r_1) + \xi/\omega}{r_1 - r_2}.
\]

The solution to (21) is

\[
V(z, x) = B(x - r_1)^{c_1}(x - r_2)^{N-c_1},
\]

where \( B = B(z) \) is an integration constant.

Observe that by its definition, \( V(z, x) \) is a polynomial in \( z \) of degree \( N \). Since \( r_1 \) and \( r_2 \) are distinct, this is possible if and only if \( c_1 \), defined in (23), is an integer in \([0, N]\). Denoting this integer by \( m \), from (23) we obtain (13). From (24) we obtain the left eigenvectors (16). Right eigenvectors \( u_m \) are also obtained from the left eigenvectors by the method of symmetrizing \( R(z) \) [14].

**Corollary 1**

\[
\xi_N(z) < \xi_{N-1}(z) < \cdots < \xi_0(z) \leq 0 \quad \text{for} \quad 0 \leq z \leq 1,
\]

and \( \xi_0(z) = 0 \) if and only if \( z = 1 \).

**Corollary 2** Equations (10) have unique real solutions in \( 0 < z < 1 \).

**Theorem 3** The mean number of customers in the system \( L \) is given by

\[
L = \frac{\rho^2(1 + c_v^2) + \xi_0''(1)h + 2p_0u_0'(1)}{2(1 - \rho)} + \rho,
\]

where \( c_v \) is the coefficient of variation of the service time and \( u_0(x) \) is normalized such that

\[
v_0(x)u_0(x) = v_0(x)e = 1,
\]

therefore

\[
u_0'(1) = \frac{\lambda(j(\alpha + \beta) - N\alpha)}{(\alpha + \beta)^2}; \quad 0 \leq j \leq N \]

\[^T \]

3.1B.2.4
Theorem 4 The steady-state LST of the virtual waiting time vector $V^*(s)$ is given by

$$V^*(s) = \begin{cases} sp_0(sI + R[H^*(s)])^{-1} & \text{if } s > 0 \\ \theta & \text{if } s = 0, \end{cases}$$

(29)

where $\theta$ is the steady-state probability vector of the Markov chain $\{J(t)\}$.

Proof. From the definition we have

$$V^*(s) = p_0 + \sum_{i=1}^{\infty} (\int_0^\infty p_i(x)e^{-sx}dx)[H^*(s)]^{-1}.$$ 

(30)

Letting $z = H^*(s)$ in (5), we obtain (29).

Remark. This is the same theorem as Theorem 5.2.1 in [8], and the formulas for the first two moments of the virtual waiting time are shown there. From the relation formula between the virtual waiting time and the actual waiting time in [16, page 129], we can also obtain the moments formulas for the actual waiting time.

5. Approximating a Packetized Voice Process

We use the IPP as a canonical process which approximates the packet arrival process from a single voice source. The usual scheme for approximating a stream by a canonical process is fitting certain characteristic quantities of the stream to those of the canonical process [10]. We discuss the following three method for deciding the 3 defining parameters of the IPP $\{\gamma, \omega, \lambda\}$. These methods require no numerical inversion technique.

1. the mean interval method: The mean on-time, off-time and the mean arrival interval during on-time of the IPP are matched with the mean talkspurt, silence duration, and the packet arrival interval during talkspurs, respectively. This means letting $\gamma = \alpha$, $\omega = \beta$, $\lambda = 1/T$.

2. 3-moments method: The first 3-moments of inter arrival time distributions of the IPP are matched with those of the packet arrival process. Let $m$, $c$, $\kappa$ be the mean, the coefficient of variation, and the third central moment $/m^3$ of the packet inter arrival time, respectively. We get:

$$\lambda = \frac{2(\kappa - 3c^2 + 1)}{(2\kappa - 3c^4 - 1)m},$$

(31)

$$\omega = \frac{3(c^2 - 1)}{(\kappa - 3c^2 + 1)m},$$

(32)

$$\gamma = \frac{9(c^2 - 1)^3}{(\kappa - 3c^2 + 1)(2\kappa - 3c^4 - 1)m}.$$ 

(33)

3. 2-moments and peakedness method: The first 2-moments of interarrival time distributions and a peakedness are matched. Let $z$ denote a peakedness value defined by (2). We get:

$$\lambda = \frac{1}{m} + \frac{(c^2 - 1)(z - 1)\mu}{c^2 + 1 - 2z},$$

(34)

$$\omega = \frac{2(z - 1)\mu}{(z - 1)(c^2 - 1)\mu + c^2 + 1 - 2z},$$

(35)

$$\gamma = \frac{2\mu^2m(c^2 - 1)(z - 1)^2}{[\mu m(c^2 - 1)(z - 1) + c^2 + 1 - 2z](c^2 + 1 - 2z)}.$$ 

(36)

In [17], Eckberg suggests that the service time of the fictitious infinite server system should be chosen to characterize the reaction time of the arrival stream with the delay system. For tractability, this paper assumes that the service time of the infinite server is the exponential distribution whose service rate $\mu$ is the same as the mean arrival rate of the stream.

Table 1 shows IPP parameters obtained by these methods. Mean packet waiting times by these approximation methods are compared with simulation results[5] in Table 2 and Fig. 3, where the service-time distribution is deterministic with mean $h = 1/3$ ms.

3.1B.2.5
6. Conclusion

This paper presents an analysis of a model for a packetized voice multiplexer based on the superposition of IPPs. Extension of this method to a statistical multiplexer with packetized voice sources and (compound) Poisson arrival data traffic is straightforward. Although this method has been applied to a packetized voice multiplexer, it can also be used for analyzing superpositions of more general traffic streams.

Acknowledgements

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References

Table 1: A COMPARISON OF THREE MATCHING METHODS.
The defining parameters and statistical properties of approximated IPPs where the matching characteristics are indicated by *.

<table>
<thead>
<tr>
<th>Mean intervals</th>
<th>Moments of the interarrival time</th>
<th>Peakedness</th>
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<tbody>
<tr>
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<td>On-time</td>
<td>Off-time</td>
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<tr>
<td>Actual voice process</td>
<td>352</td>
<td>650</td>
</tr>
<tr>
<td>Method 1</td>
<td>352*</td>
<td>650*</td>
</tr>
<tr>
<td>Method 2</td>
<td>378.8</td>
<td>625.2</td>
</tr>
<tr>
<td>Method 3</td>
<td>581.8</td>
<td>708.8</td>
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</table>

†: Mean interarrival time during on-time.

Table 2: A COMPARISON OF APPROXIMATIONS OF THE MEAN PACKET WAITING TIME.
Simulation and QNA approximation are cited from [5].

<table>
<thead>
<tr>
<th>No. of voice sources $N$</th>
<th>Traffic intensity $\rho$</th>
<th>Simulation (95% C.INT.)</th>
<th>Approximations [ms]</th>
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<td>1 2 3 QNA</td>
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3.1B.2.7