A METHOD FOR ANALYZING CIRCUIT-SWITCHED NETWORKS WITH MULTIPLE BIT RATE CLASSES

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The characteristics of individual parcels of traffic in circuit-switched networks with multiple bit rate classes are studied in this paper. Both exact and approximate methods are proposed for the analysis of overflow traffic from a first-choice trunk group. Moreover, two methods of calculating approximate individual loss probabilities on a final trunk group are presented. Numerical examples are given, in terms of which the accuracy of the methods is discussed.

1. INTRODUCTION

The main objective of ISDN is to provide various new telecommunication services in addition to the telephone service with maximum efficiency and cost effectiveness by allowing heterogeneous types of traffic to share network facilities. Various schemes of integrating services in a network have been studied. A circuit-switched network which simultaneously serves multiple classes of traffic with different bit rates is one of the configurations of ISDN. With a view to realizing such advanced networks, switching systems capable of handling traffic with different bit rates are currently developed in many locations. It should be, however, pointed out that network design, dimensioning, and network management have not yet been studied fully for networks serving such heterogeneous types of traffic. Studies on these issues are thus quite significant and urgent from the viewpoint of construction of cost effective telecommunications networks in the future. One of the important problems arising when heterogeneous types of traffic are considered is how to evaluate service qualities of individual parcels of traffic in the network. Gimpelson[1], Lindberger[5] and Ramaswami et al.[8] studied loss probabilities of individual parcels in a single trunk group. In order to evaluate the overall loss probabilities in alternate routing networks, however, it is necessary to develop an efficient method of evaluating the characteristics of traffic overflowing a trunk group.

The present paper is to propose methods for analyzing the overall loss probabilities of individual parcels of traffic in circuit-switched networks with multiple bit rate classes wherein calls are routed by an alternate routing scheme. Firstly, both exact and approximate methods for the analysis of moments of parcel overflows are presented. Secondly, two methods of calculating approximate individual loss probabilities on a final trunk group are described.

2. ASSUMPTIONS OF NETWORK MODEL

Regarding the network model studied here, we assume the following:

(1) \( K \geq 1 \) classes of calls are considered. A single call in class 1 is connected through a trunk having a bandwidth equal to a pre-defined basic bit rate (e.g. 64Kbps). A single call in class \( k, k = 2, 3, \ldots, K \) requires \( m_k \) class 1 channels for its connection where \( m_k \) is a positive integer. Assume that \( m_1 = 1 \) and \( m_1 < m_2 < \cdots < m_K \). The service time of calls in class \( k \) is exponentially distributed and the mean service time \( 1/\mu_k \) varies from class to class.

(2) All classes of calls have full access to trunk groups. No restrictions are assumed for the channel allocations (Flexible Scheme[8]). Each trunk group is a loss system; a class \( k \) call which finds less than \( m_k \) trunks idle at its arrival instant in time is blocked and overflows the trunk group.

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An alternate routing scheme is used. The calls which are blocked at a final trunk group are cleared.

In order to make the analyses of the networks feasible, the conventional technique used for telephone networks is employed: it is assumed that trunk groups can be ordered so that a trunk group can be evaluated after all the preceding trunk groups are evaluated. In this context, the major issues to be studied are how to calculate the moments of individual parcels of overflow traffic from first-choice trunk groups, and how to evaluate the loss probabilities of the parcels of traffic at final trunk groups.

3. ANALYSIS OF OVERFLOW TRAFFIC

3.1 Modeling of a First-choice Trunk Group

A trunk group consisting of \( N \) class 1 trunks is considered, to which \( K \) separate and independent streams of Poisson calls are offered. Figure 1 depicts the model. The mean interarrival time of calls in the \( k \)-th stream is \( \frac{1}{\lambda_k} \). The calls in the \( k \)-th stream are assumed to belong to bit rate class \( k \). The following notations are used below: \( A_k = \frac{\lambda_k}{\mu_k} \), \( A = \sum_{k=1}^{K} m_k A_k \), \( \nu = A / N \) and \( R_k = m_k A_k / A \).

