Control-Strategy-Based
Nonblocking Broadcast Switching Networks *

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Abstract

In a broadcast connection through a multi-stage network, an input port can be connected to more than one output port, with the restriction that at no time can an output port be connected to more than one input port. We present new constructive designs for nonblocking multi-stage switching networks for realizing broadcast connections. Our nonblocking connecting capability is based on the adherence to a connection path set-up strategy as implemented by a network control algorithm. In three-stage versions of our designs wherein there are $r$ switch modules in each of the first and third stages, $n$ input ports on each switch module in the first stage and $n$ output ports on each switch module in the third stage, the strategy demands that any broadcast connection be realized using at most $z$ middle switches, where $1 \leq z \leq \min(n - 1, r)$. We prove that if the number of switch modules in the middle stage, $m$, satisfies $m > \min(n - 1)(z + r^{1/2})$, the resulting network is nonblocking for broadcast assignments. This condition on the number of switch modules in the middle stage represents an improvement from $O(nr)$ to $O(n \log r/\log \log r)$ relative to previously known results. We present a linear algorithm for satisfying new broadcast connection requests which employs the strategy. Our results represent the best known explicit constructions with limited numbers of stages relative to both crosspoint and control algorithm complexity.

1 Introduction

In an increasing number of computing and communication environments, it is necessary to simultaneously transfer text/voice/video/graphics information from a set of transmitting devices to a set receiving devices in various combinations. This can be accomplished using an interconnection network called a multi-stage switching network. When a transmitting device simultaneously sends information to more than one receiving device, the one-to-many connection required between the transmitting device and the receiving devices is called a broadcast connection. A set of broadcast connections is referred to as a broadcast assignment. Multi-stage switching networks that can satisfy such broadcast requirements are called broadcast networks.

2 Multi-Stage Switching Networks

Multi-stage switching networks are composed of crosspoint switching elements or, more simply, crosspoints that are usually grouped together into building-block subnetworks called switch modules. In an $(N \times M)$ multi-stage switching network with $N$ input ports and $M$ output ports, the switching modules used as building blocks to implement the network might each have, for example, $n$ inputs and $m$ outputs, where $n < N$ and $m < M$. These would be referred to as $(n \times m)$ switch modules.

The connectivity available among the $n$ inputs and the $m$ outputs of the $(n \times m)$ switch modules depends upon implementation details, but a case of general interest which will be considered exclusively in the following will be that in which the switch module has sufficient crosspoint switching elements to provide broadcast capability from the $n$ inputs to the $m$ outputs in the sense that any input of the switch module can always be connected to any idle output (that is, an output from the switch module that is not currently connected to any input of the switch module). The input to output connections implemented in a switch module characterize the state of the switch module.

The $(n \times m)$ switch modules in a multi-stage switching network are interconnected by means of links. The switch modules in a multi-stage switching network are grouped together into stages such that the inputs of the switch modules of one stage are linked only to outputs of switch modules of another stage, and, similarly, the outputs of the switch modules of one stage are linked only to inputs of switch modules of another stage. Figure 1 shows a three-stage switching network with $N = M = 9$ input/output ports and comprised of three $(3 \times 4)$ input switches, four $(3 \times 3)$ middle switches, and three $(4 \times 3)$ output switches. The set of switch module states in a network characterizes the state of the network. The network of Figure 1 is shown

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in a state realizing a broadcast assignment.

In general, an \( (N_1 \times N_2) \) three-stage network has \( r_1 (n_1 \times m) \) switch modules in stage 1, \( r_2 (m \times n_2) \) switch modules in stage 2, and \( r_3 (n \times n_2) \) in stage 3. Such a multi-stage switching network is denoted as a \( v(m, n_1, r_1, n_2, r_2) \) network. For the special symmetrical case where \( n_1 = n_2 = n \) and \( r_1 = r_2 = r \), the three-stage network is denoted as a \( v(m, n, r) \) network. The three-stage network of Figure 1 is a \( v(4, 3, 3) \) network. In general, the set of input ports is denoted as \( \{1, 2, \ldots, r_1 n_1\} \) and the set of switch modules in the output stage are denoted as \( O = \{1, 2, \ldots, r_2 n_2\} \).

