Averaged Dynamics and Large Deviations Theory for the Analysis and Synthesis of Communication Network Protocols.

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Abstract.
Protocols in a communication network can be interpreted as distributed control algorithms influencing the evolution in time of the state of the network. We use this view to study the performance of algorithms over very long time intervals. This is useful when modelling the performance of a lower layer protocol as seen by a higher layer. Good performance requires the state to remain close to a locally stable equilibrium point of the often highly nonlinear averaged dynamics. We illustrate the use of large deviations theory in studying the stability and robustness of routing, admissions control and ALOHA.

1. Introduction.
This paper illustrates the use of control theory concepts such as stability and robustness in the analysis and design of communication network protocols. The tools we introduce are the analysis of the averaged dynamics and large deviations theory. The paper is mainly aimed at teletraffic or communications theorists, but it is believed that the tools will eventually prove useful to engineers designing new routing or flow control protocols. We emphasise the long term behaviour of a protocol as seen by a hierarchically higher layer. The tools we propose fit very well into the hierarchical description of a network. Large deviations results lead naturally to a concept of a finite number of aggregated states and thus to discrete event models of protocols.

Protocols can be defined as distributed algorithms [3] which determine how the state of the network evolves as a function of the randomly arriving service requests. This is similar to distributed control problems: in each processing node only limited information is available about the global state of the system. It is well known in control systems theory that it is very hard to design robustly stable decentralised controllers. Even when each control loop separately is stable, global interactions can still destabilise the system. This is particularly important when studying the behaviour over very long time intervals, for example when looking at the behaviour of a lower layer protocol, of the OSI-hierarchy, as seen by the next higher layer with a much longer time constant.

In this paper we describe the state of a communication network via a countable state Markov process with a transition matrix influenced by the protocol to be designed. This is of course the same assumption as in the classical dynamic programming approach to optimal control of queues. However due to the decentralised nature of the protocol no dynamic programming algorithm is available in this case, except under very special assumptions on the structure of the information available at each node [14]. There is thus very little hope for extending the optimal routing design via team theory [5] to general problems.

In our model the dynamics of the system, determined by the transition matrix of the Markovian state of the network, is determined on the one hand by the protocol and on the other hand by the characteristics of the random point processes of transmission requests. When the noise levels are fairly low it is possible to analyse the performance of a protocol by considering averaged state transitions. This usually leads to very complicated nonlinear deterministic models, with one of the locally stable equilibrium points corresponding to good performance of the protocol. The protocol should be designed such that the domain of attraction of this good opera-
ting point is as large as possible. Due to noise, escape from this domain to bad operating points is inevitable after sufficiently long time intervals. Large deviations theory (LDT) allows us to calculate the order of magnitude of the logarithm of the mean of this escape time. This will be explained in §2.

In §3 these concepts will be illustrated for a simple caricature of decentralised routing, while §4 considers admissions control to an SPC-switch. §4 also illustrates the robustness problem, e.g. with respect to unmodelled retrials in the SPC-switch. Bistability effects in these models, such as hysteresis, are explained directly via LDT for the Markov processes involved, not via diffusion approximations as in [13]. It is important to point out that LDT applied to a diffusion approximation does not lead to the correct estimates of the very long term behaviour [16]. In both sections LDT also indicates that at a coarser level of time and state scales the system can be described as a finite state Markov process, a very simple discrete event system.

2. Large Deviations Theory and Averaged Dynamics.

This section very briefly summarises some results from LDT as developed by Wentzell and Freidlin (see [10,8,7]). Consider a Markov process $X_t$ with a grid in $\mathbb{R}^n$ as state space. Suppose there are $I$ different point processes, with arrival rates $\lambda_i(X_t)$, influencing this Markov process $X_t$. With each arrival of type $i$ there corresponds a transition of the state from $X_{t^-}$ to $T_i(X_{t^-})$. Then the transition matrix of $X_t$ can be written as: $A = \sum_{i=1}^{I} \lambda_i(.) (T_i(.) - I)$, where $T_i$ is interpreted as an operator on $\mathbb{R}^n$ and $I$ is the identity operator. Note that changes in the protocol cause a change of the operator $T_i$ (where is an arrival routed to? does one accept an arrival?) and thus a change of $A$.

