BUFFER DIMENSIONING CRITERIA FOR AN ATM MULTIPLEXER LOADED WITH HOMOGENEOUS ON-OFF SOURCES

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In this paper we face the buffer dimensioning issue for an ATM multiplexer, whose input consists of the superposition of homogeneous ON-OFF sources, each modeled by a two-state markovian process. Our approach is based on a simple and accurate analytical model, which approximates the actual input process by means of a suitably chosen two-state Markov Modulated Poisson Process (MMPP). In this paper, firstly an efficient procedure for the computation of the loss probability of the MMPP/G/1/K queue is derived. Then, this procedure is applied to the queuing model of the ATM multiplexer. The numerical results here presented let us gain a deep insight into the multiplexer performance behaviour as the source parameters (peak bit rate, mean burst length and activity factor) are varied. This enables us to establish simple and effective buffer dimensioning criteria.

1. Introduction

In an ATM environment one of the most outstanding problems is the definition of suitable criteria to dimension the buffering resources of the network. This has to be done with the aim of satisfying some Quality of Service (QoS) requirements under loads originated from a variety of traffic sources.

In this paper we propose a solution to this problem with reference to an ATM multiplexer loaded with the superposition of homogeneous ON-OFF sources. In fact these sources are widely recognized as representative of a large class of real-life traffic sources. Beside, the traffic load originated by these sources is critical from the point of view of achieving the commonly required QoS in the ATM multiplexing, in particular an upper limit (e.g. $10^{9}$) of the cell loss probability.

The assumption of an homogeneous environment, although quite restrictive, anyway enables us to adopt an analytical model [1], which has already been proved to be very accurate, at least within a suitably restricted range of values of the source parameters. On the other hand, this approach should be considered as a first step towards a full understanding of how superposed cell stream impact on the multiplexer performance. In particular the effect of each of the source parameters (peak bit rate, activity factor and mean burst length) and of the number of superposed sources can be highlighted with respect to the buffer dimensioning.

The adopted approach approximates the input superposition process by means of a two-state Markovian Modulated Poisson Process (MMPP), as suggested in [2]. However, the identification of the MMPP parameters is here obtained via a completely different approximation method, called asymptotic matching, whose accuracy and applicability limits are widely discussed in [1].

Starting from the methodology and the results presented in [1], this paper provides a much more efficient procedure to compute the cell loss probability as a function of the source parameters. Such a procedure, based on an asymptotic approximation of the cell loss probability for increasing buffer sizes, has resulted in negligible inaccuracies for the whole range of the buffer sizes. Therefore, this analysis method is well suited to solve the buffer dimensioning problem under the above specified constraints.

The paper is organized in two main parts. In the first one (Sec. 2), the asymptotic approximation method is developed with reference to an MMPP/G/1/K queue, i.e. to a single-server finite-buffer queuing system, having general distributed service times and a multiple-state MMPP as input process. In the second part (Sec. 3), this method is applied to the specific case of an ATM multiplexer modeled as a two-state MMPP/D/1/K queue. Numerical results are presented to give design criteria for the buffer dimensioning. Finally, in the Appendix some mathematical details are dealt with.

2. Loss analysis of the MMPP/G/1/K queue

The MMPP/G/1/K queue can be described by means of a two-dimensional state variable $S(t) = [X(t), J(t)]$, where $X(t)$ denotes the number of users in the system (waiting line plus service) and can take values in the set $[0, ..., K+1]$, while $J(t)$ represents the state of the Markov process that modulates the poissonian arrivals. We will refer to the modulating Markov process as the phase process and assume that it comprises $m+1$ states, so that $J(t)$ can take values in the set $[0, ..., m]$. The classical Imbedded Markov Chain (IMC) approach can be used to determine the limiting probability distribution of the variable $S(t)$, whose state space consists of $[0, 1, ..., K+1] \times [0, 1, ..., m]$. The service completion epochs can be chosen as the set $\{t_i\}$ of time instants of the IMC. Such a choice implies that the random variable $S(t_i)$ represents a Markov chain on the space $[0, 1, ..., K] \times [0, 1, ..., m]$.