The results on nonblocking multi-stage broadcast switching networks that follow will initially be described primarily in terms of three-stage networks. It should be understood, however, that our results are applicable to \( k \)-stage networks for \( k > 3 \) by recursively using the design criteria developed on the switch modules.

### 2.1 Network control

The network controller of a multi-stage switching network executes a network control algorithm for establishing connection paths between input and output ports. There is an inherent trade-off between network control algorithm complexity and the number of switching modules used in a design of a multi-stage switching network. An awareness of this trade-off has led to the development of network structures in which providing for new connecting paths between ports can sometimes require the rearrangement of existing connection paths. In rearrangeable networks, an attempt is made to reduce the switch hardware costs of the structure at the expense of the complexity and time required to set up connection paths.

In general, it is desirable to minimize or, indeed, even eliminate the need for rearrangements to existing connections in order to satisfy a new connection request from an input port. Multi-stage switching network structures in which rearrangements of existing connections can be avoided in establishing a new requested broadcast connection path by an input port fall into two categories. The distinction between these two categories is due to Benes [1]. In strictly nonblocking broadcast networks, for any legitimate broadcast connection request from an input port to some set of output ports, it is always possible to provide a connection path through the network to satisfy the request without disturbing other existing broadcast connections, and if more than one such path is available, any of them can be selected without being concerned about satisfaction of future potential broadcast connection requests. In wide-sense nonblocking broadcast networks, it is again always possible to provide a connection path through the network to satisfy the request without disturbing other existing broadcast connections, but in this case the path used to satisfy the connection request must be carefully selected so as to maintain the nonblocking connecting capability for future potential broadcast connection requests.

Without in any way attempting to criticize the term wide-sense nonblocking, as it was chosen at a time when algorithm analysis was not well-developed, the term is nevertheless somewhat of a misnomer as it does not convey the notion of nonblocking connecting capability that it is meant to imply. This particular type of nonblocking capability might perhaps be better referred to as control-strategy-based nonblocking capability, since the network control strategy used to satisfy requests figures critically in the nonblocking connection capability provided.

Control-strategy-based nonblocking capability must be evaluated from the perspective of the complexity of the control algorithm as opposed to only a proof of its existence. It is this type of nonblocking capability we address in this paper, and we elect to refer to it as simply as non-blocking connection capability. We will show constructive multi-stage designs for such nonblocking broadcast networks for which a linear network control algorithm is given that permits a broadcast connection request from an idle input port to some set of idle output ports to be satisfied without any rearrangement of connection paths of other existing broadcast connections.

### 3 Preliminaries

#### 3.1 Fanout

Every switch module in our networks will be assumed to have broadcast capability. In a \( v(m, n_1, r_1, n_2, r_2) \) network, if a network input port is to be connected to more than one output port on the same switch module in the output stage of the network, then it is only necessary for the input port to have one connection path to that switch module. Broadcast assignments can therefore be described in terms of connections between input ports and switch modules in the output stage. An existing connection or a requested connection from an input port to output ports on \( r' \) output stage switches is said to have fanout \( r' \).
3.2 Characterizations

To characterize a broadcast assignment, for each input port \( i \in \{1, \ldots, r_1, n_1\} \), let \( I_i \subseteq O = \{1, \ldots, r_2\} \) denote the subset of the switch modules in the output stage to which \( i \) is to be connected in the broadcast assignment. For example, for the broadcast assignment shown in Figure 1, \( I_1 = \{1,2,3\}, I_2 = \{1\}, I_3 = \{2\}, I_4 = \{3\} \), and all other \( I_j = \phi \).

To characterize the state of the \( m \) switch modules in the middle stage of a three-stage switching network, let \( M_j \subseteq O = \{1, \ldots, r_2\}, j = 1, 2, \ldots, m \), denote the subset of the switch modules in the output stage to which the \( M_j \) is providing connection paths from the input ports. In other words, each \( M_j \) denotes the connection paths passing through middle switch \( j \) in terms of their output switch destinations. We will refer to the sets \( M_j \subseteq O = \{1, \ldots, r_2\}, j = 1, 2, \ldots, m \), as the destination sets of the middle switches. It should be clear that in general for any state of a three-stage switching network

\[
\sum_{i=1}^{n_1} |I_i| = \sum_{j=1}^{m} |M_j| \leq n_2 r_2.
\]

