THE SWITCHED POISSON PROCESS AND THE SPP/G/1 QUEUE

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The paper uses double Laplace transforms to obtain the distribution of the counting random variable of the switched Poisson process from the distribution of the sojourn time in the on state of an alternating Poisson process. It then presents a solution to the SPP/G/1 queue using the technique of imbedded Markov chain at the departure epochs. The direct solution obtained enables the use of the SPP as a modelling tool in the delay performance analysis of multiservice packet switching networks.

1. INTRODUCTION

The switched Poisson process (SPP) is a four parameter model for traffic in a packet switched communications network. As the two-state Markov Modulated Poisson Process (MMPP) it was proposed by Heffes and Lucantoni (1986) as a model for packetized voice and data traffic. In recent papers the author has made similar proposals. In Rossiter (1987) the important point process properties of the SPP were summarised and an alternative scheme was put forward for matching the characteristics of a superposition of SPP's. Harris and Rossiter (1987) described the extension to characterizing different classes of traffic in a multi-class, multi-commodity flow model of a packet switching network. In this paper we give details of how the properties of the SPP can be derived from the distribution of the sojourn time of an alternating Poisson process (APP). We also give a direct method of solving the SPP/G/1 queue which is fundamental to the use of the SPP as a model for traffic in a multiservice packet switching network.

The next Section of this paper introduces a result concerning continuous time stochastic processes. It shows how to define a double Laplace transform for a non-negative stochastic process \((X(t), t \geq 0)\), and the result gives a convenient method for its calculation. When the process is the sojourn time in the on state of an APP the results are particularly tractable. The third Section of the paper contains a definition of the SPP. Using the fact that a SPP is the superposition of an interrupted Poisson process (IPP) with an independent Poisson process, and using results from Section 2, the distribution of the number of events \(K(t)\) during \((0,t]\) is obtained given the state of the SPP at the time origin for the interval. The distributions at the arbitrary time and arbitrary event choices of time origin follow by taking mixtures.

The fourth Section of the paper sets up the forward Kolmogorov equations for the SPP and presents the solution in terms of generating functions. That enables solution of the SPP/G/1 queue in Section 5, where an imbedded Markov chain analysis leads to a rapid algorithm with good numerical properties. We will show that most of the quantities of interest can be obtained from the single root of a functional equation involving the service time Laplace transform. Further details of the results can be found in Rossiter (1988).
2. A DOUBLE LAPLACE TRANSFORM

To establish notation we make the following definitions. For a Borel function \( f: (0, \infty) \to \mathbb{R} \) we define the Laplace transform of \( f \), for all \( s > 0 \), by

\[
L[f(t); s] = \int_{0}^{\infty} f(t) e^{-st} dt. \tag{2.1}
\]

For a continuous time stochastic process \( \{X(t), t > 0\} \) with distribution \( \Phi(x, t) = P[X(t) \leq x], x \in \mathbb{R}, t > 0 \), the Laplace transform with respect to \( t \) of the distribution \( \Phi \) is the function \( \Phi^*: \mathbb{R} \times (0, \infty) \to \mathbb{R} \) where \( \Phi^*(x, s) = L[\Phi(x, t); s] \).

Now consider a non-negative, continuous time stochastic process \( \{X(t), t > 0\} \) with distribution \( \Phi \) and let \( \Phi^* \) be its Laplace transform with respect to \( t \). Let \( A: (0, \infty) \to \mathbb{R} \) be any Borel function such that \( E[A(X(t))] \) is finite for all \( t > 0 \). Then that expectation is a Borel function of \( t \), and its Laplace transform may be evaluated at \( s > 0 \) according to

\[
L[E[A(X(t))]; s] = \int_{0}^{\infty} A(x) \Phi^*(dx, s). \tag{2.2}
\]

