ITC19/ Performance Challenges for Efficient Next Generation Networks 919-928
LIANG X.J. and XIN Z.H.(Editors)
V.B. IVERSEN and KUO G.S.(Editors)
Beijing University of Posts and Telecommunications Press

Transient analysis of a fluid buffer driven by a birth and death process

Fabrice Guillemin[1] and Bruno Sericola[2]

[1] Fabrice Guillemin
France Telecom, 2 Avenue Pierre Marzin, 22300 Lannion, France
Fabrice.Guillemin@francetelecom.com
[2] Bruno Sericola
INRIA-IRISA, Campus de Beaulieu, 35042 Rennes, Cedex, France
Bruno.Sericola@irisa.fr

Abstract. We analyze the transient behavior of a fluid queue when the instantaneous rate of the input process is governed by a general birth and death process. These fluid flows models are widely used in the performance analysis of telecommunication systems. The transient analysis of such models has already been considered in earlier works with a finite state space background process. We consider in this paper a infinite state space driving process and we study the fluid queue by using results from spectral theory.
Keywords: Fluid buffer, spectral theory, orthogonal polynomials, transient analysis

1 Introduction

In the performance evaluation of packet telecommunication networks, fluid queues with Markov modulated input rates have been widely used to represent information flows at the burst level. By ignoring the discrete nature of packet arrivals, fluid models greatly simplify the analysis of a telecommunication system (say, a multiplexer, a switch or a router), while capturing the main performance characteristics.

There is a large number of papers dealing with the analysis of stochastic fluid flow models. Most of these papers consider such models in stationary regime. In their pioneering work, Kosten [1] and Anick *et al* [2] analyze the fluid model for several identical exponential on-off input sources. This model has been extended and studied in details, for instance, by Mitra [3, 4], Stern and Elwalid [5]. Stationary fluid queues driven by infinite state space Markov chains has also given rise to many research papers in the recent past few years. When the driving process is the $M/M/1$ queue, Virtamo and Norros [6] solve the well-known infinite differential system by studying the continuous spectrum of a key matrix; this model has also been studied by Adan and Resing [7], Barbot and Sericola [8]. More general input processes have also been considered; see for instance Van Doorn and Scheinhart [9], Sericola and Tuffin [10], Sericola [11].

The transient analysis of fluid queues fed by finite state Markov chains has been studied by using Laplace transform by Ren and Kobayashi [12, 13] for exponential on-off sources.

These studies have been extended to the Markov modulated input rate model by Tanaka *et al* in [14]. In [15], Sericola has obtained a transient solution based on simple recurrence relations, which are particularly interesting for their numerical properties. More recently, Ahn and Ramaswami [16] use an approach based on an approximation of the fluid model by the amounts of work in a sequence of Markov modulated queues of the quasi birth and death type. When the driving Markov chain has an infinite state space, the transient analysis is more complicated. Sericola *et al* [17] consider the case of the $M/M/1$ queue by using recurrence relations and Laplace transforms.

In this paper we study the transient behavior of an infinite buffer fluid queue driven by an infinite birth and death process, which controls the input rates into the buffer. By using a double Laplace transform of the joint probability density function of the buffer level and of the state of the driving process, we obtain a matrix equation, which is solved by means of spectral properties of some linear operators.

The paper is organized as follows. In Section 2, we present the model, the notation and the system of partial differential equations satisfied by the joint probability density function of the buffer level and of the state of the birth and death process. In this same section, we establish the basic matrix equation satisfied by the Laplace transforms of the joint probability density functions. Section 3 is devoted to the resolution of this matrix equation. Some concluding remarks are presented in Section 4.

2 Problem formulation and preliminary results

Throughout this paper, we consider a queue fed with a fluid traffic source, whose instantaneous transmitting rate is modulated by a general birth and death process $\{\Lambda_t\}$ taking values in $\mathbb{N}$. The input rate is precisely $r(\Lambda_t)$, where r is a given increasing function from $\mathbb{N}$ into $\mathbb{R}$.

