A type of ATM cell multiplexer is studied, where \( z \) statistically independent input links are multiplexed over \( n \) identical multiserver output groups of \( r \) links each, with \( n \cdot r = z \). The cells arriving to that system enter a common buffer from which they are randomly routed over the output groups for dispatching. Two discrete-time queueing models have been developed, allowing the proper sizing of the common memory pool and output queues in front of the multiserver groups, as well as the computation of the waiting time experienced by the cells. Numerical results are provided. Comparisons with simulations show the models to be accurate.

1. INTRODUCTION

This paper analyzes the traffic characteristics of a type of ATM cell multiplexer as depicted in figure 1. The cells coming into the multiplexer from \( z \) statistically independent input links are multiplexed over \( n \) identical multiserver output groups of \( r \) links each (\( n \cdot r = z \)). Upon arrival, the cells enter a common buffer of limited size, named Memory Pool, from which they are routed over the output groups at random, to be dispatched. There is a logical queue, the Output Queue, in front of each group. This type of multiplexer can be found in the Multi-path Self-routing Switching Network described in [1,2].

![Figure 1](image)

The purpose of this paper is to provide analytical methods allowing to gain insight into the performance of that type of multiplexers; the analysis made enables to obtain low probability tails which usually are difficult to obtain or even cannot be obtained by simulation.

The input process to the multiplexer is the superposition of the individual processes at each link. Under the assumption that these individual processes are of Bernoulli type, an exact derivation of the compound process at both the Memory Pool and the Output Queues is done. The arrival process thus found is named the Batch-Geo process.

The multiplexer is modeled as formed up of two major subsystems: the Output Queues that can be studied independently each other, and the Memory Pool whose traffic behavior depends on the combined situation of the former subsystems.

The Output Queues are modeled by feeding the Batch-Geo process into a single bulk-server FIFO queue. A detailed analysis of this queueing system is presented. From this analysis we obtain:

- the queue length and system contents distributions at random and at arrival instants,
- the distributions of the virtual waiting and of the waiting experienced by any arrival.

The common Memory Pool is viewed as finite buffer in front of \( n \) bulk-servers, the former Output Queue subsystems. The service discipline in the buffer is such that, once the cells enter in it, they are randomly assigned an Output Queue and join the FIFO (logical) queue of their assigned server. The analytical method provides:

- the distributions of contents of the Memory Pool at random instants,
- the traffic carried by the system, from which the Cell Loss Ratio, a performance parameter, is obtained.

Thus, from these models we can obtain a de-
tailed insight of the traffic performance of the system, achieve a proper sizing of the common Memory Pool as function of the Cell Loss ratio, as well as evaluate the waiting time experienced by the cells.

Finally it should be noted that the methodology here presented is general for batch arrival systems; the application to the ATM multiplexer problems may be viewed as an illustration.

2. ARRIVAL PROCESS

The first task in our methodological approach for solving the problem is to derive the processes governing the arrival of cells at the Memory Pool and Output Queues, of the above described multiplexer.

2.1. Assumptions

The basic assumptions are:

a) the arrival process of cells to any inlet of the multiplexer is a discrete time process governed by the Bernoulli law [3],
b) these inlets are independent each other,
c) the cell transmission time is herein taken as the elementary unit of time, denoted u.

Thus, the probability generating function (p.g.f.) of the number of arrivals at any inlet, in a time t, is expressed as:

\[ B(a,t) = (p \cdot a + q)^t; \quad q = 1 - p \]  

where p is the constant probability of an arrival occurring at a discrete epoch u.

2.2. Arrival process at the Memory Pool

Owing to the independence between the inlets, the arrival process at the Memory Pool, denoted N(a,t), is the z-fold convolution of processes (2.1):

\[ N(a,t) = (p \cdot a + q)^z \cdot t = o(a)^z \]  

From (2.2) we can derive the probability of no arrivals, and of one or more arrivals, in a time unit u as:

\[ \{ N=0, t=u \} = q^z, \]  
\[ \{ N>0, t=u \} = 1 - q^z \]  

that are constant and independent of previous arrivals. Furthermore, the probability \( g_i \) of getting i arrivals in a time unit, knowing there are arrivals, is:

\[ g_i = \{ N=i / N>0 \} = \frac{c^z_i \cdot p^i \cdot q^{z-i}}{1 - q^z} \]  

(2.4)

where \( c^z_i \) is the combinatorial of z over i elements. This can be interpreted as a batch arrival process, where batches occur at a rate \( 1 - q^z \), their sizes being distributed according to (2.4). The p.g.f. for the \( g_i \) is:

\[ G(a) = \frac{(p \cdot a + q)^z - q^z}{1 - q^z} \]  

(2.5)

The interarrival law can easily be deduced from (2.3). Let \( a(t) \) denote the probability of having a time interval of length t between two successive arrivals; it is expressed as:

\[ a(t) = \{ N=0, t=1 \} \cdot \{ N>0, u \} = q^z(t-1) \cdot (1-q^z) \]  

(2.6)

So far, we have found that \( a(t) \) follows a Geometric law with parameter \( q^z \).