4 Nonblocking Connecting Capability

4.1 Sufficient conditions on \( m \) for \( v(m, n, r) \) networks

Assume that a \( v(m, n, r) \) network is currently providing some set of broadcast connections from the input ports to the output ports. Suppose there is a new request for a connection from an input port \( i \) \((i \in \{1, 2, \ldots, n r\}) \) to some set of \( r' \) idle output ports \( j \) each assumed to be on a distinct output switch). We denote this new connection request \( I_i \subseteq O \), where \( r' = |I_i|, 1 \leq r' \leq r \), is the fanout of the request. We will refer to the set of middle switches with currently unused links to the input switch associated with input \( i \) as the available middle switches for this connection request.

To satisfy this request \( I_i \), we scan the available middle switches. If we find an empty middle switch (that is, a middle switch, say, \( i_1, i_2 \in \{1, 2, \ldots, m\} \), where \( M_{i_1} = \phi \), we can surely satisfy \( I_i \) through this middle switch. Similarly, if our scan determines that there are two middle switches, say, \( i_1 \) and \( i_2 \) for which

\[
M_{i_1} \cup M_{i_2} = \{1, 2, \ldots, r\},
\]

where \( M = \{1, 2, \ldots, r\} - M_{i_1} \), we can satisfy \( I_i \) through these two middle switches. In general, if there exist \( z \) \((1 \leq z \leq m) \) middle switches with

\[
\sum_{j=1}^{z} M_{i_j} = \{1, 2, \ldots, r\},
\]

we can satisfy \( I_i \) through these \( z \) middle switches.

The above naturally leads us to the following lemmas and theorems. Detailed proofs can be found in a Johns Hopkins University Technical Report [6].

Lemma 1 We can satisfy a new connection request with fanout \( r \) using some \( z \) \((z \geq 1) \) middle switches, say, \( i_1, \ldots, i_z \), from among the available middle switches if and only if the current destination sets of these \( z \) middle switches are such that

\[
\sum_{j=1}^{z} M_{i_j} = \phi
\]

Lemma 2 Assume that a \( v(m, n, r) \) network is in a state in which there exist at most \( n' \) connection paths, \( 1 \leq n' \leq n \), to each of the output switches. Then the intersection of more than \( n' \) \( M_i \)'s is empty.

For the following lemma, we will assume that a \( v(m, n, r) \) is in an arbitrary state in which there exist at most \( n' \) connection paths, \( 1 \leq n' \leq n \), to each of the output switches. Figure 1 is an example of such a state for a \( v(4,3,3) \) network where \( n' = 2 \).

Lemma 3 For all \( n' \), \( 1 \leq n' \leq n \), and for all \( z \), \( 1 \leq z \leq \min\{n', r\} \), let \( m' \) be the maximum number of middle switches whose destination sets have the following properties:

1. there are at most \( n' \) \( 1 \)'s, \( n' \) \( 2 \)'s, \ldots, \( n' \) \( r \)'s distributed among the destination sets;

2. the intersection of any \( z \) of the destination sets is not empty.

Then

\[
m' \leq n' r^{1/z}
\]

We now present fundamental results on the number of middle switches sufficient to realize a general broadcast assignment with nonblocking connecting capability.

Theorem 1 In a \( v(m, n, r) \) network, for a new connection request with fanout \( r' \), \( 1 \leq r' \leq r \), if there exist more than \( (n - 1) r'^{1/z} \), \( 1 \leq z \leq \min\{n - 1, r'\} \), available middle switches for this connection request, then there will also always exist \( z \) middle switches through which this new connection request can be satisfied.

Theorem 2 A \( v(m, n, r) \) network is nonblocking for general broadcast assignments if

\[
m > \min_{1 \leq z \leq \min\{n-1, r\}} \{(n - 1)(z + r^{1/z})\}
\]

For a given \( n \) and \( r \) in a \( v(m, n, r) \) network, we could use Theorem 2 to find \( z \) such that a minimum \( m \) can be determined for nonblocking connection capability for broadcast assignments. But it is also of interest to determine bounds on \( m \) as a function of \( n \) and \( r \).
Theorem 3 A $v(m, n, r)$ network is nonblocking for general broadcast assignments if

1. $m > (n - 1)(\log r + 2)$

or, more precisely,

2. $m \geq O\left(\frac{n \log^2 r}{\log \log r}\right)$

4.2 Some extensions

The above results for $v(m, n, r)$ networks can be generalized in two ways: first, the more general asymmetrical $v(m, n_1, r_1, n_2, r_2)$ networks can be considered; second, restricted fanout connection assignments can be considered in which each input port can have connection paths to at most $d$ middle switches, if $1 \leq d < r_2$, output switches [3, 4, 5]. Again, we will state corollaries to the above theorems that address such generalizations without proofs [6].