The loss system can be described by the Markov process \( X_t = (X_{1t}, X_{2t}, \ldots, X_{Kt}) \), \( t \in \mathbb{R}^+ \). Here \( X_{kt}, k = 1, 2, \ldots, K \) denotes the number of class \( k \) calls in progress in the facility at time \( t \). Let \( E \) denote the finite state space of the Markov process. \( E \) is a set of vectors \( (i_1, i_2, \ldots, i_K) \) where \( i_k \geq 0, k = 1, 2, \ldots, K \) and \( \sum_{k=1}^{K} m_k i_k \leq N \). Let \( \pi \) represent the steady-state distribution of the Markov process. The vector \( \pi \) satisfies the system of linear equations \( \pi Q = 0 \) and \( \pi e = 1 \) where \( Q \) denotes the infinitesimal generator of the Markov process and \( e \) represents the column vector with all its elements equal to unity. It can be proven that \( \pi \) has the product-form solution \([2]\).

3.2 Exact Solution of Overflow Traffic Characteristics

First of all, an exact solution of moments of overflow traffic from the first-choice trunk group is studied. Moment of traffic is defined according to \([9]\). Suppose that overflow calls are offered to a hypothetical infinite trunk group. Let \( X_{kt}^H \) denote the number of class \( k \) calls in progress in the hypothetical trunk group. The \( r \)-th ordinary moment of \( X_{kt}^H \) is defined as

\[
L_k^{(r)} = \sum_{i=0}^{\infty} \sum_{(i_1, i_2, \ldots, i_K) \in E} i^r Pr( X_{kt}^H = i, X_1 = i_1, X_2 = i_2, \ldots, X_K = i_K ),
\]

and is called the \( r \)-th ordinary moment of the class \( k \) parcel of overflow traffic. Here \( r \) is a positive integer.

By applying the theory in \([6]\) to the model in Figure 1, vector recursive equations for calculating the \( r \)-th ordinary moment (\( r \geq 1 \)) of individual overflow streams are derived. The following very new formulas are obtained for the first and second ordinary moments, \( L_k^{(1)} \) and \( L_k^{(2)} \), of class \( k \) overflow traffic.

\[
L_k^{(1)} = \mu_k^{-1} \pi \Lambda_k e,
\]

\[
L_k^{(2)} = \mu_k^{-1} L_k^{(1)} \Lambda_k e + L_k^{(1)},
\]

where \( k = 1, 2, \ldots, K \). \( \Lambda_k \) appearing in these equations denotes a square matrix of order \( |E| \) which describes the rate of occurrence of class \( k \) overflow calls. We call \( \Lambda_k \) the overflow rate matrix of class
For our trunk group model, $\Lambda_k$ is such a matrix that the diagonal elements corresponding to the states at which arriving class $k$ calls are blocked are all $\lambda_k$'s and all the other elements are zeros. $I_k^{(l)}$ denotes the $r$-th moment vector of class $k$ whose elements are defined as

$$I_k^{(l)}(i_1, i_2, \ldots, i_k) = \sum_{i=0}^{\infty} i^r Pr(X_k^H = i, X_1 = i_1, X_2 = i_2, \ldots, X_K = i_K).$$

It follows from the theory in [6] that the first moment vector $I_k^{(1)}$ satisfies the following equation.

$$I_k^{(1)}(\mu_k I - Q) = \pi \Lambda_k.$$

The appendix briefly describes the derivation of equations (2), (3) and (5). By using $L_k^{(1)}$ and $L_k^{(2)}$, mean $a_k$ and variance $v_k$ of class $k$ overflow traffic are given by $a_k = L_k^{(1)}$ and $v_k = L_k^{(2)} - (L_k^{(1)})^2$.

