PARCEL OVERFLOWS IN QUEUES WITH MULTIPLE INPUTS

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We provide exact, computable matrix expressions for the moments of the parcel overflow from a trunk group with multiple exponential servers, a finite queueing capacity and multiple interrupted Poisson inputs. Our analysis extends previous work to the case of queues with more than two inputs. The benefits of the exact method are illustrated by showing that significant errors can occur when parcels are aggregated and not analyzed separately.

1. INTRODUCTION

This paper provides explicit matrix expressions for the moments of the parcel overflow in a queueing system with multiple exponential servers, a finite queue, and arrivals consisting of a Poisson and multiple independent interrupted Poisson processes (IPP). (In this context, the moments of an overflow stream are the moments of the number of busy trunks on a hypothetical infinite (i.i.d. exponential) trunk group to which the overflow parcel has been offered.) The methodology can be used to determine the call congestion in teletraffic networks with alternate routing. In a network with alternate routing, parcels of traffic arrive at a first-choice destination consisting of a trunk group with (possibly) a limited capability to queue calls and overflow to a sequence of alternate destinations if the first-choice destination is busy. Due to the complexity of alternate routing networks, it is necessary to employ iterative techniques, in which each destination is analyzed in isolation. At each stage of iteration a queue with multiple inputs is analyzed, and moment-matching techniques are used to approximate the parcel overflows by interrupted Poisson processes for use in the next stage of iteration.

Teletraffic networks with alternate routing have been studied extensively [2-9]. Recently, exact expressions for the parcel blocking and overflow moments have been obtained for queues with two IPP inputs [9,16]. The overflow moments are then used to approximate multiple parcel overflows by a single IPP in the next stage of iteration. In [17], we provided an exact parcel blocking analysis for queues with more than two IPP inputs and we used a covariance matching technique from [14] to calculate the inputs for the next stage of iteration. Here we solve for the overflow moments in the multiple IPP input model. Additionally, we compare our exact parcel blocking analysis with a two-IPP approximation and show that significant errors can occur when parcels are aggregated and not analyzed separately.

This paper is organized as follows. Section 2 provides background and definitions. Section 3 discusses the aspects of the trunk group queueing model which are relevant to computation of the overflow moments. Section 4 describes the computation of the overflow moments, and Section 5 provides numerical examples and comparison with other methods.

2. BACKGROUND

For use in what follows, several definitions are required. An interrupted Poisson process (IPP) is a Poisson process which is alternately turned on for an exponentially distributed length of time and then turned off for another (independent and typically with a different mean) exponentially
distributed time. The interrupted Poisson process is a special case of the Markov-modulated Poisson process (MMPP). An MMPP is a doubly stochastic Poisson process whose instantaneous rate varies according to an irreducible m-state continuous-time Markov chain. Thus, when the underlying Markov chain is in state \( k \), arrivals occur according to a Poisson process of rate \( \lambda_k \). An MMPP is characterized by the infinitesimal generator \( Q \) of the underlying Markov process and by \( \Lambda \), an \( mxm \) diagonal matrix with diagonal elements \( \lambda_1, \lambda_2, \ldots, \lambda_m \), which will be denoted by \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_m) \). We say that the MMPP is in phase \( k \), \( 1 \leq k \leq m \), when the underlying Markov process is in state \( k \). A detailed description of the MMPP with an emphasis on applicability to modeling is given in [11]. The IPP corresponds to the special case for which \( Q \) and \( \Lambda \) have the form

\[
Q = \begin{bmatrix}
-\sigma_1 & \sigma_1 \\
\sigma_2 & -\sigma_2
\end{bmatrix}, \quad \Lambda = \begin{bmatrix}
\lambda & 0 \\
0 & 0
\end{bmatrix}.
\]

(1)

The superposition of independent Markov-modulated Poisson processes is also an MMPP [11], so that the arriving traffic to the trunk group, which is the superposition of a Poisson process and several interrupted Poisson processes, is an MMPP. This MMPP has \( 2^m \) states, which describe the total number of combinations of on-and-off states of the \( m \) input IPPs. The state space, in lexicographic order, can be conveniently described using the Kronecker sum of matrices (Bellman [12]). Briefly, the Kronecker sum of two square matrices \( L \) and \( M \) of orders \( r \) and \( s \) is given by \( L \oplus M = L \otimes I_r + I_s \otimes M \). The matrix \( I_k \) denotes the identity matrix of order \( k \), and \( \otimes \) denotes the Kronecker product. The Kronecker product, \( K \otimes P \), of two matrices is the matrix with block elements \( (K_{ij}P) \). Thus, if \( K \) and \( P \) are square matrices of orders \( r \) and \( s \), \( K \otimes P \) is of order \( rs \). The Kronecker sum of \( m \) square matrices \( L_1, \ldots, L_m \), of order \( n \) is a square matrix of order \( nm \) and is defined by

\[
L_1 \oplus L_2 \oplus \cdots \oplus L_m =
(L_1 \otimes I_s \cdots \otimes I_s) + (I_s \otimes L_2 \cdots \otimes I_s) + \cdots + (I_s \otimes I_s \cdots \otimes L_m).
\]