For if \( s > 0 \) and \( T \) is exponentially distributed with mean \( 1/s \), and is independent of \( \{X(t), t > 0\} \), we have

\[
s \int_{0}^{\infty} E[A(X(t))] e^{-st} dt = E_T[E[A(X(t)) | T = t]] = E[A(X(T))] = s \int_{0}^{\infty} A(x) \Phi^*(dx, s). \tag{2.2}
\]

The result (2.2) enables us to make the following definitions: Let \( \{X(t), t > 0\} \) be a non-negative, continuous time stochastic process with distribution \( \Phi \). The Laplace transform of \( X(t) \) is the function \( D \) given, for all \( \theta, t > 0 \), by

\[
D(\theta, t) = \int_{0}^{\infty} e^{-\theta x} \Phi(dx, t). \tag{2.3}
\]

The double Laplace transform of \( X(t) \) is the function \( D^* \) given by

\[
D^*(\theta, s) = L[D(\theta, t); s] = \int_{0}^{\infty} D(\theta, t) e^{-st} dt.
\]

for all \( \theta, s > 0 \). By (2.2), \( D^* \) is well defined since \( A(x) = e^{-\theta x} \) is integrable with respect to any probability measure on \([0, \infty)\).

An APP is a Markov process \( \{I(t), t > 0\} \) with state space \([0, 1]\). When \( I(t) = 0 \) (1) the intensity of passage is \( \omega(\gamma) \) and we say the process is in its "off" ("on") state. In Rossiter (1988) we find the double Laplace transform of the non-negative stochastic process \( \{\alpha(t), t > 0\} \), where \( \alpha(t) \) is the total time spent in the on state of an APP during an interval \((0, t]\). The distribution \( \Phi(x, t) = P[\alpha(t) \leq x] \) is obtained from Takacs (1957), conditional on the state of the process at the time origin. We let the time origin coincide with the off (on) state with probability \( P_0 (P_1) \). The conditional \( D^*_0, D^*_1 \) turn out to be given by the following expressions for all \( \theta, s > 0 \):

\[
sD^*_0(\theta, s) = \frac{s + \omega + \gamma + \theta}{s(s+\omega+\gamma) + (s+\omega)\theta}, \quad sD^*_1(\theta, s) = \frac{s + \omega + \gamma}{s(s+\omega+\gamma) + (s+\omega)\theta}. \tag{2.3}
\]
3. DISTRIBUTION OF THE COUNTING RANDOM VARIABLE OF THE SPP

The SPP can be defined as a doubly stochastic Poisson process with intensity generated by an APP. The Poisson intensity of the SPP is \( \lambda_1 (\lambda_2) \) when the APP is its on (off) state. We assume that \( \lambda_1 > \lambda_2 \geq 0 \). In order to use the theory of Section 2 to study the counting random variable of the SPP we first consider the case when no events arrive when the associated APP is in its off state. Let \( \lambda_1 = \lambda \) and \( \lambda_2 = 0 \). We then have an IPP as it is usually defined.

Let \( K(t) \) count the arriving events of the IPP and \( \alpha(t) \) be the sojourn time in the on state during \((0,t]\) of the associated APP. Then, for all \( 0 \leq x < t \)

\[
P[K(t)=k | \alpha(t)=x] = e^{-\lambda x} \frac{(\lambda x)^k}{k!}
\]

and, hence, the distribution of \( K(t) \) is given by

\[
p_k(t) = \int_0^t e^{-\lambda x} \frac{(\lambda x)^k}{k!} \phi(dx,t)
\]

for all \( k \geq 0 \), where \( \phi(.t) \) is the distribution of \( \alpha(t) \). Taking probability generating functions of (3.2) we have

\[
P_K(z,t) = \int_0^t e^{-\lambda x(1-z)} \phi(dx,t)
\]

for all \( 0 \leq z < 1 \), \( t > 0 \), a fundamental relationship which can be re-written

\[
P_K(z,s) = D[ \lambda(1-z), s ]
\]

where \( D \) is the Laplace transform of \( \alpha(t) \). Taking Laplace transforms of (3.4)

\[
P_K^*(z,s) = L[ P_K(z,t);s ] = D^* [ \lambda(1-z), s ]
\]

for all \( 0 \leq z < 1 \), \( s > 0 \), where \( D^* \) is the double Laplace transform of \( \alpha(t) \). This result for the IPP is easily extended to the SPP as follows.