Furthermore, let us define the second-order cross term $L_{k_1k_2}$ associated with class $k_1$ and $k_2$ parcels of overflow traffic where $k_1, k_2 = 1, 2, \ldots, K$ and $k_1 \neq k_2$ as

$$L_{k_1k_2} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{i_1, i_2, \ldots, i_K} i j Pr(X_{k_1}^H = i, X_{k_2}^H = j, X_1 = i_1, X_2 = i_2, \ldots, X_K = i_K).$$

It can be proven that

$$L_{k_1k_2} = (\mu_{k_1} + \mu_{k_2})^{-1} (I_k^{(1)} \Lambda_{k_2} + I_k^{(1)} \Lambda_{k_1}) e$$

holds[7]. The covariance $\text{Cov}(k_1k_2)$ of class $k_1$ and $k_2$ parcels of overflow traffic is given by $\text{Cov}(k_1k_2) = L_{k_1k_2} - L_{k_1k_2}^{(1)} L_{k_1k_2}^{(1)}$, while the peakedness factor $Z_k$ and correlation coefficient $\rho_{k_1k_2}$ of parcels of overflow are given by $Z_k = v_k / a_k$ and $\rho_{k_1k_2} = \text{Cov}(k_1k_2) / \sqrt{v_k v_{k_2}}$, respectively. Numerical examples of $Z_k$ and $\rho_{k_1k_2}$ in the case of $K = 2$ are given in Figure 2.

![Fig.2 Peakedness Factors and Correlation Coefficient of Overflow Traffic (K=2).](image)

3.3 Approximate Method for Moments of Overflow Traffic

The method presented above provides a way of calculating exact values of moments. However, difficulty arises in computation for solving the equation (5) as the values of $K$ and $N$ become large and consequently, so does the size of the state space of the loss system. Some much simpler method therefore needs to be developed. In this section a system approximation method is proposed aiming to reduce the computational efforts for computing the moments of overflow traffic.

As one of the approximate systems whose state space is independent of the value of $K$, we consider the Markov process $\hat{X} = (\hat{X}_t, t \in \mathbb{R}_+)$, associated with the number of occupied trunks $\hat{X}_t = \sum_{k=1}^{K} m_k X_{kt}$ in the trunk group. Let $\mathcal{E} = \{n : 0 \leq n \leq N\}$ be the state space of the process $\hat{X}$. Let $\hat{Q}$ and $\hat{\pi}$ represent the infinitesimal generator and the steady-state distribution, respectively, of the process. The process $\hat{X}$ makes a transition from state $n$ to state $n + m_k$ at arrival time of a class $k$ call, and a transition from state $n$ to state $n - m_k$ at departure time of a class $k$ call. Class $k$ calls overflow at the states from $N - m_k + 1$ up to $N$. The transition rate $q(n_1, n_2)$ from state $n_1$ to state $n_2$ where $n_1, n_2 \in \mathcal{E}$ is given by

\[5.1A.6.3\]
There are alternatives regarding how to determine the values of $\xi_k(n)$. The following equation is chosen here.

$$\xi_k(n) = \mu_k E[X_k \mid \sum_{b=1}^{K} m_b X_b = n],$$

(9)

where expectation $E[X_k \mid \sum_{b=1}^{K} m_b X_b = n]$ can be easily computed by using the product-form solution of $\pi$. With $\xi_k(n)$ in equation (9), it can be shown that

$$\bar{t}(n) = \sum_{(i_1,i_2, \ldots, i_K) \in E_n} \pi(i_1,i_2, \ldots, i_K),$$