It is easy to write down the averaged dynamics:

$$\dot{z}_t = \frac{E(X_{t^+} - X_t | X_t = z_t)}{dt} = \sum_{i=1}^{I} \lambda_i(z_t) (T_i(z_t) - z_t).$$

(1)

This will in general be a highly non-linear system, with many equilibrium points (zeros of the right hand side). Some of these equilibria will be locally stable, attracting trajectories, some others unstable. Consider as an example the average upward and downward drift for the ALOHA system in fig. 1, where $x_s$ is a stable and $x_u$ an unstable equilibrium point. Here $x_s$ is a good operating point with only a few retransmission mode stations. However noise will inevitably cause the state $X_t$ to grow larger than $x_u$ and from there on the average behaviour is towards infinity, the well-known ALOHA-instability.

Far away from an equilibrium point the trajectory of $X_t$ is described very well by (1). On the other hand, for small oscillations around a stable equilibrium point adding Brownian noise to (1) leads to a diffusion approximation, which locally around the stable equilibrium point is a good model for $X_t$. Neither of these approximations allow an accurate description of large excursions away from the stable equilibrium point, or of escapes from a domain of attraction.

To achieve such a large deviations description, consider the scaled process $X_{t/n}$ (notice the difference with the scaling $X_{t/n}$ of a diffusion approximation). For large $n$ this scaled process behaves with probability close to 1 like the averaged system (1). However LDT also allows us to calculate the logarithmic order of magnitude of the smallness of the probability of large deviations from this average behaviour. Let $D$ be a smooth region contained in the domain of attraction of a locally stable equilibrium point and let $x_0$ start at a point $x$ inside $D$; let $t_0$ be the first time the scaled process leaves $D$; let $\phi$ be a smooth trajectory in $\mathbb{R}^n$. Then large deviations theory gives explicit formulae for calculating limits such as the following (where $L$ and $H$ are functionals to be defined below and where $\partial D$ is the boundary of $D)$:

$$\lim n \log P(|X_{t^n}/n - \phi_0| > \delta) = \inf \int_{0}^{T} L(\phi, \dot{\phi}) dt$$

$$\lim n \log E \tau^d = \inf \{ \int_{\partial D} u(z) \, dz \}$$

The action functional $u(x)$ can be ob-
tained by solving the following Hamilton-Jacobi partial differential equation within $D$:
$$H(x, \text{grad}(u(x))) = 0. \quad (2)$$

Moreover escape from $D$ always occurs, with probability close to 1, near the point on the boundary $\partial D$ where the action functional $u(x)$ achieves its minimum, and similarly the escaping trajectory of the scaled state remains arbitrarily close to the most likely escape path which satisfies the equation
$$d\Phi_t/dt = \text{grad}(H(\Phi_t, \text{grad}(u(\Phi_t)))) \quad (3)$$

The Hamiltonian $H$ and its Legendre dual $L$ are specified by the transition matrix $A$ in the following way:
$$H(x, \alpha) = \sum_{j} \mu_j(x) \cdot \left[ \exp(\alpha_j) \cdot (T_j(x) - x) \right] - 1 \quad j$$

where the subscript $j$ indicates the $j$-th component of the change in state $T_j(x) - x$ under an arrival of type $i$. The dual of $H$ is given by
$$L(x, S) = \sup_{\alpha} \left[ \alpha^T S - H(x, \alpha) \right].$$