Corollary 1 A $v(m, n_1, r_1, n_2, r_2)$ network is nonblocking for general broadcast assignments if

$$m > \min_{1 \leq i \leq \min(n_1 - 1, r_2)} \left\{(n_1 - 1)x + (n_2 - 1)r_i^{1/2}\right\}$$

In particular, this condition can be written as

$$m > (n_1 - 1)\log r_2 + (n_2 - 1)$$

or, more precisely,

$$m > 2(n_1 - 1)\log r_2 + (n_2 - 1)(\log r_2)^{1/2}$$

Corollary 2 A $v(m, n_1, r_1, n_2, r_2)$ network is nonblocking for restricted broadcast assignments, in which each input port can be connected to at most $d$ ($1 \leq d < r_2$) output switches, if

$$m > \min_{1 \leq i \leq \min(n_1 - 1, d)} \left\{(n_1 - 1)x + (n_2 - 1)d^{1/2}\right\}$$

In particular, this condition can be written as

$$m > (n_1 - 1)\log d + (n_2 - 1)$$

or, more precisely,

$$m > 2(n_1 - 1)\log d + (n_2 - 1)(\log d)^{1/2}$$

It is interesting to note that for the restricted fanout case, when $d = 1$ we have the special case of permutation connection capability. The above result then takes on the form of the sufficient condition for strictly nonblocking connection capability of C. Clos [2].

Corollary 3 Setting $d = 1$ in Corollary 2 yields $m \geq n_1 + n_2 - 1$, which is the bound on $m$ associated with the classical Clos strictly nonblocking permutation networks [8].

5 An $O(N)$ Network Control Algorithm

The theorems in the previous section provide the basis for a network control strategy. Assuming the network satisfies the conditions of Theorem 3, this control strategy can be briefly described as follows: to satisfy each connection request, we use at most $z$ middle switches whose destination set intersections are empty from among the guaranteed more than $(n - 1)r^{1/2}$ available middle switches. Note that based on this control strategy, each of the connection paths from each input port on the input switch associated with the connection request passes through no more than $z$ middle switches, and any middle switch providing a connection path from an input port on a particular input switch is not available to the other input ports of that particular input switch. In this section, we present an $O(N)$ algorithm for satisfying a general broadcast connection request in our nonblocking $v(m, n, r)$ networks. Extensions of this algorithm to more general $v(m, n_1, r_1, n_2, r_2)$ networks and restricted fanout broadcast assignments are not difficult. Finally, this network control algorithm can be easily implemented in either software or hardware.

Given a $v(m, n, r)$ network satisfying the condition on $m$ in Theorem 2, we have some $z \in \{1, 2, \ldots, \min(n - 1, r)\}$. Given a connection request $I_i$, $i \in \{1, 2, \ldots, n_r = N\}$, with $|I_i| = r' \leq r$, we take a set of any $m' = (n - 1)r^{1/2} + 1$ available middle switches; without loss of generality, let the destination sets of these middle switches be $M_1, M_2, \ldots, M_{m'}$. The following algorithm will generate a set of middle switches through which the connection request can be satisfied.

Algorithm:

Step 1: \texttt{mid\_switch} $\leftarrow \phi$; 
for $j = 1$ to $m'$ do 
$S_j$ $\leftarrow$ $M_j \cap I_i$; 
Step 2: repeat 
find $S_k$ ($1 \leq k \leq m'$) such that $|S_k| = \min\{|S_1|, |S_2|, \ldots, |S_{m'}|\}$; 
$\text{min\_set}$ $\leftarrow S_k$; 
$\text{mid\_switch}$ $\leftarrow$ $\text{mid\_switch} \cup \{k\}$; 
if $\text{min\_set} \neq \phi$ then 
for $j = 1$ to $m'$ do 
$S_j$ $\leftarrow$ $S_j \cap \text{min\_set}$; 
until $\text{min\_set} = \phi$; 
Step 3: connect $I_i$ through the middle switches in $\text{mid\_switch}$ and update the destination sets of these middle switches.