An SPP as we have defined it is the superposition of an IPP with parameters \( \omega \) and \( \gamma \) and Poisson intensity \( \lambda = \lambda_1 - \lambda_2 \) and an independent Poisson process with intensity \( \lambda_2 \). Using this decomposition of an SPP, let \( K_1(t) \) be the number of events in the IPP, and \( K_2(t) \) be the number of events in the Poisson process in the interval \((0,t]\), where \( t > 0 \). Then the counting random variable of the SPP \( K(t) = K_1(t) + K_2(t) \) and its probability generating function is the product

\[
P_K(z,t) = P_{K_1}(z,t) P_{K_2}(z,t)
\]

for all \( 0 \leq z < 1 \), \( t > 0 \). But \( K_2(t) \) is independent of the initial state so taking Laplace transforms of (3.6) we have

\[
P_K^*(z,s) = P_{K_1}^* [ z, s+\lambda_2(1-z) ]
\]

and, by using (3.5) for the IPP with \( \lambda = \lambda_1 - \lambda_2 \), we have

\[
P_K^*(z,s) = D^* [ (\lambda_1-\lambda_2)(1-z), s+\lambda_2(1-z) ]
\]

The result (3.8) may used for a choice of time origin in either state with \( D^* \) from (2.3). The resulting expressions are given by eqns (5(i)) and (5(ii)) of Rossiter (1987). The distribution of \( K(t) \) for the arbitrary time choice of time origin (for which we replace \( K(t) \) by \( N(t) \)) is obtained by taking a mixture of the distributions for the off and on starting states using the stationary probabilities of the APP, \( p_0 = \gamma/(\omega+\gamma) \), \( p_1 = \omega/(\omega+\gamma) \) respectively.
For the arbitrary event choice of time origin (for which we replace \( K(t) \) by \( A(t) \)) we use the stationary probabilities of the Markov chain formed by the states of the APP at the arrival instants of the SPP. That chain has transition probabilities \( \varepsilon_{ij} \), \( i,j = 0,1 \) given by

\[
\begin{align*}
\varepsilon_{00} &= \frac{\lambda_2(\lambda_1+\gamma)}{(\lambda_1+\lambda_2+\lambda_1\lambda_2)}, \\
\varepsilon_{01} &= \frac{\lambda_1}{(\lambda_1+\lambda_2+\lambda_1\lambda_2)}, \\
\varepsilon_{10} &= \frac{\lambda_2}{(\lambda_1+\lambda_2+\lambda_1\lambda_2)}, \\
\varepsilon_{11} &= \frac{\lambda_1(\lambda_2+\omega)}{(\lambda_1+\lambda_2+\lambda_1\lambda_2)}. 
\end{align*}
\]

so we take \( p_0 = \frac{\lambda_2}{(\lambda_1+\lambda_2)} \), \( p_1 = \frac{\lambda_1}{(\lambda_1+\lambda_2)} \). In this way we obtain the important distributions of \( N(t) \) and \( A(t) \) for the SPP.