Let us illustrate these results for a birth-and-death process, e.g. the model of a queue with state dependent arrival and departure rates $\mu_i(x)$ and $\nu_i(x)$ resp. This corresponds to $T_i(x) = x + 1$ and $T_i(x) = x - 1$. The Hamiltonian is:
$$H(x, \alpha) = \mu_i(x) \cdot \left( \exp(\alpha) - 1 \right) + \nu_i(x) \cdot \left( \exp(-\alpha) - 1 \right) \quad (5)$$

with the following Legendre dual:
$$L(x, S) = \mu_i(x) + \nu_i(x) + S \cdot \ln(S + n) - S \cdot \ln(2 \cdot \mu_i(x)) - n \quad (6)$$

where $n = \sqrt{S^2 + 4 \cdot \mu_i(x) \cdot \nu_i(x)} \quad (7)$

The averaged dynamics $\dot{z}_i = \mu_i(z_i) - \nu_i(z_i)$ describe the most likely path returning to a stable equilibrium point. The most likely escape path away from a locally stable equilibrium point satisfies $d\phi_t/dt = \mu_i(\phi_t) - \nu_i(\phi_t)$. The unique solution of the Hamilton-Jacobi equation turns out to be
$$u(x) = \int_{x_1}^{x} \frac{\ln(\mu_i(z)/\mu_j(z))}{dz} \quad (8)$$

when there are two equilibrium points, a stable one $x_1$ and an unstable one $x_2$, $x_1 < x_2$, as in fig. 1, then one can conclude that the log of the mean time until first escape of the trajectory to infinity is given by $u(x_1)$.

Unfortunately the above theory of Freidlin and Wentzell is not strictly speaking valid when an equilibrium point occurs on the boundary of $D$. For most queueing examples the origin is a stable equilibrium point ($\mu_i(0) = 0$). Recently Dupuis, Ishii and Soner[7] have derived the proper boundary conditions for this more general case. It turns out that in the one-dimensional example above the results remain valid provided $\mu_i(0)$ is interpreted as the limit of $\mu_j(x)$. However in higher-dimensional examples it will turn out that the most likely escape path very often follows the boundary of $D$ for some time. This leads to extra boundary conditions which are crucially important. This will be illustrated in the next section.

It should also be remarked that explicit solutions as for the birth-and-death example are very rare. Consider an ALOHA-system with retransmission probability $p(x)$ and Poisson arrivals with rate $q$. Then the Hamiltonian is
$$H(x, \alpha) = -q + \ln(\exp(q \cdot \exp(\alpha)) + x \cdot p(x) \cdot (1 - p(x))^{x_1} \cdot (\exp(-\alpha) - 1) + q \cdot (1 - p(x))^x \cdot (\exp(\alpha) - 1))$$

Its Legendre dual $L(x, S)$ involves the solution $z(S)$ of an algebraic equation:
$$L(x, S) = -q + S \cdot \ln(z(S)) + \ln(\exp(q \cdot z(S)) + x \cdot p(x) \cdot (1 - p(x))^{x_1} \cdot (z(S) - 1) - 1) + q \cdot (1 - p(x))^x \cdot (z(S) - 1))$$

Action functionals and logarithmic escape rates can only be calculated numerically in this case, involving a complicated function like $z(S)$ above.

Remark. Notice that the use of LDT is different here from that in papers such as [6, 19]. There LDT is used to suggest a transformation speeding up simulations of unlikely events, by centering probability mass around the most likely escape path. Here we try to use LDT estimates directly for approximate performance evaluation, analytically or numerically.

3. A simple routing example.

Consider the following very simple routing problem, which tries to penalise rerouting in order to model (or rather caricature) bistability in a large adaptively routed telephone network (See fig. 2). There are two queues with arrival rates $\mu_i$ and departure rate $\nu_i$ resp. and with queue lengths $X_i$ at time $t$. Whenever $X_i \geq x_i$ a newly arriving service request at queue $i$ is sent to the other queue where it generates two service requests, indistinguishable from then on from normal service requests. Suppose that $\mu_i < \mu_j < 2 \cdot \nu_i$ for $i, j = 1, 2$ and $i \neq j$; then it is clear that in region I of the state space (see fig. 3) the average trajectories are attracted towards the origin, which is thus a locally stable equilibrium point. In region IV however the average trajectories are pointing towards infinity, and both queue lengths will keep growing.
fig. 2: routing model

indefinitely. The system is globally unstable. Interesting questions are therefore: How long does it take on average before the state escapes from close to the origin into region IV? What is the most likely escape path?