(2)

Using this notation, the superposition of \( m \) independent MMPPs parameterized by \( (Q_i, \Lambda_i) \), \( 1 \leq i \leq m \), may be represented by the MMPP \( (Q, \Lambda) \) with

\[
Q = Q_1 \oplus Q_2 \oplus \cdots \oplus Q_m, \quad \Lambda = \Lambda_1 \oplus \Lambda_2 \oplus \cdots \oplus \Lambda_m.
\]

(3)

3. MODEL OF THE TRUNK GROUP AND ASSOCIATED QUEUE

In this section, we formulate the queueing model which is used to determine the parcel blocking, and will be used to compute the parcel overflow moments. Arrivals to the trunk group consist of a Poisson process and multiple IPPs. The trunk group and associated queue can therefore be modeled as an MMPP/M/c/c+k queue parameterized by

a. \( \lambda \), the arrival rate of the exogenously originating calls;

b. \( \mu \), the service rate, corresponding to the inverse of the mean call holding time;

c. \( c \), the number of trunks;

d. \( k \), the queueing capacity;

e. \( m \), the number of overflow parcels arriving to the queue;

f. \( (Q_i, \Lambda_i) \), \( 1 \leq i \leq m \), the parameters of the IPPs modeling the overflow parcels, where

\[
Q_i = \begin{bmatrix}
-\sigma_{i1} & \sigma_{i1} \\
\sigma_{i2} & -\sigma_{i2}
\end{bmatrix}, \quad \Lambda_i = \begin{bmatrix}
\lambda_i & 0 \\
0 & 0
\end{bmatrix}.
\]

(4)

As noted in Section 2.1, Equation (3), the input process to the queue is then an MMPP parameterized by \( (Q, \Lambda) \) with
where $I$ is the identity matrix of order $2^m$.

The MMPP/M/c/c+k queue which models the trunk group and associated queueing capacity may be studied as a Markov process on the state space $\{(j,j'), 0 \leq j \leq c+k, 1 \leq j' \leq 2^m\}$. The number $j$ corresponds to the number of calls at the destination, and $j'$ corresponds to the state of the Markov process with infinitesimal generator $Q$. It will be convenient to let $j$ denote the set of states $\{(j,j'), 1 \leq j' \leq 2^m\}$. The infinitesimal generator of this Markov process is denoted by $\tilde{Q}$ and is given by

$$
\begin{pmatrix}
0 & Q - \Lambda & \Lambda \\
1 & \mu_l & Q - \Lambda - \mu_l & \Lambda \\
2 & 2\mu_l & Q - \Lambda - 2\mu_l & \Lambda \\
\vdots & \vdots & \vdots & \vdots \\
c & c\mu_l & Q - \Lambda - c\mu_l & \Lambda \\
c+1 & c\mu_l & Q - \Lambda - c\mu_l & \Lambda \\
c+k & c\mu_l & Q - \Lambda - c\mu_l & \Lambda \\
\end{pmatrix}
$$

The steady-state vector of $\tilde{Q}$ is denoted by $\pi = (\pi_0, \pi_1, \ldots, \pi_{c+k})$, and satisfies the equations

$$
\begin{align*}
\pi_0 (Q - \Lambda) + \pi_1 \mu &= 0 \\
\pi_{i-1} \Lambda + \pi_i (Q - \Lambda - i\mu_l) + \pi_{i+1} (i+1) \mu &= 0, \quad 1 \leq i \leq c-1, \\
\pi_{c-1} \Lambda + \pi_c (Q - \Lambda - c\mu_l) + \pi_{c+1} c \mu &= 0, \quad c \leq i \leq c+k-1, \\
\pi_{c+k-1} \Lambda + \pi_{c+k} (Q - c\mu_l) &= 0.
\end{align*}
$$