The birth and death rates of the process $\{\Lambda_t\}$ are denoted by $\lambda_i > 0$ for $i > 0$ and $\mu_j > 0$ for $j \geq 1$, respectively. The infinitesimal generator is given by the infinite tridiagonal matrix

$$\mathcal{A} = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & . & . \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & . & . \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & . \\ . & . & . & . & . \end{pmatrix}. \tag{1}$$

We assume that the birth and death process $\{\Lambda_t\}$ is ergodic, which amounts to assuming (see [18] for instance) that

$$\sum_{n=0}^{\infty} \frac{1}{\lambda_n \pi_n} = \infty \quad \text{and} \quad \sum_{n=0}^{\infty} \pi_n < \infty, \tag{2}$$

where the quantities π_n are defined by $\pi_0 = 1$ and for $n \geq 1$, $\pi_n = \lambda_0 \ldots \lambda_{n-1}/(\mu_1 \ldots \mu_n)$. Under the above assumption, the birth and death process $\{\Lambda_t\}$ has a unique invariant probability measure, denoted by $\{p(n), n \geq 0\}$ and defined by $p(n) = \frac{\pi_n}{\sum_{j=0}^{\infty} \pi_j}$ for $n \geq 0$.

Let $\{p_0(i),\ i \geq 0\}$ denote the initial distribution of the birth and death process $\{\Lambda_t\}$, i.e., $\mathbb{P}(\Lambda_0 = i) = p_0(i)$. Note that if $p_0 \equiv p$, then $\mathbb{P}(\Lambda_t = i) = p(i)$ for all $t \geq 0$ and $i \in \mathbb{N}$.

We assume that the queue under consideration is drained at constant rate $c > 0$. Furthermore, we assume that $r(i) > c$ when i is greater than a fixed $i_0 > 0$ (and that $r(i) \leq c$ for $0 \leq i \leq i_0$). In addition, the parameters c and $r(i)$ are such that

$$\rho = \sum_{i=0}^{\infty} \frac{r(i)}{c} p(i) < 1 \tag{3}$$

so that the system is stable. To avoid cumbersome special cases, we assume that $r(i) < c$ for $0 \leq i \leq i_0$, so that $r_i = r(i) - c$ is either positive or negative but always non zero. The quantity r_i is the net input rate, when the modulating process $\{\Lambda_t\}$ is in state i.

Let X_t denote the buffer content at time t. The process $\{X_t\}$ satisfies the following evolution equation: for $t \geq 0$,

$$\frac{dX_t}{dt} = \begin{cases} r(\Lambda_t) - c & \text{if } X_t > 0 \text{ or } r(\Lambda_t) > c, \\ 0 & \text{if } X_t = 0 \text{ and } r(\Lambda_t) \leq c. \end{cases} \tag{4}$$

Let $f_i(t, x)$ denote the joint density function defined by $f_i(t, x) = \mathbb{P}(\Lambda_t = i, X_t \in dx)$. As shown in [15], on top of its usual jump at point $x = 0$, when $X_0 = x_0 \geq 0$, the distribution function $\mathbb{P}(\Lambda_t = i, X_t \leq x)$ has a jump at points $x = x_0 + r_i t$, for t such that $x_0 + r_i t > 0$, which corresponds to the case when the Markov chain $\{\Lambda_t\}$ starts and remains during the whole interval $[0, t)$ in state i. We thus have,

$$\mathbb{P}(\Lambda_t = i, X_t = x_0 + r_i t) = p_0(i) e^{-(\lambda_i + \mu_i)t} \mathbb{1}_{\{x_0 + r_i t > 0\}}.$$

The measure associated with the distribution function $\mathbb{P}(\Lambda_t = i, X_t \leq x)$ can then be written as follows: for $t \geq 0$;

$$d\mathbb{P}(\Lambda_t = i, X_t \leq x) = f_i(t, x)dx + p_0(i) e^{-(\lambda_i + \mu_i)t} \delta_{x_0 + r_i t}(dx), \tag{5}$$

where $\delta_{x_0 + r_i t}(dx)$ is the Dirac mass at point $x_0 + r_i t$. We focus in the rest of paper on the probability density function $f_i(t, x)$ for $x > 0$ along with its usual jump at point $x = 0$. A direct consequence of the evolution equation (4) is the forward Chapman-Kolmogorov equations satisfied by $(f_i(t, x),\ x \geq 0, i \in \mathbb{N})$.