This is the reason why the arrival process at the Memory Pool, just derived, is called the Batch-Geo arrival process. Note that the arrival rate \( z \cdot p \) is the product of the batch arrival rate \( 1 - q^z \) by the mean batch size.

2.3. Arrival process at the Output Queue

As commented in section 1, the cells arriving to the Memory Pool are randomly routed to the Output Queues according to a uniform distribution with probability \( 1/n \). Then, the probability of getting i arrivals at any group \( N_g(i,t) \), in a time period t, is:

\[ N_g(i,t) = \sum_{j=1}^{z \cdot t} \left[ \frac{p^i}{n} \right] \cdot \frac{1}{n} \]  

(2.7)

and we can see that the p.g.f. \( N_g(a,t) \) is again the convolution of z independent Bernoulli processes of parameter \( p/n \).

We can conclude that from our starting assumption of Bernoulli arrivals at the inlets of the multiplexer, the Batch-Geo arrival process, here defined, is the process governing the arrival of cells at the two main subsystems of the multiplexer, the Memory Pool and the Output Queue.

3. MODELING THE OUTPUT QUEUE

The second step in our approach is now tackled. The Output Queue is a r-multiserver system, i.e., there are r fully accessible servers in front of a single finite queue. The service time of each server is constant, identically and independently distributed, with length the elementary time unit u.
In order to model such a complex system, we have defined a single server queueing system, \( \text{Batch-Geo/GB/1} \), where the arrival of customers (cells) occurs according to the Batch-Geo process previously described. A queue of unlimited size with FIFO discipline is assumed. This is because in our real system the queue limit must be high enough as to fulfill the \( 10^{-10} \) (or less) requirement of getting the queue fully occupied, and in that case the results one can obtain are very close to the ones corresponding to the pure waiting system. 

The Gb stands for General bulk service [4] of size \( r \). That is, when the server becomes free it will accept \( r \) customers if they are available, or if not, it will accept less than \( r \) if any are available. This type of service exactly represents our multiserver case when the service is constant of length one unit of time.

### 3.1. The Batch-Geo/GB/1 System: State Probabilities and Waiting Times

The detailed analysis of this queueing system is described in an annex to this paper. There, the stationary probabilities associated with the queue contents at departure epochs and at random instants are derived. In this section, only the results corresponding to the particular case of constant, unitary service time are summarized.

**Queue and system contents at random instants**

Let \( \{ q_n \} \) be the set of stationary probabilities associated with number of customers in the queue. The p.g.f. for this probabilities, \( Q(a) \), has the expression:

\[
Q(a) = \frac{\sum_{i=1}^{r-1} d_i \cdot q_i \cdot a^i \cdot (1 - a^r)}{L \cdot a(a) - a^r}
\]  

(3.1)

where \( d_i \) is the probability of having \( i \) customers in the queue at departure instants (see annex), and \( L \) the average length of the associated embedded Markov chain at that instants, \( a(a) = (p \cdot a + q)^r \), and \( q_i \) as defined in section 2.

Expression (3.1) must be analytic for \( |a| \leq 1 \). We have checked that the denominator has exactly \( r \) zeros within the unit circle, provided that \( z \cdot p < r \), which is just the condition for the stability of this system. This means that the \( r \) zeros of the denominator must be matched by the \( r \) zeros of numerator. Denoting by \( a_j \) the remaining \( z-r \) zeros of the denominator outside the unit circle, expression (3.1) can be rewritten as:

\[
Q(a) = K \prod_{j} (1 - a/a_j)^{-1}
\]  

(3.2)

where \( K \) is the normalizing factor. This last expression is easily inverted and then, the distribution of the number of customers in the queue is numerically obtained. A detailed discussion of how to approximate the tail distribution in more general cases is given in [5].