Even though $\tilde{Q}$ has a large number of states, $\pi$ may be efficiently computed using block Gauss-Seidel iteration. Further details on the computational aspects of this model and examples of quantities which may be calculated using $\pi$ are discussed in [17]. In this paper, we will be interested in the parcel blocking probability for input parcel $j$, $1 \leq j \leq m$, given by

$$
\gamma_{c+k}(j) = \frac{1}{\pi_{c+k} \Lambda(j) e} \left( \sum_{i=0}^{c+k} \pi_i \Lambda(j) e \right)^{-1} \pi_{c+k} \Lambda(j) e,
$$

where

$$
\Lambda(j) = I_2 \otimes I_2 \otimes \cdots \otimes I_2 \otimes \Lambda_j \otimes \cdots \otimes \Lambda_2 = I_2 \otimes I_2 \otimes \cdots \otimes I_2.
$$

3.1 The Distribution of the Parcel Overflows

The time between the overflow of calls from the trunk group and its associated queueing may be described by the MMPP parameterized by $(\tilde{Q}, \Lambda)$, where $\Lambda = \text{diag}(0, \ldots, 0, \Lambda_j)$. This can be seen by observing that calls overflow from the destination only when all trunks and all available queueing spaces are occupied, i.e., when the number in system is $c+k$. When the system state is $(c+k, r)$, calls overflow according to a Poisson process of rate $\Lambda_{rr}$, which is the $r$-th diagonal element of the matrix $\Lambda$ given in (5).

The distribution of the parcel overflow due to the $j$-th input parcel, $0 \leq j \leq m$, follows in a similar manner, and is an MMPP parameterized by $(\tilde{Q}, \Lambda(j))$, where $\Lambda(j)$ is given by $\Lambda(j) = \text{diag}(0, \ldots, 0, \Lambda_j)$, and $\Lambda(j)$ is given in (8). The MMPP $(\tilde{Q}, \Lambda(0))$ represents the overflow from the exogenously originating Poisson stream, and has $\Lambda(0) = \lambda I$. (Note that $\Lambda = \Lambda(0) \oplus \Lambda(1) \oplus \cdots \oplus \Lambda(m)$).

5.1B.3.3
4. COMPUTATION OF THE OVERFLOW MOMENTS

The exact parcel overflow distributions developed in Section 3.1 can now be used to compute the moments of the parcel overflow. As noted in the introduction, the moments of an overflow stream are defined to be the moments of the number of busy trunks on a hypothetical infinite server (i.i.d. exponential) trunk group to which the overflow parcel has been offered. Therefore, the moments of the parcel overflow from input parcel \( j \) are the moments of the number of busy servers in an MMPP/M/\( \infty \) queue with an arrival process parameterized by \( (Q, \Lambda(j)) \) and service rate \( \mu' \) (the service rate on the hypothetical infinite server trunk group). Since we require only the moments of the number of busy servers, it is convenient to directly compute the moments by differentiating the generating function of the number of busy servers.

The MMPP/M/\( \infty \) queue may be studied as a Markov process on the state space \( \{(i, j, j'), 0 \leq i \leq \infty, 0 \leq j \leq c+k, 1 \leq j' \leq 2^m\} \). The variable \( i \) corresponds to the number of busy servers in the infinite server trunk group and the variables \( (j, j') \) correspond to the state of the Markov chain \( Q \) which describes the MMPP/M/c/c+k queue. It will be convenient to let \( i \) denote the set of states \( \{(i, j, j'), 0 \leq j \leq c+k, 1 \leq j' \leq 2^m\} \).

The steady-state vector of the MMPP/M/\( \infty \) queue, \( \mathbf{x} = (x_0, x_1, \ldots) \), satisfies the steady-state equations

\[
\begin{align*}
x_0 (Q - \Lambda(j)) + x_1 \mu' &= 0, \\
x_{i-1} \Lambda(j) + x_i (Q - \Lambda(j) - i \mu') + x_{i+1} (i+1) \mu' &= 0, \quad i \geq 1 \\
x \in \mathbb{R} &= 1.
\end{align*}
\] (9)

Defining the matrix generating function \( P(z) = \sum_{i=0}^{\infty} z^i x_i \), (9) yields the following functional equation for \( P(z) \):

\[
P(z) \left[ Q - (1-z) \Lambda(j) \right] + P'(z) (1-z) \mu' = 0. \tag{10}
\]