The scalar variables used in the sequel are so denoted:

- $\Gamma(K)$ the loss probability of the MMPP/G/1/K queue;
- $A_\theta$ the mean offered load;
- $H(t)$ the Cumulative Probability Function (CPF) of the service time;
- $\tilde{H}(s)$ the Laplace-Stieltjes Transform (LST) of $H(t)$;
- $\theta$ the mean service time.

As for matricial notations, we define

- $P$ the one-step transition probability matrix of the IMC, whose $(i,j)$-th element is $P_{ij} = S_i / S_j$.

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\[ i = 0, 1, \ldots, (m + 1)(K + 1) - 1, \text{ wherein } S_i \text{ and } S_j \text{ indicate two outcomes of the random variable } S(t); \]

and the following \((m + 1) \times (m + 1)\) matrices:

- \( R \) the rate transition matrix of the phase process of the MMPP, assumed to be irreducible and ergodic;
- \( A \) the diagonal matrix, whose element \( A_{ii} \) is equal to the mean arrival rate while in phase \( i \);
- \( U \) the matrix given by \((A - R)^{-1}A\), which accounts for the evolution of \( J(t) \) during server's idle periods;
- \( A_n \) the matrix whose \((i,j)\)-th element denotes the conditional probability of reaching phase \( j \) and having \( n \) arrivals within a service time, starting from phase \( i \);
- \( M \) the matrix whose \((i,j)\)-th element gives the conditional probability of reaching phase \( j \) within a service time, starting from phase \( i \);
- \( I \) the unit matrix.

The expressions of the matrices \( A_n, A \) and \( M \) are given in [6]. Finally, we introduce some \((m + 1)\)-dimensional vector variables, denoted as follows:

- \( e \) the unit column vector;
- \( \pi_K(i) \) the row vector, whose \( j \)-th element is the limiting probability at the imbedded time instants of having \( i \) arrivals in the system and being in the phase \( j \) of the MMPP, \( i = 0, 1, \ldots, K \);
- \( q \) the row vector containing the limiting state probabilities of the phase process; obviously, it can be obtained as the unique solution of the system \( qR = 0 \) and \( qe = 1 \).

Based on the results obtained in [3], the loss probability \( \Pi(K) \) can be computed as follows:

\[ \Pi(K) = 1 - \frac{1}{A_n[1 + \pi_K(0)]AU^{-1}e} \quad (1) \]

The limiting probability distribution \( \pi_K(i) \) \( i = 0, 1, \ldots, K \) can be obtained by solving the following linear system:

\[ \begin{cases} \pi_K(0)UA_i + \sum_{v=1}^{i} \pi_K(v)A_{i,v+1} = \pi_K(i), & i = 0, \ldots, K-1 \\ \sum_{v=0}^{K} \pi_K(v) + \pi_K(0)(I - U)A(I - A + eq)^{-1} = q, \end{cases} \quad (2) \]

wherein the last vector equation includes the normalization condition.

The above linear system can be solved by successive substitutions expressing all the \( \pi_K(i) \) \( i \neq 0 \) as a function of \( \pi_K(0) \). Formally

\[ \pi_K(i) = \pi_K(0)C_i, \quad i = 0, 1, \ldots, K, \quad (3) \]

where the matrix sequence \( C_i \) does not depend on \( K \), since the structure of the first \( K \) vector equations in eq. (2) is the same for every \( K' > K \) and, in particular, for the infinite buffer queue. Hence, the Z-transform of this matrix sequence can be computed once for all [4], as \( C(z) = (zU - I)A(z)[zI - A(z)]^{-1} \).