4. FORWARD KOLMOGOROV EQUATIONS OF THE SPP

For a given choice of time origin within a SPP we have let \( I(t) \) be the state of the associated APP at time \( t \) and \( K(t) \) be the number of events during \((0,t]\), for all \( t > 0 \). Now \( \{(I(t),K(t)), t>0\} \) is a Markov process. For all \( i,j = 0,1, k = 0,1,2, ..., t > 0, \) let

\[
\begin{align*}
p_{jk}(t) &= P[I(t)=j, K(t)=k \mid I(0)=i] \quad (4.1)
\end{align*}
\]

Fix \( i = 0 \) or \( 1 \) and temporarily drop the superscript \( (i) \). Then the probabilities \( p_{jk}(t) \) satisfy the forward Kolmogorov equations:

\[
\begin{align*}
p_{00}(t)' &= \gamma p_{10}(t) - (\lambda_2+\omega) p_{00}(t) \quad (4.2.1) \\
p_{10}(t)' &= \omega p_{00}(t) - (\lambda_1+\gamma) p_{10}(t) \quad (4.2.2) \\
p_{0k+1}(t)' &= \lambda_2 p_{0k}(t) + \gamma p_{1k+1}(t) - (\lambda_2+\omega) p_{0k+1}(t) \quad (4.2.3) \\
p_{1k+1}(t)' &= \lambda_1 p_{1k}(t) + \omega p_{0k+1}(t) - (\lambda_1+\gamma) p_{1k+1}(t) \quad (4.2.4)
\end{align*}
\]

for all \( k = 0,1,2,... \). We define the generating functions

\[
\begin{align*}
p_{0k}(t)(z) &= \sum_{k=0}^{\infty} p_{0k}(t) z^k \\
p_{1k}(t)(z) &= \sum_{k=0}^{\infty} p_{1k}(t) z^k
\end{align*}
\]

Then, (4.2) can be solved for \( i = 0,1 \) and the results summarised as follows:

\[
\begin{align*}
p_{0k}(t)(z) &= A_1(z) e^{-n_1 t} - A_2(z) e^{-n_2 t} \quad (4.4.1) \\
p_{1k}(t)(z) &= B_1(z) e^{-n_1 t} + B_2(z) e^{-n_2 t} \quad (4.4.2)
\end{align*}
\]

where the coefficients are functions of \( z \) given, for all \( 0 \leq z < 1 \), by

\[
\begin{align*}
A_1^{(0)} &= \frac{1}{\sqrt{\nu}} \left[ \frac{\lambda_1}{\lambda_1(1-z)+\gamma-n_1} \right] \\
A_2^{(0)} &= \frac{-1}{\sqrt{\nu}} \left[ \frac{\eta_2-\lambda_1(1-z)-\gamma}{n_2} \right] \\
B_1^{(0)} &= \frac{\omega}{\sqrt{\nu}} = -B_2^{(0)} \quad A_1^{(1)} = \frac{\gamma}{\sqrt{\nu}} = A_2^{(1)} \\
B_1^{(1)} &= \frac{1}{\sqrt{\nu}} \left[ \frac{\lambda_2}{\lambda_2(1-z)+\omega-n_1} \right] \\
B_2^{(1)} &= \frac{1}{\sqrt{\nu}} \left[ \frac{\eta_2-\lambda_2(1-z)-\omega}{n_2} \right]
\end{align*}
\]

with \( \nu = \nu(z) \), \( n_1 = n_1(z) \) and \( n_2 = n_2(z) \) given by (9) and (10) of Rossiter (1987).

The generating functions given by (4.4) and (4.5) were derived from the backward equations by van Hoorn and Seelen (1983). Their first and second partial derivatives with respect to \( z \) as \( z \rightarrow 1 \) are given in Rossiter (1988).

5. THE SPP/G/1 QUEUE

Consider a single server queue with a switched Poisson arrival process. There
are an infinite number of waiting places and the service discipline is first in, first out. Let the service time of each customer have a distribution \( F \) with mean \( 1/\mu \) and Laplace transform \( \mathcal{L}(F(s), s > 0) \).

