STATISTICAL MULTIPLEXING OF MARKOV MODULATED SOURCES: THEORY AND COMPUTATIONAL ALGORITHMS

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We give theory and algorithms for exactly calculating the stationary state distribution of a statistical multiplexing system with \( K \) sources. Each source is an \( N \)-state Markov chain and its state determines the Poisson rate of packet generation. The efficient analyses of such models are important in the design of Broadband-ISDN applications involving statistical multiplexing of many bursty sources. Our algorithm for computing the spectral expansion of the state distribution has complexity \( O(K^{3(N-1)}) \) for \( N \) fixed and \( K \) large. The theory gives the (exact) decomposition of the eigenvalue problem of the entire system into many small eigenvalue problems, the Kronecker-product form to the eigenvectors and a recursive algorithm for (exact) aggregation. The rate matrix in matrix-geometric theory is efficiently computed from our spectral representation.

1 INTRODUCTION

We give a theory and an efficient algorithm for computing the stationary probability distributions of a multiplexing system in which information from \( K \) independent sources is multiplexed for transmission over a channel. Each source is a \( N \)-state continuous-time Markov chain with generator \( M \). When a source is in state \( i \) it generates discrete units of information, say packets, in a Poisson stream at a rate \( \lambda_i \) (\( 1 \leq i \leq N \)). This process is called a Markov Modulated Poisson Process (MMPP). The service time of each packet is an independent, exponentially distributed random variable with mean \( 1/\mu \). When the channel is busy packets from all sources wait in a common buffer. The size of the buffer is assumed infinite.

The model is an established paradigm and yet its practical importance in communications is still growing. New impetus for the analysis of statistical multiplexing is coming from the imminence of B-ISDN which proposes to have sources of varying degree of burstiness share the bandwidth. Rules for either accepting or blocking calls with bursty traffic require analysis. New impetus for the analysis of statistical multiplexing are reported in [7,8,9]. These studies have compared, among others, the fluid approximation [6] and the techniques in [17]. Comparative studies of models and analytic techniques for statistical multiplexing are reported in [7,8,9]. These studies have compared, among others, the fluid approximation [6] and the techniques in [10,11].

The MMPP has been widely used to model traffic in communication systems [12,10]. The Interrupted Poisson Process (IPP), a special case of MMPP, has been widely used to model traffic streams in [13,14]. Neuts' matrix geometric method applies to the model in this paper [15]. The limitations of this method, which are due to slow convergence in bursty environments and in high traffic intensities, have been noted previously (see for example [16]). In addition, the complexity grows rapidly with increased size of state space, a problem exacerbated in B-ISDN applications. Another approach is based on generating functions [17]. In large problems this approach requires the calculation of roots of polynomials of very high degree.

The direct antecedents of the present work are the results of Anick et al. [6], Kosten [18], Mitra [19] and Stern and Elwalid [20] on stochastic fluid models. Various results presented in this report are new and we have been able to apply them to fluid models. This report is necessarily brief and [21] may be consulted for further information.

Our approach is based on computing the spectral expansion of the equilibrium state distribution of the system. By exploiting the structure of the problem, we have obtained explicit decompositions of the eigensystem and from it a provably efficient algorithm. Our analysis has two phases: the first phase analyzes the unaggregated system, i.e., where the description of the state of the sources is oblivious of the fact that the \( K \) sources are statistically identical; the latter is taken into account in the second phase on the aggregated system. The crucial decompositions and Kronecker-product forms are uncovered in the unaggregated form, while all the calculations are made in the aggregated form.