Finally, we introduce some variables, denoted as follows:

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\[ A(z) = \sum_{n=0}^{\infty} z^n A_n = \int_0^\infty e^{[R + (z-1)A^k]} dH(t) \quad (4) \]

On the basis of eqs. (2) and (3), it follows that:

\[ \pi_K(0) \left[ \sum_{v=0}^{K} \pi_K(v)(I - U)A(I - A + eq)^{-1} \right] = q \quad (5) \]

\[ C_{k+1} = [C_i - UA_i - \sum_{v=1}^{i} C_v A_{v,v+1}] A_{i0}^{-1}, \quad i \geq 1 \quad (6) \]

where the matrix sequence in eq. (6) can be initialized by letting \( C_i = (I - UA_i)A_{i0}^{-1} \). The matrix within square brackets in eq. (5) is non singular, because the solution of the finite buffer queue always exists and is unique. Then the following algorithm can be carried out to compute the loss probability:

1. compute the matrices \( C_i \) for \( i = 1, \ldots, K \) according to the recursion defined in eq. (6) and let \( C_0 = I \);
2. compute the vector \( \pi_K(0) \) from eq. (5);
3. compute the loss probability according to eq. (1).

The drawbacks of such an algorithm are mainly due to its computational burden, that gets intolerable for large buffer sizes. Moreover, it is quite difficult to invert the above procedure, in order to find the buffer size required to achieve a given value of the loss probability.

It is therefore felt the need for a simpler though still accurate characterization of the loss probability. This can be obtained by means of a suitable approximation, which can be derived from the expression of \( \pi_K(0) \) given by eq. (5). Such an approach will be referred to as asymptotic approximation, since it tends to the exact value for increasing buffer sizes.

It can be seen that the matrix within square brackets in eq. (5) can be so written:

\[ D_K = \sum_{v=0}^{K} \pi_K(v)(I - U)A_{i0}^{-1}A_{i0}^{-1}(I - eq) \quad (7) \]

The Z-transform \( D(z) \) of the matrix sequence \( D_K \) is equal to:

\[ D(z) = \frac{C(z) + UA + \theta A^{-1}AM^{-1}(I - eq)}{I - z} \]

where \( \theta \) is the least root greater than 1. The poles of \( C(z) \) are obviously given by the roots of det\([zI - A(z)]\). In Appendix A it is shown that \( C(z) \) has exactly \( m \) real simple poles within the unit circle of the \( z \)-plane and \( m + 1 \) real simple poles outside. Let us denote the first ones with \( \eta_i \) \( i = 1, \ldots, m \) and the latter ones with \( \zeta_i \) \( i = 0, 1, \ldots, m \), where \( \zeta_0 \) is the least root greater than 1. It is also shown that these poles can be computed as the roots of the equations:

\[ z = \tilde{H}(1 - x_i(z)), \quad i = 0, 1, \ldots, m \]

wherein \( x_i(z) \) are the eigenvalues of the matrix \( S(z) = R + (z - I)A \).

Our approximation consists in expressing the \( D(z) \) as a sum of simple rational functions, each corresponding to one of its simple poles, so that the sequence \( D_K \) is given by a superposition of discrete exponential modes. If \( D(z) \) were a rational function of \( z \), this approach would be exact. According to these considerations, we obtain:
wherein \( Q(1) = C(1) + U A^{-1} \Theta^{-1} A M^{-1}(I-\epsilon q) \) and the matrices \( Q(\cdot) \) are the residuals corresponding to the poles of \( D(\cdot) \). The expressions of the matrices \( Q(\cdot) \) are given in Appendix A. It is to be noted [5] that:

\[
\lim_{K \to \infty} \pi_k(0)Q(\eta_i) = 0, \quad i = 1, \ldots, m, \tag{9}
\]

so that, when considering the equation \( \pi_k(0)D_K = q \) for large \( K \), apart from the constant term \( Q(1) \), the second term on the right hand side of eq. (8) dominates and the term under the summation sign has a comparatively negligible value. Therefore \( \pi_k(0) \) tends exponentially to its limiting value at a rate equal to \( \zeta_p \). As a consequence, \( \Pi(K) \) tends to \( 0 \), as \( K \to \infty \), with the same decay rate. More formally, it can be shown [6] that \( \Pi(K) = C \zeta_p^K + o(\zeta_p^K) \), as \( K \to \infty \).