End

It is not difficult to show that the complexity of the our algorithm is $O(m'r')$. Taking $x = \log r$, we get that the complexity is $O(nr')$ or $O(N)$ for satisfying one input port broadcast connection request. It can also be shown that the complexity of our algorithm for satisfying an entire broadcast assignment is $O(nN)$ [6].
6 Asymptotic Crosspoint Growth

Let $G_k(N)$ denote the minimum number of crosspoints of our $(N \times N)$ broadcast network with $k$ stages. The following theorem gives an upper bound on $G_k(N)$ for the case of fixed $k$.

**Theorem 4** For each fixed integer $k \geq 1$,

$$G_{2k+1}(N) \leq c_{2k+1}N^{1+\frac{1}{k+1}}(\log N)^{\frac{k+2}{k+1} - \frac{1}{k+1}}$$

where, $c_3 = \frac{5}{3}$ and $c_{2k+1} = \frac{k}{k+1}(2 + \frac{k+1}{k+1} - \frac{1}{2} c_{2k-1})$.

From the above theorem, we actually considered all subnetworks at each stage as symmetrical networks which can be easily implemented in practice. If we allow these subnetworks to be asymmetrical, the upper bound in (6) can be reduced to

$$G_{2k+1}(N) = O(N^{1+\frac{1}{k+1}}(\log N)^{\frac{k+3}{k+1}})$$

7 Comparisons

Hwang and Jajszczyk's [7, 8, 9] nonblocking multi-connection broadcast networks require that $m = O(n^r)$ even for the case where the input set is a singleton (that is, the broadcast case). Thus, since our networks only require that $m = O(n\log r/\log \log r)$, extensive comparison of our nonblocking broadcast networks with that Hwang and Jajszczyk's network is not very illuminating as our designs clearly require significantly fewer middle switches and therefore have an overwhelming advantage in crosspoint growth.

Even though our networks are nonblocking, our results warrant comparisons not only with other nonblocking broadcast switching networks, but with some rearrangeable networks as well. Recall that our nonblocking three-stage network has $O(N^{1/2}\log N)$ crosspoints per output port. Richards and Hwang [10, 11] have proposed a multi-stage rearrangeable broadcast network for video teleconferencing applications. Their rearrangeable broadcast network utilizes a fixed fanout of each input port to $M$ of the input switches in the first or input stage. The proven two-stage version of the Richards/Hwang network has $O(N^{3/4})$ crosspoints per output port and the proven three-stage version has $O(N^{9/14})$ crosspoints per output port. Thus our nonblocking three-stage network demonstrates superior crosspoint growth for these cases.

Richards and Hwang conjecture that the two-stage versions of their rearrangeable networks can be constructed with $O(N^{3/3})$ crosspoints per output and that three-stage versions of their rearrangeable networks (which result after one recursive decomposition of their two-stage design) can be constructed with $O(N^{7/3})$ crosspoints per output port. Regardless of the whether or not the conjecture is correct, our nonblocking three-stage network again exhibits superior crosspoint growth relative to their claimed two-stage rearrangeable construction. However, Richards/Hwang's conjectured three-stage rearrangeable network must be considered more carefully. The conjectured three-stage rearrangeable design relies upon the an unproven claim that, $n_2$, the number of output ports per output switch that can be accommodated in two-stage versions of their rearrangeable networks is $O(M^2)$. However, Richards/Hwang only prove that $n_2$ is between $O(M^2)$ and $O(M^3)$. They cannot guarantee that $n_2$ actually reaches $O(M^2)$. This is critical, for if $n_2$ can only reach $O(M^k)$ where $k$ is slightly less than 3, say $k = 3 - \epsilon$, for an arbitrary small $\epsilon > 0$, the resulting number of crosspoints in their three-stage network would be $O(N^{1+\epsilon})$ for some $\epsilon' > 0$. Such crosspoint growth would be inferior to that of our three-stage nonblocking network.