The output of the first phase are the eigenvalues and eigenvectors \((z_{jk}, \phi_k)\), where the dimension of the eigenvectors is \( NK \), which is the number of unaggregated states of the sources. The output of the second phase is the aggregated spectral expansion

\[
\pi^{(A)}(n) = \sum_{\tau} a_{\tau} z_{\tau}^n \xi_{\tau}, \quad (n = 0, 1, 2, \cdots) \tag{1}
\]

where \( \pi^{(A)}(n) = \{\pi(n, \sigma)\} \) is of dimension \( L \), where

\[
L = \begin{pmatrix}
K + N - 1 \\
N - 1
\end{pmatrix} \tag{2}
\]
and \( \pi(n, \sigma) \) denotes the probability that the buffer content is \( n \) and the aggregate state of the sources is \( \sigma \). It is shown that the the eigenvalues of the aggregated source-buffer system, \( \{s_T\} \) in (1), are the roots of a family of \( L/N! \) explicit polynomials in which each polynomial is of degree \( 2N! \). Also, a simple recursive algorithm is devised to obtain exactly the aggregate eigenvector \( \xi \) from \( \phi \). The coefficients \( a_T \) in (1) are obtained by solving a system of \( L \) normalization equations. The complexity of the entire computational procedure is \( O(L^3) \). For the case of particular practical interest where \( N \) is small and \( K \) large, the complexity is \( O(K^{3(N-1)}) \), i.e., polynomial in the large parameter \( K \).

### 2 DECOMPOSITIONS

Let the state of source \( i \) be denoted by \( s(i) \) where \( s(i) \in \{1, 2, \ldots, N\} \). The unaggregated state of the sources is given by \( s = (s(1), s(2), \ldots, s(K)) \). We let the state space of the unaggregated source process be

\[
\mathcal{H}_{N,K} \overset{\text{def}}{=} \{ k \mid k \in \mathbb{Z}^K(1 \leq k(i) \leq N) \}.
\]

Let \( \pi(n) \) denote the lexicographic arrangement of \( \{\pi(n; k)\} \),

\[
\pi(n) = (\pi(n; 1), \pi(n; 2), \ldots, \pi(n; N))
\]

The generator of the unaggregated source process is

\[
Q = M \oplus M \oplus \cdots \oplus M, \quad (4)
\]

a \( K \)-fold Kronecker sum [15] on \( M \), and the diagonal rate matrix \( R \) is

\[
R = \Lambda \oplus \Lambda \oplus \cdots \oplus \Lambda, \quad (5)
\]

where \( \Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_K) \). The balance equations of the composite source and buffer system are

\[
0 = \pi(n)(Q - R) + \mu\pi(n + 1) \quad (n = 0)
\]

\[
= \pi(n - 1)R + \pi(n)[Q - (R + \mu)] + \mu\pi(n + 1)
\]

\[
(n \geq 1)
\]

The independent solutions have the form

\[
\pi(n) = z^n\phi.
\]

Here \( z \) is a root of the characteristic polynomial \( |C(z)| \) where

\[
C(z) = \mu z^2 + z [Q - (R + \mu)] + R,
\]

and \( \phi C(z) = 0 \).

Thus each \( z \) is an eigenvalue of the unaggregated system and \( \phi \) is the associated eigenvector. In general they are complex; however, they are all real for time-reversible sources.

The following **ergodicity condition**, which is assumed throughout, allows us to be specific about the number of stable eigenvalues, i.e., satisfying \( |z| < 1 \). Let \( u \) be the stationary probability vector of an individual source, i.e., \( u \cdot M = 0 \), and \( \langle u, 1 \rangle = 1 \), where \( \langle \cdot, \cdot \rangle \) represents the inner product of vectors and \( 1 \) is the vector in which all elements are unity. Then the ergodicity condition is satisfied

\[
\rho < 1,
\]

where \( \rho = \lambda/\theta, \theta = \langle u, 1 \rangle \), and \( \lambda = \mu/K \).

The solution to system balance equations given above has been shown by Neuts [15] to have the matrix-geometric form, i.e., for an appropriate vector \( b \) and matrix \( X \),

\[
\pi(n) = bX^n \quad (n = 0, 1, 2, \ldots)
\]

The "rate matrix" \( X \) can be shown to be composed from the elements of our spectral representation thus,

\[
X = \Phi^{-1}Y \Phi
\]

where \( Y = \text{diag}\{z_1, z_2, \ldots\} \) and \( \Phi \) is the matrix with rows \( \phi_1, \phi_2, \ldots \). A similar representation holds in the aggregated framework as shown in Section 6. The above implies that if the ergodicity condition is satisfied then the number of stable eigenvalues is \( |\mathcal{H}_{N,K}| = N^K \) and that there exists an equal number of eigenvalues for which \( |z| \geq 1 \), including one at 1.