It is easy to write down the Hamiltonian and its Legendre dual inside region I:

$$H_I(x,a) = \mu_{1,1} \cdot (\exp(a_1) - 1) + \mu_{2,1} \cdot (\exp(a_2) - 1)$$

$$L_I(x,S) = \sum (\mu_{1,1} + \mu_{2,1}) S_i \cdot \ln(S_i + n_i) - S_i \cdot \ln(2 \cdot \mu_{1,1}) - n_i$$

where $n_i$ is defined by appropriately indexing the parameters in (8). This dual can be written down explicitly because only horizontal and vertical transitions of the states occur, decoupling the maximisations over $a_1$ and over $a_2$. The same decoupling resulting in an explicit solution occurs in region II, III and IV, but there the explicit form of $L$ is more complicated because it involves the solution of a third order equation.

Solving the Hamilton-Jacobi equation (2) in the whole state-space $\mathbb{R}^2$, however is tricky as pointed out in [7]. At the boundaries $x_i = 0$ and at the interior boundaries $x_i = \bar{x}_i$ local Hamiltonians have to be calculated. On boundary I e.g.

$$H_I(x,a) = \mu_{1,1} \cdot (\exp(a_1) - 1) + \mu_{2,1} \cdot (\exp(a_2) - 1) + \mu_{2,1} \cdot (\exp(a_2) - 1)$$

The action functional is specified by the viscosity solution of the equation (2) with a boundary condition roughly specified by

$$\min(H_I(x,\text{grad}(u)), H_I(x,\text{grad}(u))) = 0$$

( for a rigorous statement see [7]). We conjecture that because all transitions on the boundary are either parallel or orthogonal to this boundary, the action functional, i.e. the viscosity solution of the Hamilton-Jacobi equation, will simply be the sum of two action functionals like (8), at least in the part of region I close to the origin:

$$u(x) = u_1(x) + u_2(x)$$

where $u_i(x)$ is obtained by properly indexing the rates in (8).

On the interior boundaries similar modifications of the Hamilton-Jacobi equation are necessary, e.g. on $x_i = x_i$ $u$ satisfies

$$\min(H_I(x,\text{grad}(u)), H_{II}(x,\text{grad}(u))) = 0$$

to account for the possible discontinuity of the derivatives of $u$ on this boundary. The effects of this boundary condition will propagate back into region I except where $u$ is smaller than the minimal value $u$ achieves on the interior boundary (the arced region in fig. 3).

Therefore we conjecture that the action functional is as follows: let $u$ be as in (2) and let

$$\mathcal{U} = \min(u(x_i,0), u(0,x_i))$$

Then the action functional is $\min(\mathcal{U}, u(x))$. Thus we conjecture that the most likely escape path follows one of the boundaries, boundary 1 in fig. 3, then follows the adjoining interior boundary and finally follows a most likely
trajectory in region IV. Moreover we conjecture that the last two parts of the escape trajectory will be followed infinitely fast (in the LDT time scale n.t of course) so that the mean escape time is asymptotically equal to $\bar{U}$. A proof of this would require showing that the conjectured functional is the unique viscosity solution to the Hamilton-Jacobi equation with all its boundary conditions.