The \( i \)th factorial moment of the number of busy servers is given by \( f_i = P^{(i)}(1) \mathbf{e} \), where \( P^{(i)}(z) \) denotes the \( i \)th derivative of \( P(z) \), and \( P^{(0)}(1) = P(1) = \pi \). From (10), we obtain

\[
P^{(i)}(1) = i P^{(i-1)}(1) \Lambda(j) (i \mu' - Q)^{-1}.
\]

It can be shown that \( (i \mu' - Q)^{-1} \mathbf{e} = i \mu'^{-1} \mathbf{e} \), which yields

\[
\begin{align*}
f_1 &= P'(1) \mathbf{e} = (\pi \Lambda(j) \mathbf{e}) \mu'^{-1}, \\
f_i &= P^{(i)}(1) \mathbf{e} = (i-1)! \mu'^{-1} \pi \Lambda(j) \prod_{r=1}^{i-1} r \mu' - Q \Lambda(j) \mathbf{e}, \quad i \geq 2.
\end{align*}
\] (11)

Remarks:

1. Noting that \( \pi \Lambda(j) \mathbf{e} \) is the average rate of overflow due to the \( j \)th input parcel, we observe the analogy between the factorial moments of the MMPP/M/\( \infty \) queue and the factorial moments of the M/M/\( \infty \) queue which are of given by \( (\lambda \mu'^{-1})^i \).

2. See [15] for a different approach of obtaining the overflow moments.

Although the factorial moments appear complicated to compute, they can be computed iteratively in the same manner as the steady-state vector \( \pi \) in equation (6). Computation of \( f_1 \) is trivial. The second factorial moment is given by \( f_2 = \mu'^{-1} \pi \Lambda(j) \mathbf{e} \), where \( \mathbf{e} \) = \( \pi \Lambda(j) (\mu' - Q)^{-1} \). The vector \( \mathbf{e} \) can be obtained by solving the system of equations

5.1B.3.4
\[ \nu (\mu' I - \bar{Q}) = \pi \bar{\Lambda}(j), \quad \nu e = f_1. \]

Partitioning \( \nu = (\nu_0, \nu_1, ..., \nu_{c+k}) \), we obtain the following system of equations which has the same structure as (6) and can therefore be solved in the same manner using Gauss-Seidel iteration:

\begin{align*}
\nu_0 \left( \bar{Q} - \bar{\Lambda}(j) - \mu' I \right) + \nu_1 \mu &= 0, \\
\nu_{-1} \bar{\Lambda}(j) + \nu_j \left( \bar{Q} - \bar{\Lambda}(j) - i \mu I - \mu' I \right) + \nu_{j+1} (i+1) \mu &= 0, \quad 1 \leq i \leq c-1, \\
\nu_{-1} \bar{\Lambda}(j) + \nu_j \left( \bar{Q} - \bar{\Lambda}(j) - c \mu I - \mu' I \right) + \nu_{j+1} c \mu &= 0, \quad c \leq i \leq c+k-1, \\
\nu_{c+k-1} \bar{\Lambda}(j) + \nu_{c+k} \left( \bar{Q} - c \mu I - \mu' I \right) &= -\nu_{c+k} \Lambda(j), \\
\nu e &= f_1.
\end{align*}

The third factorial moment can be obtained in a similar manner as the second factorial moment. We have \( f_3 = 2 \mu^{-1} z \bar{\Lambda}(j) e, \) where \( z = \nu \bar{\Lambda}(j) (2 \mu' I - \bar{Q})^{-1} \), and the vector \( z \) satisfies

\[ z \left( 2 \mu' I - \bar{Q} \right) = \nu \bar{\Lambda}(j), \quad z e = \begin{cases} 1 \text{ for } i = 0, \end{cases} f_2, \]

and can also be computed using Gauss-Seidel iteration.

The factorial moments in (11) can now be used to approximate each parcel overflow distribution by an IPP. For example, the parameters of the IPP which is obtained from the three-moment match [1] are given by

\[ \lambda = \mu \frac{\delta_0 \delta_1 + \delta_1 \delta_2 - 2 \delta_0 \delta_2}{2 \delta_1 - \delta_0 - \delta_2}, \quad \sigma_2 = \mu' \frac{\delta_0 (\lambda' - \delta_1)}{\mu}, \quad \sigma_1 = \mu' \frac{\lambda' - \delta_0}{\delta_0}. \]

where \( \lambda, \sigma_1 \) and \( \sigma_2 \) are as in equation (1) and \( \delta_j = \frac{f_{j+1}}{f_j}, \) \( j = 0, 1, 2, \) and \( f_0 = 1. \)

5. NUMERICAL EXAMPLES AND COMPARISON WITH OTHER METHODS

In this section we compare our method for evaluating a queue with \( n > 2 \) IPP inputs to using the two-IPP results of [9,16] and aggregating \( n-1 \) inputs as one IPP in order to analyze the other IPP. For the exact parcel blocking model, the \( n \) identical inputs were generated using a three-moment match on the overflow from a primary trunk group with Poisson input, service rate 1, and no queueing capacity. (The results of Section 4 were particularized to the special case of Poisson input).

