NEW APPROXIMATION FOR GENERAL DISTRIBUTIONS BY N-STAGE MARKOV PROCESSES
BASED ON CONTINUED FRACTIONS

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This paper presents a new stage method for approximating a non-Markovian distribution which is commonly used in traffic and reliability design of telecommunications systems. The proposed method requires only a few stages for approximating the distribution with an arbitrary coefficient of variation.

1. INTRODUCTION

In the traffic and reliability design for telecommunications systems, the "stage method"[1] is commonly used. In this method, a non-Markovian distribution is approximated by a Markovian distribution such as Erlang or hyper-exponential. However, the number of stages become extremely large when the coefficient of variation for the distribution is small. In such cases, the stage method is not sufficient for practical use.

This paper proposes a new stage method for approximating a general (non-Markovian) distribution. The proposed method is to approximate the Laplace transform for the original function instead of to approximate the original function directly. The approximation function is generally restricted to the non-negative function class in the stochastic model. However, the approximation function in this method is allowed to the real function class considering the stochastic model as the flow model. First of all, the Taylor series of the Laplace transform for the original functions is derived and it can be expressed in continued fractions. Then, the approximation Laplace function is obtained by terminating the continued fractions in finite terms. Finally, the approximation Laplace function is represented by an n-stage Markov process with negative branching probabilities. In this method, only a few stages are needed for approximating a general distribution with an arbitrary coefficient of variation.
2. NOTATION

Let us define the following notation.

\( g(t) \): The probability density function approximated here.

\( g^*(s) \): The Laplace transform of \( g(t) \).

\( \mu_k \): The \( k \)-th moment of \( g(t) \).

\( \Delta \) \[ \frac{d^k}{ds^k} g^*(s) \big|_{s=0} = \int_0^\infty tkf(t) \, dt \ , \]

\( \gamma_k \) \[ \frac{(-1)^k}{k!} \frac{d^k}{ds^k} g^*(s) \big|_{s=0} \, , \]

\( f_n(t) \): The approximation function of \( g(t) \) which has \( n \)-stages.

\( f_n^*(t) \): The Laplace transform of \( f_n(t) \).

The continued fractions can be described conveniently in the following form:

\[
\frac{A_1}{B_1 + \frac{A_2}{\frac{A_3}{B_3 + \cdots}}} = \frac{A_1}{B_1 + \frac{A_2}{\frac{A_3}{B_3 + \cdots}}}
\]

3. REPRESENTATION OF THE LAPLACE TRANSFORM BY CONTINUED FRACTIONS

The Taylor series for \( g^*(s) \) can be represented by

\[ g^*(s) = 1 + \gamma_1 s + \gamma_2 s^2 + \cdots + \gamma_k s^k + \cdots \quad . \ (1) \]

Equation (1) can be expressed in continued fractions form as follows;

\[ g^*(s) = \frac{1}{2} + \frac{d_1 s}{1} + \frac{d_2 s}{1} + \cdots \quad , \ (2) \]

where

\[ d_{2n-1} = -\frac{H_n^{(1)} H_{n-1}^{(0)}}{H_{n-1}^{(1)} H_n^{(0)}} \ , \quad (3) \]

\[ d_{2n} = -\frac{H_{n+1}^{(0)} H_{n-1}^{(1)}}{H_n^{(0)} H_n^{(1)}} \ , \quad n=1,2,\cdots . \quad (4) \]
and $H_n^{(k)}$ (Hankel's determinant) is defined by \[2\]

\[
H_n^{(k)} = \begin{vmatrix} \gamma_k & \gamma_{k+1} & \cdots & \gamma_{k+n-1} \\ \gamma_{k+1} & \gamma_{k+2} & \cdots & \gamma_{k+n} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{k+n-1} & \gamma_{k+n} & \cdots & \gamma_{k+2n-2} \end{vmatrix}
\] (5)

The n-stage approximation function $f_n^*(s)$ for $g^*(s)$ is obtained by terminating the continued fractions $g^*(s)$ in $2n$ terms. $f_n^*(s)$ can then be represented by the rational function,

\[
f_n^*(s) = \frac{p_{2n}(s)}{q_{2n}(s)} = \frac{a_{n-1}s^{n-1} + \cdots + a_1s + a_0}{s^n + b_{n-1}s^{n-1} + \cdots + b_1s + b_0}
\] (6)

where $p_{2n}(s)$ and $q_{2n}(s)$ are recursively given by

\[
p_m(s) = sd_{m-1}p_{m-2}(s) + p_{m-1}(s), \quad m=0,1,2,\ldots
\] (7)

\[
q_m(s) = sd_{m-1}q_{m-2}(s) + p_{m-1}(s), \quad m=0,1,2,\ldots
\] (8)

\[
p_0 = 0, p_1 = 1, q_0 = 0, q_1 = 1.
\]

**Remarks.**

1) From the characteristics of the continued fractions [2], it follows that:

\[
\left(\frac{d}{ds}\right)^i g^*(s) \bigg|_{s=0} = \left(\frac{d}{ds}\right)^i f_n^*(s) \bigg|_{s=0} , i=0,1,\ldots,2n-1.
\] (9)

Hence, the $i$-th moment of $g(t)$ equals to the $i$-th moment of $f_n(t)$ for $i=0,1,\ldots,2n-1$.