In (6) consider the following particular form:

\[
\phi = u_1 \otimes u_2 \otimes \cdots \otimes u_K
\]

On substituting in (8) the above and the expressions in (4) and (5) for \( Q \) and \( R \), we find that (8) is satisfied if and only if there exists a set of \( K \) numbers \( \nu_1, \nu_2, \ldots, \nu_K \) which, with pair \((z, \phi)\) satisfy the equations

\[
u_iA(z, \nu) = 0, \quad (1 \leq i \leq K)
\]

where \( A(z, \nu) = \nu z^2 + z[M - (\Lambda + \nu)] + \Lambda \).

The above is is a system of \( K \) coupled eigenvalue problems where each is \( N \)-dimensional. It will be useful to view the equation \( |A(z, \nu)| = 0 \) as the following equation in \( \nu \) with parameter \( z \):

\[
|zI - B(z)| = 0,
\]

where \( B(z) = \frac{1}{1 - z}M + \frac{1}{z}\Lambda \).

The function on the left side of (16) is the characteristic polynomial (of degree \( N \)) in \( \nu \) of the matrix \( B(z) \). The solutions in \( \nu \) are denoted by \( \eta_j(z) \) \( (1 \leq j \leq N) \). These are, in general, continuous functions of complex variables with singularities at 0 and 1. Notice that each \( k \in \mathcal{H}_{N,K} \) gives an equation

\[
\sum_{i=1}^{K} \eta_i(z) = \mu,
\]

whose solutions \( z \) are solutions to the coupled eigenvalue problem in (13)-(15). (We attempt to conform to the practice of denoting the elements of a vector \( k \) by \( k(1), k(2), \ldots \). The subscript on a vector is reserved for indexing the vector.) Typically there are many \( k \) (later to be aggregated into one class) that give the same left hand side to the equation and therefore the same solutions \( z \). These redundancies are eliminated in
the aggregated version of (18).

2.1 Aggregate Source State

Let \( \sigma \) denote an aggregate source state and \( S_{N,K} \) the state space where,

\[
S_{N,K} \equiv \{ \sigma | \sigma \in \mathbb{Z}^N, \ 0 \leq \sigma(i) \text{ and } \sum_{i=1}^{N} \sigma(i) = K \}.
\]

Note that \( |S_{N,K}| = L \). We interpret \( \sigma(i) \) to be the number of states in state \( i \) (\( 1 \leq i \leq N \)). The set of unaggregated source states which are lumped into a particular aggregate source state \( \sigma \) is

\[
A(\sigma) = \left \{ k \in \mathcal{H}_{N,K} \big| \sum_{i=1}^{K} 1(k(i) = j) = \sigma(j) \ (1 \leq j \leq N) \right \}.
\]

where \( 1(\cdot) \) is the indicator function whose value is either 0 or 1. The cardinality of \( A(\sigma) \) is

\[
C(\sigma) = K!/[\sigma(1)! \cdots \sigma(N)!]. \quad (19)
\]

3 AGGREGATED SYSTEM

Our starting point is (18). We note that for any \( \sigma \in S_{N,K} \), all \( k \in A(\sigma) \) give rise to a common equation (18) and therefore also a common set of roots. Hence the aggregated counterpart to (18) is

\[
\sum_{j=1}^{N} \sigma(j) g_j(z) = \mu \ (\sigma \in S_{N,K}) \quad (20)
\]