Proposition 1 (Chapman-Kolmogorov equations). *The functions $(x, t) \to f_i(t, x)$ for $i \in \mathbb{N}$ satisfy the differential system (in the sense of distributions):*

$$\frac{\partial f_i}{\partial t} = -r_i \frac{\partial}{\partial x} \left(\left(\mathbb{1}_{\{i > i_0\}} + \mathbb{1}_{\{i \leq i_0\}} \mathbb{1}_{\{x > 0\}} \right) f_i \right) - (\lambda_i + \mu_i) f_i + \lambda_{i-1} f_{i-1} + \mu_{i+1} f_{i+1}, \tag{6}$$

with the convention $\lambda_{-1} = 0$, $f_{-1} \equiv 0$ and $f_i(t, x) = 0$ for $x < 0$.

Note that the differential system (6) holds for the density probability functions $f_i(t, x)$. This is the main difference with the differential system considered in [9, 19] and governing the probability distribution functions $\mathbb{P}(X_t \leq x, \Lambda_t = i)$, $i \geq 0$. The differential system (6) is actually the equivalent of the Takács' integro-differential formula for the $M/G/1$ queue [20]. The resolution of this differential system is addressed below.

Introduce the double Laplace transform

$$F_i(s,\xi) = \int_{0^-}^{\infty}\int_{0^-}^{\infty} e^{-st-\xi x} f_i(t,x)dtdx = \int_0^{\infty} e^{-st}\mathbb{E}\left({}^{-\xi X_t}\mathbb{1}_{\{\Lambda_t=i\}}\right)dt$$

and define the functions $f_i^{(0)}(\xi)$ and $h_i(s)$ for $i \in \mathbb{N}$ as follows

$$f_i^{(0)}(\xi) = \int_0^{\infty} e^{-x\xi}\mathbb{P}\{\Lambda_0 = i, X_0 \in dx\} \quad \text{and} \quad h_i(s) = \int_0^{\infty} e^{-st}\mathbb{P}\{\Lambda_t = i, X_t = 0\}dt.$$

The functions $f_i^{(0)}$ are related to the initial conditions of the system and are known functions. On the contrary, the functions h_i for $i \leq i_0$ are unknown and have to be determined by taking into account the dynamics of the system. In the following, we set $h_i(s) = 0$ for $i > i_0$; the buffer cannot be empty when the net input rate is positive.

By taking Laplace transforms in equation (6), we obtain the following result.

Proposition 2. *Let $F(s,\xi)$, $f^{(0)}(\xi)$, and $h(s)$ be the infinite column vectors, which ith components are $F_i(s,\xi)/\pi_i$, $f_i^{(0)}(\xi)/\pi_i$, and $h_i(s)/\pi_i$, respectively. Then, these vectors satisfy the matrix equation*

$$(s\mathbb{I} + \xi R - A)F(s,\xi) = f^{(0)}(\xi) + \xi R h(s), \tag{7}$$

where $\mathbb{I}$ is the identity matrix, A is the infinite matrix defined by equation (1), *and R is the diagonal matrix with diagonal elements r_i, $i \geq 0$.*

When we consider the stationary regime of the fluid queue, we have to set $f^{(0)}(s) \equiv 0$ and eliminate the term $s\mathbb{I}$ in equation (7), which then becomes

$$(\xi R - A)F(\xi) = \xi R h, \tag{8}$$

where $F(\xi)$ is the vector, which ith component is $\mathbb{E}\left[e^{-\xi X_t}\mathbb{1}_{\{\Lambda_t=i\}}\right]/\pi_i$. This is the Laplace transform version of equation (12) in the paper by Van Doorn and Scheinhardt [9]. (We take in this paper the Laplace transform of the probability density function $f_i(t,x)$ instead of the Laplace transform of the probability distribution function $\mathbb{P}\{\Lambda_t = i, X_t \leq x\}$.) The resolution of equation (8) can be found in [9].