2) In the stage method, $2^R$-stages are required to approximate the distribution with the coefficient of variation of $g$. Hence, the number of stages in the stage method is much larger than that in our method when the coefficient of variation is small.

4. REPRESENTATION OF MARKOVIAN PROCESSES BY RATIONAL FUNCTIONS

The approximation rational function $f_n^*(s)$ in equation (3) can be represented by an $n$-stage Markovian process with negative branching probabilities in Fig.1. Generally, the branching probabilities should be all positive in the physical meaning in stochastic models. However, this method is approximately considered the stochastic model as a flow model in which negative branching probabilities are permitted.
Let \( h_i(t) \) denote the transition probability from state 0 to state \( i \) in 
\( (t, t+\text{dt}) \). The Laplace transform \( h_i^*(s) \) for \( h_i(t) \) is given by

\[
A^*(s) \begin{bmatrix} h_1^*(s) \\ h_2^*(s) \\ \vdots \\ h_{n+1}^*(s) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},
\]

where

\[
A^*(s) =
\begin{bmatrix}
1 & -\frac{\lambda_2}{s+\lambda_2} & -\frac{\lambda_3}{s+\lambda_3} & \cdots & -\frac{\lambda_{n-1}}{s+\lambda_{n-1}} & -\frac{\lambda_n}{s+\lambda_n} & 0 \\
-\frac{\lambda_1}{s+\lambda_1} & 1 & 0 & \cdots & 0 & 0 & 0 \\
0 & -\frac{\lambda_2}{s+\lambda_2} & 1 & \cdots & 0 & 0 & 0 \\
0 & 0 & -\frac{\lambda_3}{s+\lambda_3} & \cdots & 0 & \cdots & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & -\frac{\lambda_{n-1}}{s+\lambda_{n-1}} & -\frac{\lambda_n}{s+\lambda_n} & 0 & 1 \\
-\frac{\lambda_1}{s+\lambda_1} & -\frac{\lambda_2}{s+\lambda_2} & -\frac{\lambda_3}{s+\lambda_3} & \cdots & -\frac{\lambda_{n-1}}{s+\lambda_{n-1}} & -\frac{\lambda_n}{s+\lambda_n} & s+\lambda_{n-1} \gamma_{ne} 1
\end{bmatrix}
\]

From the definition,

\[
f_n^*(s) = h_{n+1}^*(s).
\]
From (10), (11), (12),

\[
\begin{vmatrix}
  s^+ \lambda_1 & -\lambda_2 & \gamma_{21} & -\lambda_3 & \gamma_{31} & \cdots & -\lambda_n & \gamma_{n1} & 1 \\
  -\lambda_1 & s^+ \lambda_2 & 0 & \cdots & 0 & 0 \\
  0 & -\lambda_2 & s^+ \lambda_3 & \cdots & 0 & 0 \\
  0 & 0 & -\lambda_3 & \gamma_{34} & \cdots & 0 & 0 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  0 & 0 & 0 & \cdots & s^+ \lambda_n & 0 \\
  -\lambda_1 & \gamma_{1e} & -\lambda_2 & \gamma_{2e} & -\lambda_3 & \gamma_{3e} & \cdots & -\lambda_n & \gamma_{n\lambda} & 0
\end{vmatrix}
\]

\[
f_n^*(s) = \frac{s^+ \lambda_1 - \lambda_2 \gamma_{21} - \lambda_3 \gamma_{31} - \cdots - \lambda_n \gamma_{n1}}{s^+ \lambda_1 - \lambda_2 \gamma_{21} - \lambda_3 \gamma_{31} - \cdots - \lambda_n \gamma_{n1} 0}
\]

\[
= \left( (\gamma_{1e} \lambda_1) s^{n-1} + (\gamma_{1e} \lambda_1 - \sum_{1<i<n} \lambda_i + \gamma_{12} \gamma_{2e} \lambda_1 \lambda_2) s^{n-2} + \right.
\]

\[
\left. + (\gamma_{1e} + \gamma_{12} \gamma_{2e} + \cdots + \gamma_{12} \gamma_{3e} + \cdots + \gamma_{n-1} \gamma_{ne}) \lambda_1 \cdots \lambda_n \right) /
\]

\[
\left( s^n + (\sum_{1<i<n} \lambda_i) s^{n-1} + (\sum_{1<i<j<n} \lambda_i \lambda_j - \gamma_{12} \gamma_{2e} \lambda_1 \lambda_2) s^{n-2} + \right.
\]

\[
\left. + (1 - \gamma_{12} \gamma_{2e} + \gamma_{12} \gamma_{23} \gamma_{3e} - \cdots - \gamma_{12} \gamma_{3e} - \cdots - \gamma_{n-1} \gamma_{n\lambda}) \lambda_1 \cdots \lambda_n \right)
\]