Now consider the following family of polynomials of \( z \) parametrized by \( \sigma \in S_{N,K} \):

\[
P(z; \sigma) = \sum_{\omega \in \Omega_N} \left \{ \omega(\sigma), g(z) \right \} - \mu \quad (21)
\]

where \( \omega(\sigma) \) is a permutation on the \( N \)-tuple \( \sigma \) and \( \Omega_N \) is the set of all permutations \( \omega \) on \( N \)-tuples. In (21), \( g(z) = \{ g_1(z), \ldots, g_N(z) \} \). These polynomials should be viewed as the symmetrization of the set of equations in (20) since \( P \) is symmetric in \( \sigma \) and any solution \( z \) of (21) is also a solution of some equation from the set in (20). Conversely, any solution of (20) is also a root of \( P(z; \sigma) \) for an appropriate \( \sigma \). We have proven [21]

Theorem 1 (i) \( P(z, \sigma) \) is a polynomial in \( z \) of degree \( 2N! \) and has real coefficients.

(ii) \( P(z, \sigma) = P(z, \omega(\sigma)) \) for all \( \omega \in \Omega_N \), i.e., \( P \) is symmetric in \( \sigma \). Hence there are only \( L/N! \) such distinct polynomials.

(iii) The union of the roots of all the distinct polynomials give all the eigenvalues of the aggregated system.

(iv) The set of roots of \( P(z; \sigma) \) may be grouped into pairs and each pair is associated with a distinct permutation of \( \sigma \).

A sketch of the constructive proof of the polynomial structure of \( P(z; \sigma) \) is as follows. The expression in (21) is a symmetric polynomial in \( g_1, g_2, \ldots, g_N \). Hence from the “Fundamental Theorem on Symmetric Polynomials”, \( P(z; \sigma) \) is a polynomial in the elementary symmetric polynomials of \( g_1, g_2, \ldots, g_N \). These elementary symmetric polynomials may be obtained by noting that they are also the coefficients of the characteristic polynomial of \( B(z) \). Finally, the factor \( \{z(1-z)^{N_1}\} \) in (21) allows the elementary symmetric polynomials to be expressed as polynomials in \( z \). Notice that in effect what has been achieved is a closed-form factorization of the characteristic polynomial of the aggregate source-buffer system, which is of degree \( 2L \), into \( L/N! \) factors where each factor is an explicit polynomial of degree \( 2N! \). The theorem and algorithm for synthesizing the polynomials constitute an important extension of the factorization results in [6] and [19].

We illustrate the above procedure for the case of \( N = 2 \) in which

\[
M = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (22)
\]

\[
P(z; \sigma) = \gamma^2 - (\gamma + \sigma_1 + \sigma_2)\{-(\alpha + \beta)z + (\lambda_1 + \lambda_2)(1-z)\} + \sigma_1\sigma_2\{-(\alpha + \beta)z + (\lambda_1 + \lambda_2)(1-z)\}^2 - (\sigma_1 - \sigma_2)^2(1-z)\{(\alpha\lambda_2 + \beta\lambda_1)z - \lambda_1\lambda_2(1-z)\} \quad (23)
\]

where \( \gamma = z(1-z)\mu \). Each member of the above family of polynomials \( \{P(z; \sigma)\} \) is of degree \( 4 \). It can be shown that they are ordered by a simple function of their parameter \( \sigma \): for \( -\infty < z < \infty \),

\[
\sigma'(1)\sigma''(2) \geq \sigma'(1)\sigma'(2) \Rightarrow P(z; \sigma') \geq P(z; \sigma'') \quad (23)
\]

3.1 Extremal Polynomial

The polynomial obtained when \( \sigma = Ke_j \) in (21), where \( e_1 = (1, 0, \ldots, 0) \) and generally \( e_j(n) = \delta_{jn} \), is called the extremal polynomial.

Proposition 1 The extremal polynomial \( P(z; Ke_j) \) is obtained thus from \( |A(z, \theta)| \), the characteristic polynomial of a system with a single source and service rate \( \theta = \mu/K \):

\[
P(z; Ke_j) = K^{N_1}|A(z, \theta)|^{(N-1)} \quad (24)
\]

By repeating the argument in Section 2 which enumerated the number of eigenvalues inside the unit circle, we have as a corollary to above proposition,

Proposition 2 When the ergodicity condition in (9) is satisfied the extremal polynomial has \( N! \) roots with modulus \( < 1 \) and an equal number with modulus \( \geq 1 \), including one at 1.