Figure 3.1 shows the variation of the first moment of this distribution as a function of the load. The advantage of the multiserver approach in the case of high load (beyond 0.7) is shown there.

The system state probabilities \( \{ p_k \} \) can be obtained from the expression (see annex):

\[
P(a) = Q(a) \cdot (p \cdot a + q)^z
\]  

(3.3)

**System state probabilities at arrival instants**

In this system, the state we can find on arrival of a batch (or the first cell of a batch), that is a random instant, is different from the system state found upon arrival of any arbitrary cell, because in this last case the system state is modified by the preceding cells of the same batch. Therefore, to characterize the system upon arrival of any cell, we should account for the position of that cell in its batch. If \( \{ r_k \} \) is the set of state probabilities at arrival instants, we can write:

\[
k+1 \sum_{j=1}^{k} L_j \cdot P_{k-j+1} \quad k \geq 0;
\]  

(3.4)

where \( L_{j} \) is the probability for a cell to be in position \( j \) within a batch. \( L_{j} \) is proportional to the probability that the batch size exceeds \( j \), and can be evaluated as:

\[
L_j = K \sum_{n=j}^{z} q_n \quad 1 \leq j \leq z;
\]  

(3.5)

The value of the normalizing factor \( K \) is the
inverse of the average batch size.

A close form for both \( r_k \) and \( L_j \) can be deduced from (3.4) and (3.5), by means of the corresponding p.g.f., \( R(a) \) and \( L(a) \), as:

\[
L(a) = \frac{a(1 - \sigma(a))}{1 - a(1 - \alpha)} \quad (3.6)
\]

\[
R(a) = a^{-1}L(a) \cdot P(a) \quad (3.7)
\]

Similarly, the distribution of the queue contents found at arrival instants, can be expressed as:

\[
Q^a(\alpha) = a^{-1}L(\alpha) \cdot Q(\alpha) \quad (3.8)
\]

3.2. Waiting Times

The virtual waiting time in our particular case, where the service is constant of length one time unit, is directly related with the queueing probabilities at random instants \( \{q_k\} \), derived before. If \( w(t) \) is the density function for this virtual waiting, we can write:

\[
w(t=0) = q_o + \cdots + q_{r-1} \quad (3.9)
\]

\[
w(t=i) = q_{i \cdot r} + \cdots + q_{(i+1) \cdot r-1} \quad i \geq 1
\]

In a similar way, the cell waiting time distribution, \( w_a(t) \), that significantly characterizes the performance of the multiplexer, is derived from the queueing probabilities on arrival as:

\[
w_a(t=0) = q_o^a + \cdots + q_{r-1}^a \quad (3.10)
\]

\[
w_a(t=i) = q_{i \cdot r}^a + \cdots + q_{(i+1) \cdot r-1}^a \quad i \geq 1
\]

\[\text{Figure 3.2}\]

The first moment of this distribution, as a function of the load, is plotted in figure 3.2. The above comment on the advantage of the multiserver case applies also here.

4. MODELING THE MEMORY POOL

The Memory Pool is modeled as a Batch-Geo/Db/n/M queueing system where there are \( n \) servers of the bulk type described above, in front of a common finite buffer of size \( M \). The customers arrive at that buffer following the Batch-Geo law. The queueing discipline is such that once the cells enter the buffer, they are randomly assigned to a server, and join the FIFO logical queue (the Output Queue) of their assigned server. It is assumed that cells leave the Memory Pool just in the instant their service starts.

Since the arrival of batches occur at random instants, we can consider the embedded Markov at that instants, to find out the joint steady state probabilities of the whole system. Let \( \{ P_j \} \) be the set of these probabilities, \( \{ j \} \) being the set of vectors \( \{j_1, \ldots, j_n\} \) representing the specific states of the individual Output Queues; we can approximate the state equations as:

\[
P_j = \sum_{a(t)} \sum_{i=1}^{z} \sum_{k=1}^{t} \sum_{b_{a(t)}} \sum_{(k)} B_{k}(k_1, k_2, \ldots, k_n)
\]

where:

\[
\{k\} \text{ represents the set of allowed partitions } \{k_1, \ldots, k_n\} \text{ of } k.
\]

\( \{j\} \) is determined by the following conditions:

\[
 j_1 = \min \{ (i_1-r)+k_1 - (t-1) \cdot r, m+z \}
\]

\[
 j_n = \min \{ j_{n-1} + m+z - \sum_{n=0}^{1-1} j_n \}
\]

and

\[
B_{k}(k_1, k_2, \ldots, k_n) = \frac{1}{k_1! \cdot k_2! \cdots k_n!} n^k
\]

The above conditions on \( j_1 \) imply a simplification of the extreme cases (MP is full). It has no impact in the results for the cases studied.