Lea [12] has shown a multi-stage rearrangeable broadcast network which cascades a so-called consecutive spreading network with a rearrangeable permutation network to achieve broadcast capability. The resulting rearrangeable multi-stage structure has $O(\log N)$ stages and $O(N\log N)$ crosspoints. Clearly, Lea's rearrangeable network exhibits superior crosspoint growth to our nonblocking network. However, the fact that to achieve the $O(N\log N)$ crosspoint growth the number of stages in Lea's rearrangeable network must grow at a rate of $O(\log N)$ with the number of input ports and output ports represents a significant restriction to its application. Since a constant-stage network cannot be provided with this approach, installations which will undergo eventual port size growth must be initially installed with their worst-case staging requirements. This would result in a considerable cost and size overhead for expandability. Since we can provide our nonblocking broadcast networks with limited-stage configurations, for example, 3 or 5 stages, we do not suffer this same overhead penalty. Finally, delays through a version of Lea's network of practical size could be significant, and rearrangements, when they occur, can be prohibitive, involving as many as half the connections already existing through the network.

Kumar's five-stage rearrangeable network [13, 17] has $5N^{3/2}$ crosspoints per input port or output port, while a five-stage version of our nonblocking network has $2N^{1/3}(\log N)^{5/3}$ crosspoints per input port or output port. Clearly, the $O(N^{1/3}(\log N)^{5/3})$ crosspoint growth in our nonblocking broadcast networks is superior to Kumar's $O(N^{1/3})$ crosspoint growth in his rearrangeable broadcast networks. Furthermore, rearrangement in Kumar's network can be extensive sometimes requiring $O(N)$ rearrangements per connection request.

Dolev, Dwork, Pippenger and Wigderson [14] have given a minimum possible upper bound of $O((N\log N)^{3+1/6})$ on the number of crosspoints needed for $k$-stage rearrangeable broadcast networks (which they refer to as generalized connectore). Subsequently, Feldman, Friedman and Pippenger [15], gave an upper bound $O(N^{3+1/6}(\log N)^{1-1/6})$ on the number of crosspoints in $k$ stage nonblocking broadcast networks. Since the latter is a superior bound, it therefore also represents the best upper bound on crosspoints for rearrangeable nonblocking networks. Clearly,
our results are not in agreement with this latter upper bound. But there are no known constructions satisfying these bounds nor are there efficient control algorithms. Dolev, Dwork, Pippinger and Widgerson [14] also gave an explicit construction for \( (3k - 2) \)-stage rearrangeable broadcast network \((k \geq 1)\) with \(g_{3k-2}(N) = O(N^{1+1/2k})\) crosspoints, where \(g_i\) is the number of crosspoints in an \(i\)-stage network. From Theorem 4 we have that the number of crosspoints of our \((2k + 1)\)-stage network is \(G_{2k+1}(N) = O(N^{1+1/2k+1}(\log N)^{2^{2k-1} - 2^{k+1}})\). In other words, for any limited \(i\)-stage network, \(g_i = O(N^{1+1/2k})\) and \(G_i = O(N^{1+1/2k}(\log N)^{2^{2k-1} - 2^{k}})\). Obviously, for any \(i > 1\), we have an improved result. Feldman, Friedman and Pippenger [15], also gave explicit constructions for non-blocking broadcast networks with two-stages with \(O(N^{S/3})\) crosspoints, and with three-stages with \(O(N^{11/7})\) crosspoints. However, there are no efficient control algorithms for these constructions. Again our \(O(N^{3/2}(\log N)\) crosspoints is an improvement with the added benefit of a linear control algorithm.

Kirkpatrick, Klawe, and Pippenger [16] gave a constructive result for multi-stage rearrangeable broadcast networks where, from the perspective of a three-stage network, \(n \geq (n_1 - 1)\log 2r_3 + 2n_2\). Recall that by Corollary 1, our result is \(m \geq 2(n_1 - 1)(\log r_3/\log \log r_3) + (\log r_3)^{1/2}(n_2 - 1) + 1\). Therefore, in any case we would have slightly fewer crosspoints. In addition, we give a linear time control algorithm while Kirkpatrick, Klawe, and Pippenger [16] did not specify a control algorithm, and their proof implies only an exponential time control algorithm.

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References