The two-IPP approximation requires that \( n-1 \) of the inputs be approximated by a single IPP. A variety of moment-matching techniques could be used for this purpose, including the three-moment match [1], the two-moment match [1], or [14]. In [9,16] the three-moment match was suggested and it is that method which we consider here. Because the input parcels are assumed to be independent, the \( i \)-th central moment of the superposition of parcel overflows is the sum of the \( i \)-th central moments of the component streams. Therefore, the superposition is approximated by a single IPP with the first three central moments \( m_r = \sum m_r(j), \quad r = 1, 2, 3, \) where the sum is taken over all but one of the input parcels and \( m_r(j) \) is the \( j \)-th central moment of the \( j \)-th input parcel.

The error introduced by aggregating \( n-1 \) IPPs by a single IPP is investigated in Figures 1 and 2. In Figure 1, the IPP inputs to the exact queueing model were generated using a three-moment match on the overflow from 60 servers with an offered load of 60 erlangs, producing a peakedness of 4.1, or from 200 servers with an offered load of 220 erlangs, producing a peakedness of 5.6. (The peakedness of a traffic stream is defined as the ratio of the variance to the mean of the
number of busy servers on an infinite (i.i.d. exponential) server group with that stream as input; it is derivable from (11). Since all inputs to the overflow trunk group are identical and independent, the peakedness of the superposed input traffic to the overflow group is the same as the peakedness of the individual inputs. In Figure 2, the input IPPs are generated from the overflow from 60 erlangs (as in Figure 1). However, the inputs are normalized so that \( \lambda' = \lambda/n \), \( \sigma_1' = \sigma_1/n \), \( \sigma_2' = \sigma_2/n \), where \( \lambda, \sigma_1, \) and \( \sigma_2 \) denote the IPP parameters obtained for the unnormalized inputs. Therefore, the total arrival rate to the overflow queue is held constant as \( n \) increases and the superposed arrival process tends to Poisson as \( n \to \infty \). In both Figures 1 and 2 the service rate was normalized to be 1 and the queue has no Poisson input (\( \lambda = 0 \)) and no queuing capacity (\( k = 0 \)).

In all cases, the accuracy of the two-IPP approximation was evaluated by computing the percent error in the parcel blocking. We found that

a. For a fixed peakedness, e.g., without normalizing the inputs, the percent error in the two-IPP approximation increases with \( n \). This is to be expected, since an IPP (which is a renewal process) is being used to approximate the superposition of \( n-1 \) IPPs (which is non-renewal). The comparison of the exact and approximate methods indicates that the superposition becomes "further from renewal" as \( n \) increases.

b. When the inputs are normalized, the percent error in the parcel blocking increased with \( n \) for those values which were examined. Since the superposition of the normalized inputs tends to Poisson, the percent error in the parcel blocking must begin to decrease and tend to zero for sufficiently large \( n \).

c. The percent error in the two-IPP approximation increases with the peakedness of the input stream.

d. The effect of the number of superposed streams on the error seems to be stronger than the effect of peakedness (see Figure 2).

e. The percent errors tend to be smaller for very small or very large values of the parcel blocking. This is to be expected since both the exact and approximate parcel blockings tend to 0 (1) as the arrival rate to the overflow queue tends to 0 (\( \infty \)).

6. CONCLUSIONS

We have provided exact, computable matrix expressions for the moments of the parcel overflow from a trunk group with multiple exponential servers, a finite queuing capacity and multiple IPP inputs. This analysis, in conjunction with [17], extends previous work to the case of queues with more than two inputs. We compared our exact analysis with a two-IPP approximation and observed significant errors in the approximation. Although we used a specific method for approximating \( n-1 \) IPPs with a single IPP (the three-moment match), similar results hold for other approximation techniques. This is to be expected since there is inherent error in approximating a non-renewal process (the superposition of \( n-1 \) IPPs) by a renewal process.

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Figure 1. Percent error in parcel blocking for unnormalized inputs.

Figure 2. Percent error in parcel blocking for normalized inputs.
REFERENCES