Let us outline the steps to be carried out for the asymptotic approximation. Given the arrival process characteristics through \( R \) and \( A \) and the service time \( CPF \), we need to:

1) compute all the eigenvalues \( x_i(z) \) of the matrix \( S(z) = R + z(A-I)A \);
2) find \( \zeta_p \), (i.e. the unique real root greater than 1 of the equation \( z = H(-x_0(z)) \)) and \( \eta_i \), (i.e. the unique real roots of the equation \( z = H(z) \)) such that \( \eta_i < 1 < \zeta_p \) for \( i = 1, \ldots, m \);
3) compute the matrices \( Q(\cdot) \) of eq. (8) according to the expressions given in Appendix A; that requires in turn the computation of the right and left eigenvectors of \( S(\zeta_p) \), \( S(\zeta_p) \) and \( S(\eta_i) \), for \( i = 1, \ldots, m \);
4) compute the matrix \( U = (A-R)^{-1}A \);
5) compute the matrix \( D_K \) according to eq. (8), and then \( \pi_k(0) \) from eq. (5);
6) compute the loss probability from eq. (1).

It is clear that the computational burden depends mostly on the first three steps, apart from the inversion of \( (m+1)x(m+1) \) matrices required in the steps 4) and 5).

In some particular cases the expression of the eigenvalues and eigenvectors of the matrix \( S(z) \) are explicitly known. This is true, for example, when \( m = 1 \), i.e. if we consider a two-state MMPP. Also the rest of the procedure is particularly simple in this case, since only 3 roots need to be determined numerically, while matrix inversions are elementary.

3. Analysis of the ATM multiplexer and buffer design criteria

We consider an ATM multiplexer loaded with a superposition of homogeneous ON-OFF sources, characterized by: i) the normalized peak bit rate \( \gamma \) defined as the ratio of the source peak bit rate \( P_p \) to the net MUX output capacity; ii) the activity factor \( p \), defined as the ratio of the average bit rate to \( P_p \); iii) the normalized mean burst length \( b \) defined as the mean number of cells generated during an active period. The superposition of these input sources is modeled by a two-state MMPP, whose parameters are determined according to the matching method proposed in [1].

Assuming a buffer size equal to \( K \) cells, the ATM multiplexer is modeled as a two-state MMPP/D/1/K queue. The validity limits of such a model are discussed in [1], where it is shown that \( b \) and \( p \) are the most critical parameters. In fact, decreasing values of \( b \) or increasing values of \( p \) lead to increasing inaccuracies of the model results, as compared to simulation results.

Throughout the present paper, the considered source parameter values are such that the chosen model satisfactorily agrees with the simulation results (see [1]).

The analysis here presented has been carried out with a twofold aim: i) to validate the approximate procedure for the computation of the cell loss probability; ii) to establish buffer dimensioning criteria for a broad range of values of the source parameters and for a given value of the cell loss probability.

The cell length has been fixed to 53 bytes, with a 48 bytes payload. The multiplexer output link capacity has been assumed equal to 150 Mbit/s, thus yielding a net capacity of about 135.85 Mbit/s, when the cell overhead is taken into account.

We choose the following values for the reference source parameters: \( \gamma = 0.0736 \) (corresponding to \( F_p = 10 \) Mbit/s), \( p = 0.1 \) and \( b = 340 \) cells. With respect to such values, we have varied each parameter, one at a time, considering less critical sources.

In fig. 1 the behaviour of the function \( \Pi(K) \) is shown for various values of \( A_o \) considering the reference source parameters. This graph has already been presented in [1], where it has been obtained using the exact solution of the queue. It is reported here just to provide a starting point for the analysis carried out in this work. In the same figure some simulation results are shown as well. The more than satisfactory agreement between the analytic and simulation data is to be pointed out.

![Graph](image-url)
influenced by the correlations among the cell interarrival times or equivalently by the value of $b$.

In both regions the behaviour of $\Pi(K)$ exhibits an exponential decay with different rates. The crosspoint of these two exponential functions will be referred to as the cell loss probability breakpoint, whose abscissa will be denoted by $K_0$.

The next graph is concerned with the comparison between the exact analytical solution of the MMPP/D/1/K queue and its asymptotic approximation. In fig. 2 the function $\Pi(K)$, computed in both ways, is plotted for $A_o = 0.8$ and for different source parameter values.

$$A_o = 0.8$$

x asymptotic approximation

Fig. 2 - $\Pi$ vs $K$: comparison between exact solution and asymptotic approximation for various values of the source parameters.