(10)

where $E_n = \{(i_1,i_2, \ldots, i_K) : \sum_{b=1}^{K} m_b i_b = n\} \subset E$. It should be noted that the same formulas as in (2),(3),(5) and (7) hold also for this approximate system $X$ so that the approximations $\hat{L}_k, \hat{L}_k^{(2)}$ and $\hat{L}_{k,k_2}$ to the exact solutions $L_k^{(1)}, L_k^{(2)}$ and $L_{k,k_2}$ can be computed. In particular, note that $\hat{L}_k^{(1)} = L_k^{(1)}$ holds for all $k$. The equation corresponding to (5) is given by

$$\hat{L}_k^{(1)} (\mu_k I - \hat{Q}) = \bar{t} \hat{A}_k.$$

Here $\hat{A}_k$ denotes the overflow rate matrix of class $k$ traffic of the approximate system. It is a square matrix of order $N + 1$ whose last $m_b$ diagonal elements are all $\lambda_b$'s. $\hat{L}_k^{(1)}$ must be solved to calculate the approximate second moments $\hat{L}_k^{(1)}$ and $\hat{L}_{k,k_2}$. The major advantages of this approximate system are that the order of the equation (11) is $N + 1$ independently of $K$, and additionally that its coefficient matrix is a band matrix with bandwidth of $2 \max_k (m_b) + 1$. It follows that equation (11) can be solved with a considerably small amount of computation time even when $K$ and $N$ take large values.

Examples of the relative errors $r_k$ of the approximate peakedness factor value $\hat{Z}_k$ to the exact values $Z_k$ in the case of $K = 2$ are plotted in Figure 3. For this example, the relative error $r$ of the approximate $\hat{Z}$ to the exact peakedness factor value $Z$ of the total overflow traffic has similar characteristics to $r_k$.

Based upon various sample runs that we tried, it is concluded that the accuracy of the approximation to peakedness factor for the case of $K = 2$ is fairly good, especially when the disparity between $\mu_k$'s is small.

4. APPROXIMATE METHODS FOR INDIVIDUAL LOSS PROBABILITIES

4.1 Cluster Network Model

To dimension a network, it is necessary to evaluate the performance of network clusters. It is thus required to develop a method of calculating individual loss probabilities at a final trunk group. The cluster network model consisting of $J$ first-choice trunk groups and a single final trunk group as shown in Figure 4 is considered. Figure 4 also gives the notations employed. Parcels of traffic are distinguished by label $jk$. Here $j$ designates a stream of calls belonging to the bit rate class which is
designated by $k$. Let us define $A_j = \sum_{b=1}^{K} m_k A_{jk}$. $V_j = A_j / N_j$, $R_{jk} = m_k A_{jk} / A_j$ for all $j = 0, 1, ..., J$ and $k = 1, 2, ..., K$. Let $Z_{jk}$ denote the peakedness factor of the parcel of traffic $jk$ offered to the final trunk group, that is, $Z_{jk} = v_{jk} / a_{jk}$. The total loss $\bar{a}$ and individual overall loss probabilities $B_{jk}$ are defined as $\bar{a} = \sum_{j=0}^{J} \sum_{k=1}^{K} \bar{a}_{jk}$ and $B_{jk} = \bar{a}_{jk} / A_{jk}$ for all $j$ and $k$.

Let $Z_{jk}$ denote the peakedness factor of the parcel of traffic $jk$ offered to the final trunk group, that is, $Z_{jk} = v_{jk} / a_{jk}$. The total loss $\bar{a}$ and individual overall loss probabilities $B_{jk}$ are defined as $\bar{a} = \sum_{j=0}^{J} \sum_{k=1}^{K} \bar{a}_{jk}$ and $B_{jk} = \bar{a}_{jk} / A_{jk}$ for all $j$ and $k$.

Fig. 4 Cluster Network Model.

4.2 Two Approximate Methods

The first method proposed (method 1) is to approximate each parcel of overflow traffic by an independent IPP (Interrupted Poisson Process)[3], and solve a system of linear equations to compute the steady-state probabilities associated with the final trunk group. It is considered that, as the parcels of traffic are less correlated, a better accuracy of approximation will be achieved. However, it is seen that the method is applicable only to small-scale trunk groups, since a computational problem arises due to the large dimensionality of the state space.