4 TIME-REVERSIBLE SOURCES

In this section we specialize the results in the previous Section to the class of time-reversible sources, which include those for which the rates are modulated by birth-
death processes. For time-reversible sources the matrix \( M \) can be symmetrized by the similarity transformation 
\[
\tilde{M} = W^{1/2}MW^{-1/2}
\]  
(25)
Here \( W = \text{diag}\{w\} \). A consequence of time-reversibility is that every system eigenvalue \( z \) is real. Moreover, every eigenvalue \( g_j(z) \) \( (1 \leq j \leq N) \) of \( B(z) \) in (17) is also real. Each function \( g_j(z) \) is continuous and continuously differentiable (except at the singular points 0 and 1). At each \( z \) at which \( B(z) \) has distinct eigenvalues the derivative of \( g_j(z) \),
\[
g_j'(z) = \frac{1}{(1-z)^2} \langle \bar{u}_j(z), z \bar{u}_j(z) \tilde{M} \rangle - \frac{1}{z^2} \langle \bar{u}(z), z \bar{u}(z) \Lambda \rangle < 0
\]
where, \( \bar{u}_j(z) = u_j(z)W^{-1/2}, \langle \bar{u}_j, \bar{u}_j \rangle = 1 \) and \( u_j(z) \) is an eigenvector of \( B(z) \). The functions \( g_j(z) \) are indexed to be increasing with \( j \) when \( |z| \) is large. A sketch of these functions is in Fig. 4.1. Related functions which arise in fluid models are studied in [20].

We now present a Newton-type algorithm for computing the eigenvalues of the aggregated system. This is an alternative to the method given in Section 3 and is advantageous when \( N \) is large and the polynomials are hard to construct. Letting \( z^{(n)} \) be the \( n \)-th iterate for the solution \( z \) of (20), Newton’s method is defined as
\[
z^{(n+1)} = z^{(n)} - \frac{\sum_{j=1}^{N} \sigma(j)g_j(z^{(n)}) - \mu}{\sum_{j=1}^{N} \sigma(j)g_j'(z^{(n)})}
\]  
(26)
The set \( \{g_1(z), \ldots, g_N(z)\} \) is the eigenvalues of \( B(z) \), see (17). The derivatives \( \{g_1'(z), \ldots, g_N'(z)\} \) are calculated by using the formula given earlier.

4.1 Dominant Eigenvalue

Proposition 3 For any \( K \), the maximum, \( r \) (minimum, \( r_1 \)) of the aggregated system’s eigenvalues that lie in the interval \((-1, 1)\) is the maximum (minimum) of the roots of the extremal polynomial \( P(z; KE) \) that lie in the interval \([0, 1]\).

The proof is in [21]. The eigenvalue \( r \) is called the dominant eigenvalue. The above proposition implies that it is also the dominant eigenvalue of a system with a single source and service rate \( \theta \). The dominant eigenvalue is one of the most important parameters in characterizing system performance. This above result extends similar results in [6] and [19].

For the IPP case \((\lambda_1 = 0 \text{ in } (22))\) the extremal polynomial corresponds to \( \sigma = (0, K) \). It follows from (23) that \( r_1 = 0 \) and the dominant eigenvalue is
\[
r = \frac{1}{2} \left[ 1 + \frac{\lambda_2 + \alpha + \beta}{\theta} \right] - \frac{1}{2} \left[ 1 + \frac{\lambda_2 + \alpha + \beta}{\theta} \right]^2 - 4 \lambda_2 (\theta + \alpha) \theta^2
\]

5 EIGENVECTORS

Recall that the eigenvectors of the unaggregated system were postulated in (12) to have the Kronecker product form with \( N \)-dimensional vector components \( \{u_j\} \). It is worth noting here that once an eigenvalue \( z \) has been computed, it is straightforward to obtain the corresponding vectors \( u_j(z) \) \( (1 \leq j \leq N) \): first \( B(z) \) is constituted as in (17), then the solutions \( \nu_1, \nu_2, \ldots, \nu_N \) are computed by solving (16) and, finally, \( u_j(z) \) \( (1 \leq j \leq N) \) is obtained from (13) as the null vector of \( A(z; \nu_j) \).