The SPP/G/1 queue has been studied by van Boorn and Seelen (1983), and as a particular case of the MMPP/G/1 queue by Heffes and Lucantoni (1986). Our solution is an extension of the classical M/G/1 analysis. We let \( I_n \) be the state of the SPP and \( Q_n \) be the number of customers in the queue immediately after the \( n \)th customer departure instant. Then \( \{(I_n, Q_n)\}_{n=1}^{\infty} \) is a Markov chain with state space \( \{0,1\} \times \{0,1,\ldots\} \). We wish to determine the probabilities

\[
\lim_{n \to \infty} P(I_n=0, Q_n=j) \quad \lim_{n \to \infty} P(I_n=1, Q_n=j)
\]

assuming ergodicity of the chain, and we let \( \pi_j = \pi_{0j} + \pi_{1j} \), for all \( j \geq 0 \).

In order to discuss the transition probabilities of the chain we introduce some more notation. Jointly considering the changes of phase of the SPP and the number of arrivals during one service time, for all \( i = 0,1, j \geq 0 \), let

\[
a^{(i)}_j = \int_0^\infty p^{(i)}_{0j}(t) F(dt) \quad b^{(i)}_j = \int_0^\infty p^{(i)}_{1j}(t) F(dt)
\]

and let these joint probabilities have generating functions

\[
P^{(i)}_0(z) = \sum_{j=0}^\infty a^{(i)}_j z^j \quad P^{(i)}_1(z) = \sum_{j=0}^\infty b^{(i)}_j z^j
\]

for all \( 0 \leq z \leq 1 \). Then, for \( i = 0,1 \) we have

\[
P^{(i)}_0(z) = A^{(i)}_1(z) F[\eta_1(z)] - A^{(i)}_2(z) F[\eta_2(z)]
\]

\[
P^{(i)}_1(z) = B^{(i)}_1(z) F[\eta_1(z)] + B^{(i)}_2(z) F[\eta_2(z)]
\]

for all \( 0 \leq z \leq 1 \), from (4.4). Also let \( P^{(i)}(z) = P^{(i)}_0(z) + P^{(i)}_1(z), i = 0,1 \), and the limiting probabilities of the chain have generating functions

\[
\pi_0(z) = \sum_{j=0}^\infty \pi_{0j} z^j \quad \pi_1(z) = \sum_{j=0}^\infty \pi_{1j} z^j
\]

with \( \pi(z) = \pi_1(z) + \pi_2(z) \), for all \( 0 \leq z \leq 1 \). Then the stationary transition probabilities of the chain have the form

\[
P(k,i) = P(I_{n+1}=k, Q_{n+1}=i \mid I_n=0, Q_n=i) = \sum_{j=0}^\infty P^{(i)}_0(j) \pi_{0j} + P^{(i)}_1(j) \pi_{1j}
\]

for all \( k, i, j = 0,1,2, \ldots \) are given by

\[
P(0,0) = \epsilon_{00} a^{(0)}_0 + \epsilon_{01} a^{(1)}_0 \quad P(0,1) = a^{(0)}_1 + 1 \leq i \leq j + 1
\]

\[
P(1,0) = \epsilon_{10} a^{(0)}_1 + \epsilon_{11} a^{(1)}_1 \quad P(1,1) = a^{(1)}_2 + 1 \leq i \leq j + 1
\]

\[
P(0,0) = \epsilon_{00} b^{(0)}_0 + \epsilon_{01} b^{(1)}_0 \quad P(0,1) = b^{(0)}_1 + 1 \leq i \leq j + 1
\]

\[
P(1,0) = \epsilon_{10} b^{(0)}_1 + \epsilon_{11} b^{(1)}_1 \quad P(1,1) = b^{(1)}_2 + 1 \leq i \leq j + 1
\]

where the \( \epsilon_{ij} \) are given by (3.9). The system of equilibrium equations for the Markov chain \( \{(I_n, Q_n)\}_{n=1}^{\infty} \) may be written as