To overcome the difficulty in computation, the heuristic method (method 2) which is much simpler than the above method is proposed. The method is based upon the idea of transforming a given model of the final trunk group with multiple bit rate classes to an equivalent model with only bit rate class 1, and subsequently applying the established methods for conventional telephone networks. The steps in the proposed method are as follows.

**STEP1** For each class $k$ traffic in the final trunk group, calculate mean $a_{jk}$ and variance $v_{jk}$ of its equivalent class 1 traffic by the following transformation:

$$a_{jk}^* = m_k a_{jk}, \quad v_{jk}^* = m_k^2 v_{jk}.$$  \hspace{1cm} (12)

Calculate mean $\bar{a}$ and variance $\bar{v}$ of total traffic offered to the final trunk group by using $\bar{a} = \sum_{j=0}^{J} \sum_{k=1}^{K} a_{jk}^*$ and $\bar{v} = \sum_{j=0}^{J} \sum_{k=1}^{K} v_{jk}^* + \sum_{j=1}^{J} \sum_{k=1}^{K} q_{jk} (m_k m_{kl} - m_{kl})^2$.  \hspace{1cm} (13)

**STEP2** Using $(\bar{a}, \bar{v})$, calculate mean $\bar{a}$ of total overflow traffic from the final trunk group by such a method as Kuczura’s IPP method [3], Wilkinson’s ERT method [9] and Hayward’s formula.

**STEP3** Let $a_{jk}^* = \sum_{j=0}^{J} a_{jk}^*$, $v_{jk}^* = \sum_{j=0}^{J} v_{jk}^*$ and $\bar{a}_{jk} = \sum_{j=0}^{J} \bar{a}_{jk}$ for all $k$. Decompose mean $\bar{a}$ into mean $\bar{a}_k$ of total class $k$ lost traffic by using Kuczura’s decomposition method[3]:

$$\bar{a}_k = \bar{a} \frac{v_{jk}^*}{\sum_{i=1}^{K} v_i} \quad \text{for all} \quad k.$$  \hspace{1cm} (13)

**STEP4** Decompose mean $\bar{a}_k$ into means $\bar{a}_{jk}$ of individual parcels of traffic by using a modified version of Lindberger’s decomposition equation[4]:

$$\bar{a}_{jk} = \bar{a}_k \left[ \frac{a_{jk}^*}{\bar{a}_k} + \left( 1 - \frac{a_{jk}^*}{\bar{a}_k} \right) \frac{v_{jk}^*}{v_k^*} \right], \quad \text{for all} \quad j, k.$$  \hspace{1cm} (14)

where

$$w_k = \frac{1 + \eta_k}{1 + \eta_k B}, \quad B = \frac{\bar{a}}{\bar{v}}, \quad \eta_k = \frac{\max_j (Z_{jk})}{\min_j (Z_{jk})} - 2.$$  \hspace{1cm} (15)
The inclusion of STEP3 and the expression for $B$ in equation (15) were decided according to our experimental experience. The expression for $\eta_4$ in equation (15) is employed to avoid the occurrence of such a contradiction as was pointed out in [4]. Table 1 shows numerical examples of $\bar{a}$ and $B_{jk}$ in the case of a simple network model with one first-choice trunk group ($J = 1$) and two bit rate classes ($K = 2$). In Table 1, the exact values of the overall loss probabilities obtained by solving a system of linear equations by means of the SOR (Successive Over-relaxation) method are provided for comparison. The superscripts 1 and 2 of $\bar{a}$ and $B_{jk}$ in Table 1 designate the exact method, and approximate method 1 and 2, respectively. At STEP2 in method 2, an IPP method with two-moment matching was used for the examples. The exact values of the moments of overflow traffic from the first-choice trunk group are used in both methods 1 and 2. On a FACOM M380S computer, methods 1 and 2 required a CPU time of 1.79 sec. and 0.20 sec. respectively for these examples, while the CPU time spent by the exact solution method was in the range 10 to 20 min. Based on our sample runs, it is seen that the error of the approximations is highly dependent on load parameter $v$'s and $R$'s and that generally, as offered load increases, the accuracy increases. It was confirmed that method 1 is superior with regard to accuracy, especially with respect to $B_{0k}$. Method 2 was, however, found to be quite useful for rough estimation of overall loss probabilities.