Whether the above conjecture is true or not, it is clear from the analysis that no algorithm rerouting arrivals of stream $i$ whenever $X_i(t)$ is too large will make the system stable. It is equally clear that rerouting only for an intermediate range of queue lengths will lead to bistability, with a second undesirable locally stable equilibrium point appearing. To ensure stability it will however suffice to exchange information about the other queue length as soon as the state comes close to the critical boundary of the system (the boundary of the arc ed region in fig. 3; notice that this is not the same as the boundary of the domain of attraction of the origin). Thus LDT has given us a minimal requirement on the information exchange for a good protocol.

Remark: Notice that the complicated interior boundary conditions above could be avoided if the rerouting was done on the basis of a smooth rerouting probability $p(x_i)$ instead of the above step function. However then the Hamiltonian would be very complicated in the transition region near the boundary, and even harder to solve. Notice also that if the smooth rerouting function tends to the step function the results will behave continuously according to the LDT. This is the approach of [15] to the problem of boundary conditions.

To extend the above routing problem to routing in a large communication network, e.g. a DNHR telephone network, one would have to consider many servers which on overflow reroute to one (possibly randomly chosen) other server in their neighbourhood. This would give a caricature of an adaptively routed network in the same vein as the models considered in [2, 12, 17]. While the dimensionality of the equations increases dramatically, it will still be true that there will be one most likely escape path which initially follows the edges of the state space. A good routing protocol should be designed so that $\bar{U}$ is as large as possible.

It is also clear from the above model that, except during a transient of negligible duration in the fast LDT time scale, a server in the above model can be in either of two states: close to the origin (good operating point, region I) or saturated (region IV near the upper right hand corner). One can easily show that transitions from good to saturated state occur (again on the fast LDT time scale) after an exponentially distributed time with mean $\bar{U}$. As seen from the point of view of a higher layer (e.g. the network manager) each server might thus be modelled as a two-state Markov process. Of course transitions for different servers are coupled so that the overall finite-state Markov process is still quite complicated. This coupling is exactly what the manager in the higher layer of the hierarchy should be concerned with, not the internal transitions in each server.

4. Admissions control at an SPC-switch.

In this last section we very briefly indicate the robustness problem which inevitably results from unmodelled dynamics. Consider an SPC-switch for which a stabilising admissions controller has been designed [4]. Such controllers are designed to avoid a decrease in throughput due to premature hang-up when the call set-up phase becomes too long as a result of processor overload. This design however does not take into account the retrials. The following very simple model illustrates that the positive feedback of retrials inevitably destabilises the system: the more retrials there are, the higher the arrival rate and the more overloaded the processor becomes.

![fig. 4: SPC-model with retrials.](image-url)

Consider the system of fig. 4 where call set-up is modelled via a single server with service rate $\mu(x_i)$ decreasing for long queue lengths $x_i$. New calls arrive according to a Poisson process with rate $\lambda$. Calls rejected in the admissions switch (a fraction $p(x_i)$) and a queue length dependent fraction $q(x_i)$ of the completed calls go to an M/H/$\infty$ queue (with mean service time $1/\mu$) modelling the retrials (with $x_i$ denoting the number of users waiting to
Whether it is feasible to solve this via the performance of a protocol over very long periods of time. Whether the Poisson clumping heuristic of [1] can be used to improve the results from estimates of the order of magnitude of the logarithm of the likelihood of very rare events to estimating actually the probabilities themselves requires further research.

Having obtained albeit approximate expressions for the performance in terms of the dynamics of the state processes, it becomes possible to exploit the analogy between protocols and control theory for optimal or at least robustly stable protocol design.

5. Conclusions.

This paper has attempted to show that large deviations theory can be helpful in the analysis of the performance of a protocol over very long periods of time. Whether the Poisson clumping heuristic of [1] can be used to improve the results from estimates of the order of magnitude of the logarithm of the likelihood of very rare events to estimating actually the probabilities themselves requires further research.

Having obtained albeit approximate expressions for the performance in terms of the dynamics of the state processes, it becomes possible to exploit the analogy between protocols and control theory for optimal or at least robustly stable protocol design.

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6. References.