The asymptotic approximation appears to perform very well for all the considered source parameters values, so that it is not possible to distinguish between the exact and approximate values from the plot. It has been found that the relative error implied by the asymptotic approximation is already negligible (e.g. less than $10^{-4}$) for buffers sizes of a few cells (e.g. 5 cells).

Therefore the asymptotic approximation makes available a highly accurate fast computing tool to solve for $\Pi(K)$. In fact the CPU time required to compute a whole curve is lowered by some orders of magnitude (a few seconds instead of more than an hour for a buffer range from 0 to 300, using a desktop PC). Moreover, the exact approach incurs in numerical and data storage problems for large buffer sizes, whereas the asymptotic approximation has no such problems.

From now on the analytical data will be obtained exclusively by means of the asymptotic approximation.

From the data presented in fig. 2 some further comments emerge about the dependence of $\Pi(K)$ upon the source parameters. It can be seen that:

i) the value of $\gamma$ influences the value of $\Pi(K_o)$ and therefore it causes the loss curve to shift downwards as $\gamma$ decreases, maintaining almost the same decay rates in the cell and burst regions;

ii) the value of $b$ affects only the decay rate of the burst region, which increases for decreasing values of $b$, whereas $K_0$, $\Pi(K_o)$ and the behaviour of the curve in the cell region remain essentially the same;

iii) a change in $p$ results in a different value for both $\Pi(K_o)$ and the burst region decay rate, higher values of $p$ corresponding to better loss performance.

The next three figures refer to the buffer dimensioning problem. In particular they plot the minimum required buffer size to achieve a value of $\Pi(K)$ not greater than $\Pi_{max} = 10^{-9}$ as a function of $A_o$, for various values of the source parameters.

With respect to the reference source, in fig. 3 the normalized peak bit rate $\gamma$ spans from 0.00736 to 0.0736 (corresponding to values of $F_p$ comprised between 1 and 10 Mbit/s), in fig. 4 the normalized mean burst length $b$ is varied in the range $85 + 340$ (corresponding to values of the mean burst length ranging from about 4 to about 16 kbytes) and, finally, in fig. 5 the activity factor $p$ takes values ranging from 0.1 to 0.5.

As for fig. 3, the effect of the two regions of the cell loss probability is evident. Until $A_o$ is low enough so that the constraint $\Pi(K) \leq \Pi_{max}$ is fulfilled remaining in the cell region, the required buffer size is independent of the value of $\gamma$ and very small increments of this size cause substantial gains of the affordable $A_o$. For higher values of $A_o$, we move to the burst region; hence the curves exhibit a sharp knee and the required buffer size grows up very fast.

For a given value of the maximum tolerable transit delay $D_{max}$ through the ATM multiplexer and therefore for a given maximum buffer size, the admitted value of $A_o$ can be easily derived. Reminding that $p$ equals 0.1 and therefore a peak bandwidth allocation entails an output link utilization as small as 10%, the advantage brought about by the dynamic multiplexing appears to be quite significant. In fact, the output link utilization ranges from 30%, for $D_{max} = 1$ ms and the worst case value of $\gamma = 0.0736$, to percentages as high as 70%, for $D_{max} = 2$ ms and $\gamma = 0.00736$.

Similar comments apply to the curves of fig. 4. In this case, only for very low values of $A_o$ the requirement $\Pi(K) \leq \Pi_{max}$ is met remaining in the cell region and therefore independently of the value of $b$. As a consequence, for reasonable values of $A_o$, the required QoS can only be reached
within the burst region, where small multiplexing gains require large buffer size increments.

Min. required buffer size under the constraint $\Pi \leq 10^{-9}$ vs. $A_o$, for various values of $b$.

As for fig. 5, it is not interesting to consider cases where $A_o < p$. In fact, in such a situation the overall instantaneous bit rate of the superposed sources can never overcome the output link capacity and therefore the cell loss is due only to random fluctuations in the cell arrival process. On the other hand, in such a situation a straightforward peak bandwidth allocation would be possible, thus making this case uninteresting. This explains the starting point of the curves contained in fig. 5.