(13)

The parameters in Fig.1 can be obtained by the coefficient in equation (6) which corresponds to the coefficient in equation (13). Especially, if we set

\[
\lambda_1 = 1/K, \quad \lambda_2 = 3/K, \quad \lambda_i = (2i-1)/K, \quad K = n^2/b_{n-1}.
\]

For example, in the case of unit distribution with a mean of 1, the Taylor series for the Laplace transform of it, \(g^*(s)\), can be represented by

\[
g^*(s) = 1 - s + \frac{1}{2} s^2 + \frac{1}{6} s^3 + \frac{1}{24} s^4 + \frac{1}{120} s^5 + \cdots
\]

From equation (6), the 3-stage approximation function \(f_3^*(s)\) for \(g(s)\) is given by

\[
f_3^*(s) = \frac{3s^2 - 24s + 60}{s^3 + 9s^2 + 36s + 60}.
\]

\(f_3^*(s)\) can be represented by the 3-stage Markovian process with negative branching probabilities in Fig.2.
5. NUMERICAL EXAMPLES

In this section, the proposed approximation is numerically validated by comparing it to the exact solution and to the approximation by the stage method. Table 1 shows the call congestion probability of the D/M/S/S model which is already solved exactly by the imbedded Markov chain method. Table 2 shows the call congestion probability of the M/D/1/2 model which is also solved exactly. It is observed that our results are sufficiently accurate, and our 3-stage result is more accurate than the results obtained by the stage method having 100 or 10000 stages.

Table 1. Call congestion probabilities of D/M/S/S.

<table>
<thead>
<tr>
<th>offered traffic[erl]</th>
<th>24</th>
<th>28</th>
<th>32</th>
<th>36</th>
<th>40</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Exact</strong></td>
<td>26</td>
<td>6639</td>
<td>217373</td>
<td>1720873</td>
<td>5549006</td>
<td>19466636</td>
</tr>
<tr>
<td><strong>Our</strong></td>
<td>26</td>
<td>6644</td>
<td>217397</td>
<td>1720898</td>
<td>5549017</td>
<td>19466639</td>
</tr>
<tr>
<td><strong>approx. 2-stage</strong></td>
<td>19</td>
<td>6023</td>
<td>211891</td>
<td>1710522</td>
<td>5541115</td>
<td>19465240</td>
</tr>
<tr>
<td><strong>Stage</strong></td>
<td>26</td>
<td>6645</td>
<td>217456</td>
<td>1721160</td>
<td>5549406</td>
<td>19466903</td>
</tr>
<tr>
<td><strong>method</strong></td>
<td>30</td>
<td>7174</td>
<td>215715</td>
<td>1749627</td>
<td>5588974</td>
<td>19493117</td>
</tr>
<tr>
<td><strong>10000-stage</strong></td>
<td>$\times 10^{-8}$</td>
<td>$\times 10^{-8}$</td>
<td>$\times 10^{-8}$</td>
<td>$\times 10^{-8}$</td>
<td>$\times 10^{-8}$</td>
<td>$\times 10^{-8}$</td>
</tr>
<tr>
<td><strong>100-stage</strong></td>
<td>$\times 10^{-8}$</td>
<td>$\times 10^{-8}$</td>
<td>$\times 10^{-8}$</td>
<td>$\times 10^{-8}$</td>
<td>$\times 10^{-8}$</td>
<td>$\times 10^{-8}$</td>
</tr>
</tbody>
</table>

(* Number of servers = 42)
Table 2. Call congestion probabilities of M/D/1/2.

<table>
<thead>
<tr>
<th>offered traffic[erl]</th>
<th>40</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact</td>
<td>9627448</td>
<td>26894142</td>
</tr>
<tr>
<td>Our approx. 4-stage</td>
<td>9627447</td>
<td>26894129</td>
</tr>
<tr>
<td>3-stage</td>
<td>9627547</td>
<td>26896541</td>
</tr>
<tr>
<td>2-stage [4]</td>
<td>9589041</td>
<td>26666666</td>
</tr>
<tr>
<td></td>
<td>$\times 10^{-8}$</td>
<td>$\times 10^{-8}$</td>
</tr>
</tbody>
</table>

6. CONCLUSIONS

This paper presents a new stage method for approximating an arbitrary distribution. The proposed method requires only a few stages for obtaining a distribution with an arbitrary coefficient of variation and the accuracy of this approximation is quite well, so it is sufficient for practical use.

References