\[
\begin{align*}
\pi_{0j} &= P(0,0) \pi_{00} + \sum_{i=1}^{j+1} P(0,i) \pi_{0i} + P(1,0) \pi_{10} + \sum_{i=1}^{j+1} P(1,i) \pi_{1i} \\
\pi_{1j} &= P(0,0) \pi_{10} + \sum_{i=1}^{j+1} P(0,i) \pi_{1i} + P(1,0) \pi_{00} + \sum_{i=1}^{j+1} P(1,i) \pi_{11}
\end{align*}
\]
and taking generating functions of (5.8) gives

\[
\begin{align*}
n_0(z) &= \left[ \varepsilon_{00} p_0^{(0)}(z) + \varepsilon_{01} p_0^{(1)}(z) \right] \pi_{00} + \frac{p_0^{(0)}(z)}{z} \left[ \pi_0(z) - \pi_{00} \right] \\
&\quad + \left[ \varepsilon_{10} p_1^{(0)}(z) + \varepsilon_{11} p_1^{(1)}(z) \right] \pi_{10} + \frac{p_1^{(0)}(z)}{z} \left[ \pi_1(z) - \pi_{10} \right] \\
n_1(z) &= \left[ \varepsilon_{00} p_0^{(0)}(z) + \varepsilon_{01} p_1^{(1)}(z) \right] \pi_{00} + \frac{p_0^{(0)}(z)}{z} \left[ \pi_0(z) - \pi_{00} \right] \\
&\quad + \left[ \varepsilon_{10} p_1^{(0)}(z) + \varepsilon_{11} p_1^{(1)}(z) \right] \pi_{10} + \frac{p_1^{(0)}(z)}{z} \left[ \pi_1(z) - \pi_{10} \right]
\end{align*}
\]

(5.9.1)

(5.9.2)

The equations in (5.9) after considerable re-arrangement yield the following solution for \(n_0(z)\) and \(n_1(z)\) expressed in terms \(n_{00}\) and \(n_{10}\)

\[
\begin{align*}
n_0(z) &= \left[ \pi_{00} u_0^{(0)}(z) + \pi_{10} u_1^{(0)}(z) \right] / D(z), \\
n_1(z) &= \left[ \pi_{00} v_0^{(0)}(z) + \pi_{10} v_1^{(0)}(z) \right] / D(z)
\end{align*}
\]

(5.10)

for all \(0 \leq z \leq 1\), provided \(D(z) \neq 0\), where

\[
D(z) = \left[ z - p_0^{(0)}(z) \right] \left[ z - p_1^{(1)}(z) \right] - p_1^{(0)}(z)p_0^{(1)}(z)
\]

(5.11)

\[
U(z) = \left[ z - p_0^{(1)}(z) \right] p_0^{(0)}(z) + p_1^{(0)}(z)p_0^{(1)}(z)
\]

(5.12.1)

\[
V(z) = \left[ z - p_0^{(0)}(z) \right] p_1^{(1)}(z) + p_1^{(0)}(z)p_1^{(1)}(z)
\]

(5.12.2)

\[
U_0(z) = \varepsilon_{01} z^2 p_0^{(1)}(z) + (\varepsilon_{00} z - 1) U(z)
\]

(5.13.1)

\[
U_1(z) = \varepsilon_{11} z - 1 z p_0^{(1)}(z) + \varepsilon_{10} z U(z)
\]

(5.13.2)

\[
V_0(z) = \varepsilon_{00} z - 1 z p_0^{(1)}(z) + \varepsilon_{01} z V(z)
\]

(5.13.3)

\[
V_1(z) = \varepsilon_{10} z^2 p_0^{(0)}(z) + (\varepsilon_{11} z - 1) V(z)
\]

(5.13.4)

After some simplification we find \(D(z) = D_1(z)D_2(z)\) where \(D_i(z) = z - F[\eta_i^{(0)}(z)]\) for all \(0 \leq z \leq 1\), \(i = 1,2\), and that \(T_0 = U_0 + V_0\) and \(T_1 = U_1 + V_1\) may be written in terms of \(P_0^{(0)}(z)\) and \(P_1^{(1)}(z)\). The generating function of the stationary distribution of queue length at the departure epochs can be expressed as

\[
\pi(z) = \frac{D(z)}{\pi_{00} T_0^{(0)}(z) + \pi_{10} T_1^{(0)}(z)}
\]