Table 1 Numerical Examples of Overall Losses

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<tr>
<th>$R_{ij}$</th>
<th>$\bar{a}_1$</th>
<th>$\bar{a}_2$</th>
<th>$B_{11}$ (% of $\eta_4$)</th>
<th>$B_{12}$ (% of $\eta_4$)</th>
<th>$B_{11}$ (% of $\eta_4$)</th>
<th>$B_{12}$ (% of $\eta_4$)</th>
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5. CONCLUSIONS

A circuit-switched network with multiple bit rate classes and alternate routing scheme was studied. By the methods proposed here, it has become possible to evaluate the characteristics of overflow traffic from first-choice trunk groups and approximate overall loss probabilities of individual parcels of traffic in that type of network. It was seen that the approximate methods described here have fairly good performance. Improvements, however, will be necessary to make more accurate evaluations available for every traffic parameter value.

Furthermore, it should be remarked that the formulas presented for moments of overflow traffic can be immediately extended to the case of general Markovian queueing systems and therefore, the theory provides a powerful means for traffic analyses of other types of ISDNs.

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Consider the Markov process \( X_t^H = (X_t^H, X_1, X_2, \ldots, X_K), t \in \mathbb{R}_+ \). Let \( i = (i_1, i_2, \ldots, i_K) \) be the current state of the process \( X_t^H \) where \( i \) is a nonnegative integer and \((i_1, i_2, \ldots, i_K) \in \mathbb{E}\). Let \( \pi^H \) denote the steady-state distribution of the process \( X_t^H \). By lexicographically ordering the states \( i \in \mathbb{E} \), the infinitesimal generator \( Q^H \) of the process \( X_t^H \) is expressed as a block-tridiagonal matrix:

\[
Q^H = \begin{bmatrix}
(Q - \Lambda_k) & \Lambda_k & 0 & 0 & 0 & \cdots \\
\mu_k & (Q - \Lambda_k - \mu_k I) & \Lambda_k & 0 & 0 & \cdots \\
0 & 2\mu_k & (Q - \Lambda_k - 2\mu_k I) & \Lambda_k & 0 & \cdots \\
0 & 0 & 3\mu_k & (Q - \Lambda_k - 3\mu_k I) & \Lambda_k & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix} \tag{A.1}
\]

By using the block-partitioned structure, the system of linear equations \( \pi^H Q^H = 0 \) can be rewritten as

\[
\mu_k \pi^H_1 + \pi^H_0 \left( Q - \Lambda_k \right) = 0, \quad \text{for } i = 0, \tag{A.2}
\]

\[
(i+1) \mu_k \pi^H_{i+1} - i \mu_k \pi^H_i + \pi^H_i \left( Q - \Lambda_k \right) + \pi^H_{i+1} \Lambda_k = 0, \quad \text{for } i \geq 1, \tag{A.3}
\]

where \( \pi^H_i \) represents the \( i \)-th subvector of \( \pi^H \). Multiplying the \( i \)-th equation in (A.2) and (A.3) with \( (i+1)^r, \ r = 1, 2 \) and then summing the equations over \( i \) leads to

\[
1^{(1)} \left( \mu_k I - Q \right) = \pi \Lambda_k, \tag{A.4}
\]

\[
1^{(2)} \left( 2\mu_k I - Q \right) = 1^{(1)} \left( 2\Lambda_k + \mu_k I \right) + \pi \Lambda_k. \tag{A.5}
\]

By postmultiplying the last two equations with the vector \( e \) and using \( L^{(r)}_k = 1^{(r)}_k e \) and \( Qe = 0 \), we obtain equation (2) and (3). Reference [6] gives the equation for an arbitrary positive integer \( r \).

REFERENCES