This figure also shows that the highest multiplexing gain with respect to the peak bandwidth allocation can be obtained with low activity factor values. In fact, if sources, which have a nearly CBR behaviour, are multiplexed, there is little to be gained by a dynamic multiplexing, even though the output link utilization may be high.

Finally, we turn our attention to the behaviour of the value of $K_o$. The above results indicate that $K_o$ does not depend appreciably on $b$, whereas it varies both with $\gamma$ and $p$.

In fig. 6 $K_o$ is plotted vs. $\gamma$ for various values of $p$. The plot clearly shows that $K_o \to \infty$ as $\gamma$ tends to 0, higher values of $p$ resulting in a faster increase of $K_o$.

This graph therefore gives a quantitative support to the well known observation that the superposition of sources having a peak bit rate equal to a small fraction of the output link capacity tends to behave like a poissonian process.

4. Conclusions

In this paper we have studied the loss performance of an ATM multiplexer loaded with the superposition of homogeneous ON-OFF sources. The ATM multiplexer has been modeled as a MMPP/D/1/K queue, according to the technique proposed in [1].

A very efficient procedure for the computation of the cell loss probability of the MMPP/G/1/K queue has been presented: in spite of being an approximate approach, this procedure yields extremely accurate results.

The availability of an accurate and efficient analytical tool to evaluate the cell loss probability has made possible to assess the value of the buffer size required to meet the constraint imposed by the loss performance in the ATM multiplexing.

We point out that the efficient solution method here presented for the MMPP/G/1/K queue can be also applied using any other matching technique, as that proposed in [8], which overcomes the applicability limits inherent in the matching technique here adopted.

Appendix A

Let $S(z) = R(1-z^{-1})A$ and let $x_i(z), u_i(z)$ and $v_i(z)$ ($i = 0, 1, \ldots, m$) be the eigenvalues, the right eigenvectors and the left eigenvectors of $S(z)$, respectively. On the basis of the definition of $S(z)$ and of eq. (4), it is easily found that the eigenvalues $\lambda_i(z)$ of $A(z)$ are given by $\lambda_i(z) = \tilde{H}(x_i(z))$, $i = 0, \ldots, m$.

We assume that the eigenvalues are indexed so that $\lambda_0(z) > Re[x_i(z)] \geq \ldots \geq Re[x_m(z)]$, or equivalently $\lambda_0(z) > |x_i(z)| \geq \ldots \geq |x_m(z)|$, where $\lambda_0(z)$ is the Perron-Frobenius eigenvalue.
of $A(z)$ and is therefore real and positive, which implies that $x_d(z)$ is real as well. Finally, from [2.9] it can be derived that

$$\chi_d(I) = \Theta \chi_d(I) = q \Theta \chi_d(I) = \Theta q \chi_d = A_o.$$  

If $T(z)$ denotes the matrix whose columns are orderly equal to the right eigenvectors of $S(z)$ and $X(z)$ denotes a diagonal matrix, whose $i$-th diagonal element equals the $i$-th eigenvalue of $S(z)$, it is known that $T^(-1)(z)$ is just the matrix whose rows are orderly equal to the left eigenvectors of $S(z)$ and that $S(z) = T^(-1)(z)X(z)T(z)$. Then simple algebraic manipulations show that:

$$C(z) = UA^{-1}\Theta^{-1} \sum_{i=0}^{m} d_i(z)u_i(z)\nu_i(z), \quad (A.2)$$

wherein $d_i(z)$ is equal to:

$$d_i(z) = \frac{\Theta \chi_i(z)\tilde{H}[x_i(z)]}{z\tilde{H}[x_i(z)]}, \quad i = 0,1,...,m. \quad (A.3)$$

It can be seen that $z$ can be safely set to 1 in all the $d_i(z)$, apart from $d_{d}(z)$, which requires the application of the L'Hospital theorem to be computed in $z = 1$. The result is:

$$d_d(1) = \lim_{z \to 1} d_d(z) = \frac{\Theta \chi_d(I)}{1-\Theta \chi_d(I)} = \frac{A_o}{1-A_o}. \quad (A.4)$$

Moreover, a straightforward consequence of the fact that $S(I) = R$ is that $x_d(I) = 0$, $u_d(I) = e$ and $\nu_d(I) = q$. Hence, since in general $\nu_i(z)u_i(z) = \delta_i, i_d = 0,1,...,m$, it descends that $\nu_i(1) = 0$, $i = 1,...,m$. Then, on the basis of eqs. (A.2) and (A.4), it follows that $C(I)e = UA^{-1}\Theta^{-1}e \frac{A_o}{1-A_o}$.