3 Resolution of the basic matrix equation

3.1 Resolution of the matrix equation

In this section, we show how equation (7) can be solved. For, we analyze the structure of this equation and in a first step, we prove that the functions $F_i(s,\xi)$ can be expressed in terms of the function $F_{i_0}(s,\xi)$. (Recall that the index i_0 is the greatest integer such that $r(i) - c < 0$ and that for $i \geq i_0 + 1$, $r(i) > c$.). The proof greatly relies on the spectral properties of some operators defined in adequate Hilbert spaces.

In the following, we use the polynomials $Q_n(s;x)$, $n \geq 0$, defined by the recursion: $Q_0(s;x) \equiv 1$, $Q_1(s;x) = (s + \lambda_0 - |r_0|x)/\lambda_0$ and for $n \geq 1$,

$$\frac{\lambda_n}{|r_n|}Q_{n+1}(s;x) + \left(x - \frac{s+\lambda_n+\mu_n}{|r_n|}\right)Q_n(s;x) + \frac{\mu_n}{|r_n|}Q_{n-1}(s;x) = 0. \tag{9}$$

The polynomials $Q_n(s;x)$ for $n \geq 0$ form an orthogonal polynomial system and the polynomials $\left\{\frac{\lambda_1\ldots\lambda_n}{|r_1\ldots r_n|}Q_n(s;-z), n \geq 0\right\}$ are the successive denominators of the continued fraction

$$\mathcal{F}^e(s;z) = \cfrac{1}{z + \frac{s+\lambda_0}{|r_0|} - \cfrac{\frac{\mu_1\lambda_0}{|r_0 r_1|}}{z + \frac{s+\lambda_1+\mu_1}{|r_1|} - \cfrac{\frac{\mu_2\lambda_1}{|r_2 r_1|}}{z + \frac{s+\lambda_2+\mu_2}{|r_2|} - \ddots}}} \tag{10}$$

which is itself the even part of the continued fraction defined for $s \geq 0$ by

$$\mathcal{F}(s;z) = \cfrac{\alpha_1(s)}{z + \cfrac{\alpha_2(s)}{1 + \cfrac{\alpha_3(s)}{z + \cfrac{\alpha_4(s)}{1 + \ddots}}}}, \tag{11}$$

where the coefficients $\alpha_k(s)$ are such that $\alpha_1(s) = 1$, $\alpha_2(s) = (s+\lambda_0)/|r_0|$, and for $k \geq 1$,

$$\alpha_{2k}(s)\alpha_{2k+1}(s) = \frac{\lambda_{k-1}\mu_k}{|r_{k-1}r_k|}, \quad \alpha_{2k+1}(s) + \alpha_{2(k+1)}(s) = \frac{s+\lambda_k+\mu_k}{|r_k|}.$$

It can be shown (see [21] for details) that the continued fraction $\mathcal{F}(z)$ is a converging Stieltjes fraction (in particular $\alpha_k(s) > 0$ for $k \geq 0$). Hence, there exists a unique bounded, increasing function $\psi(s;x)$ such that

$$\mathcal{F}(s;z) = \int_0^\infty \frac{1}{z+x} d\psi(s;x).$$

In addition, the polynomials $Q_n(s;x)$ are orthogonal with respect to the measure $d\psi(s;x)$ and satisfy the orthogonality relations

$$\int_0^\infty Q_i(s;x)Q_j(s;x)d\psi(s;x) = \frac{|r_0|}{|r_i|\pi_i}\delta_{i,j},$$

where $\delta_{i,j}$ is equal to 0 if $i \neq j$ and to 1 if $i = j$. Note that $\int_0^\infty d\psi(s;x) = 1$.