(5.14)

for all \(0 \leq z \leq 1\), where \(\pi_{00}\) and \(\pi_{10}\) are the probabilities that a departure leaves the system empty with the SPP in its off and on states respectively. To obtain the two unknowns \(\pi_{00}\) and \(\pi_{10}\) we first let \(z \uparrow 1\) in (5.14) using L' Hopital's rule. We get the following relationship:

\[
\pi_{00} \left( t_0'/d' \right) + \pi_{10} \left( t_1'/d' \right) = 1
\]

(5.15)

where

\[
t_0' = T_0'(1^-) = \frac{\left( \lambda_1^{(0)} + \lambda_2^{(0)} \gamma \right) \left( \lambda_1^{(0)} + \omega + \gamma \right)}{\left( \omega + \gamma \right) \left( \lambda_1^{(0)} + \lambda_2^{(0)} \gamma + \lambda_1^{(0)} \lambda_2^{(0)} \gamma \right)} \left[ 1 - F(\omega + \gamma) \right]
\]

(5.16.1)

\[
t_1' = T_1'(1^-) = \frac{\left( \lambda_1^{(1)} + \lambda_2^{(1)} \gamma \right) \left( \lambda_2^{(1)} + \omega + \gamma \right)}{\left( \omega + \gamma \right) \left( \lambda_1^{(1)} + \lambda_2^{(1)} \gamma + \lambda_1^{(1)} \lambda_2^{(1)} \gamma \right)} \left[ 1 - F(\omega + \gamma) \right]
\]

(5.16.2)

\[
d' = D'(1^-) = (1 - \rho) \left[ 1 - F(\omega + \gamma) \right]
\]

(5.17)

with \(\rho = \left( \lambda_1^{(0)} + \lambda_2^{(0)} \gamma \right) / [\mu(\omega + \gamma)]\), the occupancy of the queue. To obtain a second relationship between \(\pi_{00}\) and \(\pi_{10}\) we consider the roots in \((0, 1)\) of the denominator of (5.14). We claim there is a unique root \(z^*\) if \(\rho < 1\). Then,
since \( D(z^*) = 0 \), \( z^* \) must also be a root of the numerator of (5.14), ie,
\[
\pi_0 T_0(z^*) + \pi_{10} T_1(z^*) = 0
\]  - (5.18)

Now, (5.14) and (5.18) are linearly independent and can be solved to give
\[
\pi_{00}^{-1} = \frac{t'_0}{d'} - \frac{t'_1}{d'} \frac{T_0(z^*)}{T_1(z^*)} \quad \pi_{10}^{-1} = \frac{t'_1}{d'} - \frac{t'_0}{d'} \frac{T_1(z^*)}{T_0(z^*)}
\]  - (5.19)

The root \( z^* \) can be found very quickly using the bisection method. Having obtained \( \pi_{00} \) and \( \pi_{10} \) we remark that \( \pi_0 = \pi_{00} + \pi_{10} \) is the probability that a departure leaves the system empty. Using the Theorem of 5-2 of Cooper (1972) we also have that \( \{ \pi_j \}_{j=0}^\infty \) is the distribution of the number of customers in the system at an arrival epoch. The moments follow by differentiation of (5.14). The Laplace transforms of the virtual and actual waiting time distributions can be obtained using \( \pi_{00} \) and \( \pi_{10} \) by using Steps 9. and 10. of the algorithm of Heffes and Lucantoni (1986). Van Hoorn and Seelen (1983) give a closed form expression for the mean queue length at a virtual time instant, which involves the departure instant emptiness probability. The modelling procedure suggested in Rossiter (1987), together with the solution obtained above, enables the use of the SPP as a model for packet traffic within the network context.

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7. REFERENCES