In order to find the matrix $C(I)$, we can exploit this result as follows. Multiplying both sides of the expression of $C(z)$ by $U^T A(z)$ and setting $z = 1$, it can be obtained that $C(I)(I-A) = (U-I)A$. Since $AR = RA, MR = RM, A-I = \Theta MR$ and $U-I = UA^T R$, it follows that $[C(I)+UA^{-1}\Theta^{-1}A^M]R = 0$. This in turn implies that the matrix in square brackets can be written as the product of a column vector $y$ by the row vector $g$, since $g$ is the unique solution of the system $gR = 0$, up to a multiplication factor. Formally, $C(I)+UA^{-1}\Theta^{-1}A^M = yg$, which implies that $y = (I-A)^{-1}UA^T \Theta^{-1}g$. Hence, reminding that $Q(I) = C(I)+UA^T \Theta^{-1}A^M(I-eq)$, it can be derived that:

$$Q(I) = \frac{A_o}{1-A_o}UA^{-1}\Theta^{-1}eq. \quad (A.5)$$

As for the matrices $Q(\eta_i)$ and $Q(\zeta_i)$, they can be computed for example from the spectral form of the matrix $C(z)$, given in eq. (A.4), yielding:

$$Q(\eta_i) = \frac{UA^{-1}u_\eta(\eta_i)\nu_\eta(\eta_i)x_\eta(\eta_i)}{(1-\eta_i) [x_\eta(\eta_i)\tilde{H}[x_\eta(\eta_i)]] - 1} \quad (A.6)$$

for $i = 0,1,...,m$ and

$$Q(\zeta_i) = \frac{UA^{-1}u_\zeta(\zeta_i)\nu_\zeta(\zeta_i)x_\zeta(\zeta_i)}{(1-\zeta_i) [x_\zeta(\zeta_i)\tilde{H}[x_\zeta(\zeta_i)]] - 1} \quad (A.7)$$

for $i = 0,1,...,m$.

Taking the derivative of both sides of the identity $[R+IA(z)]u(z) = x_d(z)u(z)$ and multiplying both sides of the resulting identity by $\nu_i(z)$, it can be obtained that $x_d(z) = \nu_i(z)u_i(z)$, $i = 0,1,...,m$. Therefore, we need not compute the derivatives of the eigenvalues at $\zeta_i$ and $\eta_i$ explicitly, since the knowledge of the eigenvectors is sufficient. Moreover, in case of deterministic service times, $\tilde{H}[x_\eta(\eta_i)]$ and $\tilde{H}[x_\zeta(\zeta_i)]$ reduce to $\Theta \zeta_i$ and $\Theta \eta_i$, respectively.

As for the poles of $C(z)$, they obviously coincide with the roots of $det(zI-A) = \prod_{i=0}^{m}(z-\chi_i(z))$. The eigenvalues $\chi_i(z)$ have the following properties: i) they are convex functions of $z$; ii) for $z \geq 0$ they are strictly positive; iii) $\chi_d(z) > |\chi_i(z)|$, for $i = 1,...,m$; iv) $\chi(I) = 1$. Simple geometrical considerations based on these properties show that the equations $z = \chi_i(z)$ have two distinct real roots, which satisfy $\eta_i < \eta_i < \zeta_i < \zeta_i < \zeta_i$, wherein $\zeta_i$ and $\eta_i$ are the roots of $z = \chi_d(z)$ and $\eta_i$ and $\zeta_i$ are the roots of $z = \chi_d(z)$, for $i = 1,...,m$.

References