To find the system’s aggregated eigenvectors, we begin by assuming that the eigenvalues of the aggregated system and associated indices \( \sigma \in S_{N,K} \) have been computed. The unaggregated eigenvectors indexed by \( k \), for all \( k \in A(\sigma) \), are
\[
\phi_k = \phi_k(1) \otimes \phi_k(2) \otimes \cdots \otimes \phi_k(K),
\]  
(27)
where
\[
\phi_k(i) \in \{u_1(z\sigma), \ldots, u_N(z\sigma)\} \quad (1 \leq i \leq K)
\]  
(28)
and \( u_j(z\sigma) \) has multiplicity \( \sigma(j) \) \( (1 \leq j \leq N) \). Any pair of unaggregated eigenvectors in the same aggregate class differ only in the order in which the Kronecker product in (27) is taken (Kronecker products are not commutative.) This gives the following result.

Proposition 4 For each \( \sigma \in S_{N,K} \), and all \( k \in A(\sigma) \)
\[
\sum_{l \in A(\tau)} \phi_k(l) \text{ has a common value, say } \xi(\tau)
\]  
(29)
The vectors \( \{\xi(\sigma)\} \) are the eigenvectors of the aggregate source-buffer system.

To evaluate \( \xi(\sigma) \) first define its multivariate generating function,
\[
G(\sigma; y) = \sum_{r \in S(\sigma)} y_1^r \cdots y_N^r \xi(\tau)
\]  
(30)
Alternatively we have
\[
G(\sigma; y) = \prod_{i=1}^{K} (y, \phi_i(k)),
\]  
(32)
where \( k \) is any element of \( A(\sigma) \). From (31),
\[
G(\sigma; y) = \prod_{j=1}^{N} (y, u_j(z\sigma))^{\sigma(j)},
\]  
(33)
We now have a procedure for calculating \( \xi(\tau) \): expand (33) as in (30) and obtain the appropriate coefficient. We illustrate the above for the case \( N = 2 \):
\[
\nu_{1,2} = \frac{1}{2} \left[ \frac{\lambda_1 + \lambda_2}{z} - \frac{\alpha + \beta}{1 - z} \right]
\]
\[ \pm \frac{1}{2} \sqrt{ \left[ \frac{\lambda_1 + \lambda_2}{z} - \frac{\alpha + \beta}{1 - z} \right]^2 - 4 \left[ \frac{\lambda_1 \lambda_2}{z^2} - \frac{\alpha \lambda_2 + \beta \lambda_1}{z(1 - z)} \right] } \]
if we let \( u_j = (1, u_j(2)) \) then
\[ u_j(2) = \frac{1 - z}{\beta} \left[ v_j + \frac{\alpha}{1 - z} - \frac{\lambda_1}{z} \right] \quad (j = 1, 2) \]  

(34)

Finally,
\[ \xi \sigma(\tau_1, \tau_2) = \sum_{l=0}^{\sigma_1} \left( \frac{1}{l} \right) \left( \tau_1 - l \right) \left\{ u_1(2) \right\}^{\sigma_1 - l} \left\{ u_2(2) \right\}^{\sigma_2 - \tau_2 + l} \]

5.1 Recursive Algorithm

From (29) it is clear that typically there are many terms in the composition of \( \xi \sigma(\tau) \). Therefore, the efficiency of our method is dependent on the following algorithm for aggregation. See [21] for proof.