The equations are solved for \( P_j \) in an iterative way, starting from an initial value \( p_{0j} \) taken as the product of the state probabilities of the Output Queues. For the studied cases, sufficient accuracy has been obtained after few iterations.

Once the joint steady state probabilities at random instants have been calculated, the global distribution of the system and queue contents are easily obtained, as well as the traffic carried by the system. Then, the cell loss ratio, used as a criterion to properly size the Memory Pool, is determined as the complement to one of the carried-to-offered traffic ratio.
The numerical results so obtained are in good agreement with simulations. Table 4 shows a comparison of the mean and variance of the MP contents for a multiplexer with 16 inlets \((z=16)\), and 16 outlets for an offered load of 0.7 erlangs. The outlets are grouped into two multiserver groups in one case \((n=2, r=8)\), and into four groups \((n=4, r=4)\) in the other case. In both cases the size of the MP buffer has been set to 19.

Table 4: Comparison between theory and simulation. Load=0.7 erlang, \(z=16\), MP size=19

<table>
<thead>
<tr>
<th></th>
<th>Avg. MP contents</th>
<th>Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Model</td>
<td>0.2593</td>
</tr>
<tr>
<td>n=2</td>
<td>Simul.</td>
<td>0.2514 ± 0.0098</td>
</tr>
<tr>
<td>r=8</td>
<td>Conv.</td>
<td>0.2593</td>
</tr>
<tr>
<td>n=4</td>
<td>Model</td>
<td>1.5034</td>
</tr>
<tr>
<td>r=4</td>
<td>Simul.</td>
<td>1.4832 ± 0.0609</td>
</tr>
<tr>
<td></td>
<td>Conv.</td>
<td>1.5034</td>
</tr>
</tbody>
</table>

Figure 4.1 shows the negative cumulative distribution of the Memory Pool contents at random instants, for the same two cases.

Figure 4.1

5. CONCLUSIONS

A discrete-time arrival process, designated Batch-Geo, intended to characterize the traffic flow offered to a type of ATM multiplexer has been presented. Two analytical queueing models, the Batch-Geo/Gb/1 and the Batch-Geo/Db/n/M have been developed in order to analyze the performance of the multiplexer and provide a criterion for the proper dimensioning of the buffer of such a multiplexer. The main results are summarized below:

a) Characterization of the cell arrival: Both, the count process and the interarrival process have been derived.

b) For the first model, stationary queue length and system state probabilities, and waiting time distributions, at random and at arrival instants, have been obtained.

c) The distribution of the system contents for the second model has been derived.

A computer program has been developed that provides numerical results for the above items. The obtained accuracy is in good agreement with simulation. This tool provides the low probability tails required to analyze the behavior and to properly engineer those systems.

Further investigation is required in order to either certify the validity of the Bernoulli arrival law or to find other more general laws that may better represent the reality. Then, extensions of this model to include those general laws have to be developed.

ANNEX: Batch-Geo/Gb/1 MODEL

Model assumptions:

- The arrival of customers is according to the Batch-Geo process.
- There is an infinite FIFO queue in front of a single server.
- The server accepts a bulk of customers of maximum size \(r\).
- The service is distributed according to a general law \(H(\cdot)\) (density function \(h(\cdot)\), with moments \(h^{(k)}\)).

Queue length at departure instants

Consider the embedded Markov chain at instants \(t_0, t_1, \cdots\), when the customers leave the system. Let \(\{d_n\}\) denote the steady state probabilities associated with the queue length at those instants. The \(\{d_n\}\) obey the equations below:

\[
\begin{align*}
\sum_{t=1}^{\infty} & \sum_{i=1}^{r} \sum_{t=1}^{n-r+i} \frac{r}{g_i \cdot N(n,t-\tau)} \cdot h(t-\tau) + \\
\sum_{i=r+1}^{n+r} \sum_{t=1}^{\infty} & \sum_{i=1}^{r} \sum_{t=1}^{n-r+i} \frac{r}{g_i \cdot N(n-i+r,t-\tau)} \cdot h(t-\tau) + \\
\sum_{t=1}^{\infty} & \sum_{i=1}^{r} \sum_{t=1}^{n-r+i} \frac{r}{g_i \cdot N(n-r+i+r,t)} \cdot h(t) + \\
\sum_{t=1}^{\infty} & \sum_{i=1}^{r} \sum_{t=1}^{n-r+i} \frac{r}{g_i \cdot N(n-i+r,t)} \cdot h(t) + \\
\sum_{t=1}^{\infty} & \sum_{i=1}^{r} \sum_{t=1}^{n-r+i} \frac{r}{g_i \cdot N(n-i+r,t)} \cdot h(t)
\end{align*}
\]