**Theorem 2** For each \( \sigma \in S_{N,K} \),
\[ \xi \sigma(\tau) = H_N(\sigma; \tau), \quad (\tau \in S_{N,K}) \]

(35)

where, for \((1 \leq i \leq N)\),
\[ H_i(s; t) = \sum_{j=1}^{N} H_i(s - e_i; t - e_j)u_i(j) + H_{i-1}(s; t) \]

(36)

and the initial conditions are
\[ H_0(s; t) = \begin{cases} 1 & \text{if } s = t = 0 \\ 0 & \text{otherwise} \end{cases} \]

For each \( i \), the recursion in (36) is executed for \( s \leq \sigma \) and \(|t| = |s|\).

6 BUFFER CONTENT STATISTICS

The stationary probability vector of the unaggregated source process is \( \mathbf{p} = \mathbf{w} \otimes \mathbf{w} \otimes \mathbf{w} \cdots \otimes \mathbf{w} \). The elements of its aggregated counterpart \( \mathbf{p}^{(A)} \) are (for \( \sigma \in S_{N,K} \))
\[ p^{(A)}(\sigma) = c(\sigma)(w(1)^{e(1)}w(2)^{e(2)}\cdots w(N)^{e(N)}) \]

The spectral expansion of the stationary probability vector of the aggregated source-buffer system \( \pi^{(A)}(n) \),
\[ \pi^{(A)}(n; \sigma) = \sum_{\tau \in S_{N,K}} \alpha_\tau \epsilon_\tau \xi_\tau(\sigma) \]  

(38)

must satisfy the normalizations conditions,
\[ p^{(A)}(\sigma) = \sum_{n=0}^{\infty} \pi^{(A)}(n; \sigma) \quad (\sigma \in S_{N,K}) \]

In matrix form,
\[ \mathbf{p}^{(A)} = a[I - Z]^{-1} \Xi \]

(39)

The rows of the \( L \times L \) matrix \( \Xi \) are \( \{\xi_\tau\} \) and \( Z = \text{diag}(z_\sigma) \).

An important function in applications is the complementary buffer content distribution,
\[ G(n) = \Pr[\text{buffer content} \geq n] \]

(40)

\( G(n) \) closely approximates the overflow probability (commonly referred to as "grade of service") of a system with finite buffer of size \( n \), when the given overflow probability is small.

Letting \( q = a[I - Z]^{-1} \) and \( d(\sigma) = (\xi_\tau, 1) \) we obtain
\[ G(n) = \sum_{m=0}^{\infty} (\pi^{(A)}(m), 1) = (qZ^\sigma \Xi, 1) \]

\[ = \sum_{\sigma \in S_{N,K}} q(\sigma) d(\sigma) z_\sigma^n \quad (n = 0, 1, 2, \ldots) \]

The vector \( q \) is obtained by solving a system of \( L \) linear equations.

Notice that if
\[ \mathbf{X} = \Xi^{-1}Z\Xi \]

(41)

then for \( n = 0, 1, 2, \ldots \)
\[ p^{(A)}(n) = \pi^{(A)}(I - \mathbf{X})\mathbf{X}^n \]

(42)

The matrix \( \mathbf{X} \) will be recognized to be Neuts' rate matrix in the aggregated framework.

7 Numerical Investigation

We report on applications of the algorithms developed in this paper to the statistical multiplexing of IPP sources. Fig. 7.1 shows the effect on \( G(n) \) of varying the mean cycle time \( T \) (defined as the sum of mean burst period \( 1/\beta \) and mean silent period \( 1/\alpha \)) while the ratio \( \alpha/\beta \) is held fixed. Observe that decreasing \( T \), i.e., increasing the "jitteriness" of the arrival process results in a decrease in buffer requirement for a given grade of service. As \( \alpha \) and \( \beta \rightarrow \infty \) all correlations in the arrival process are removed and the system behavior approaches that of \( M/M/1 \).

Fig. 7.2 illustrates the effect of source burstiness (peak rate/mean rate) on the system performance. Four curves are displayed as the burstiness is increased while the traffic intensity \( \rho \) is held fixed. Note the expected increase in \( G \) due to increasing burstiness.
REFERENCES