where \(N(i,t)\) is the probability of \(i\) arrivals during a service period, \(g_i\) is the probability of getting a batch of size \(i\), and \(a(t)\) the
interarrival density function.

The p.g.f. for these probabilities is:

\[
d_o \cdot (1 - G(\alpha)) + \sum_{i=1}^{r-1} (d_o g_i + d_1) \cdot (\alpha^i - \alpha^r)
\]

\[
D(\alpha) = \frac{r-1}{S(\sigma(\alpha)) - \alpha^r} \cdot S(\sigma(\alpha))
\]

The normalizing condition imposes that:

\[
d_o'G'(1) + \sum_{i=1}^{r-1} (d_1 g_i + d_1) \cdot (r-i) = r \cdot (1 - R_0)
\]

where \( R_0 = \frac{z \cdot p \cdot h}{r} \) is the load of the server.

Queue length at random instants

The function we are looking for is the joint density function \( q_n(x) \), when at a random instant there are \( n \) customers in queue (not including the customer in service), and the time needed to finish the current service is \( x \).

First, we find the density function \( V_n(x,y;t) \), when at instant \( t \) between two departure points, the server is busy, the length of the current service is \( y \), the residual time is \( x \), and the number of customers in the queue is \( n \).

This function \( V(\cdot) \) is defined by:

\[
V_0 = d_o \cdot q^x \cdot t, \quad y = x = 0, \quad t \geq 0
\]

For \( 0 \leq x < y < t + x, \ t > 0 \) and \( n \geq 0 \)

\[
V_n(x,y,t) = d_o \sum_{\tau=1}^{t} \left[ \sum_{i=1}^{r} g_i \cdot N(n,t-\tau) + \sum_{i=r+1}^{n+r} g_i \cdot N(n-i+r,t-\tau) \right] h(y)
\]

For \( 0 < t + x = y, \ x \geq 0, \ t \geq 1 \) and \( n \geq 0 \)

\[
V_n(x,y;t) = \left[ \sum_{i=1}^{r} g_i \cdot N(n,t) + \sum_{i=r+1}^{n+r} g_i \cdot N(n-i+r,t) \right] h(y)
\]

and \( V_n(x,y;t) = 0 \) otherwise.

Taking p.g.f. in \( n, x, y \) and \( t \), and disregarding \( y \) and \( t \) we obtain:

\[
V_0 = d_o \cdot \frac{1}{1 - q^x}
\]

\[
D(\alpha) S(s) - S(\sigma(\alpha))
\]

Now, the stationary probabilities \( \{ q_n(x) \} \) previously defined are obtained applying the key renewal theorem [6]:

\[
P_0 = \frac{1}{L} \cdot \frac{d_1}{1 - q^x}
\]

\[
Q(\alpha,s) = \frac{1}{L} \cdot \frac{D(\alpha)}{S(\sigma(\alpha))} \cdot \frac{S(s) - S(\sigma(\alpha))}{s - \sigma(\alpha)}
\]

where \( L \) is the average time interval between departures given by:

\[
L = h + V_0 \text{ or } L = h/(1-P_0)
\]

In particular the p.g.f. for the queue contents disregarding the residual service is:

\[
Q(\alpha,s) = \frac{1}{L} \cdot \frac{D(\alpha)}{S(\sigma(\alpha))} \cdot \frac{1 - S(\sigma(\alpha))}{1 - \sigma(\alpha)} + P_0
\]

The same techniques have been applied to obtain the system state probabilities at any random instant \( \{ p_i \} \); the result, in terms of p.g.f., being:

\[
P(\alpha) = Q(\alpha) \cdot S(\sigma(\alpha)) + P_0
\]

The model can be restricted to a finite queue case by just imposing the proper limits to the summations in (A.1) and (A.4). In this case we get a finite set of equations that has to be numerically solved. If the model is restricted to a unitary length constant service, the equations become much simpler and hence, more accurate ad-hoc numerical solution methods may be applied.

REFERENCES