Under condition (2), it can be shown (see [22] for details) that for all $s \geq 0$, the operator defined by the infinite matrix $R^{-1}(s\mathbb{I} - A)$ is self-adjoint in the Hilbert space

$$H = \left\{ (f_n) \in \mathbb{C}^{\mathbb{N}} : \sum_{n=0}^{\infty} |f_n|^2 |r_n| \pi_n < \infty \right\},$$

equipped with the scalar product $(f,g) = \sum_{n=0}^\infty f_n \bar{g}_n |r_n|\pi_n$. The operator $R^{-1}(s\mathbb{I} - A)$ is non-negative (i.e., $(R^{-1}(s\mathbb{I} - A)f, f) \geq 0$ for all $f \in H$) and its spectrum is equal to the support of the measure $d\psi(s;x)$. In the following, we denote by e_i the vector with all entries equal to 0 except the ith one equal to 1.

In a first step, we show how we can compute F_i for $i \le i_0$ by means of F_{i_0+1}. Let $\check{\mathbb{I}}$, $\check{A}$ and $\check{R}$ denote the finite matrices obtained from $\mathbb{I}$, A and R, respectively by deleting the lines and the columns with an index greater than i_0. Denoting by $\check{F}$, $\check{h}$, and $\check{f}^{(0)}$ the finite column vectors, which ith components are F_i/π_i, h_i/π, and $f_i^{(0)}/\pi_i$, respectively for $i = 0, \dots, i_0$, equation (7) can be written as

$$\left(\xi\check{\mathbb{I}} + \check{R}^{-1}(s\check{\mathbb{I}} - \check{A})\right)\check{F} = \check{R}^{-1}\check{f}^{(0)} + \xi\check{h} + \frac{\lambda_{i_0}}{r_{i_0}\pi_{i_0+1}}F_{i_0+1}e_{i_0}.$$

(The matrix $\check{R}$ is invertible since $r(i) < c$ for all $i \le i_0$.)

Let $\zeta_k(s)$ for $k = 0, \dots, i_0$ denote the zeros of the polynomial $Q_{i_0+1}(s; x)$, which are real, positive and simple from the general theory of orthogonal polynomials [23]. Let $\check{Q}(s; \zeta_k(s))$ denote the vector, which ith entry is $Q_i(s; \zeta_k(s))$ for $i = 0, \dots, i_0$. Then , the vectors $\check{Q}(s; \zeta_k(s))$, $k = 0, \dots, i_0$, are eigen vectors of the finite matrix $\check{R}^{-1}(s\check{\mathbb{I}} - \check{A})$ and form an orthogonal basis for the vector space spanned by the vectors e_i for $i = 0, \dots, i_0$. Their weight function, denoted by $\psi_0(s, x)$, is discrete with a finite number of atoms and satisfies

$$\int_0^\infty \frac{1}{z+x}d\psi_0(s;x) = -\frac{P_{i_0+1}(s;z)}{Q_{i_0+1}(s;x)},$$

where $P_n(s; z)$, $n \ge 0$, are the polynomials of the first kind associated with the polynomials $Q_n(s; x)$; those polynomials satisfy the recurrence relation (9) with the initial conditions $P_0(s; z) = 0$ and $P_1(s; z) = |r_0|/\lambda_0$. Via standard manipulations, we obtain the following result.

Lemma 1. *The functions F_i for $i \le i_0$ are related to the function F_{i_0+1} as follows: for $\xi \ne \zeta_k(s)$, $k = 0, \dots, i_0$,*

$$F_i = \frac{\pi_i}{r_0}\sum_{j=0}^{i_0}(f_j^{(0)}(\xi) + r_j\xi h_j(s))\int_0^\infty \frac{Q_j(s;x)Q_i(s;x)}{\xi - x}d\psi_0(s;x)$$
$$+ \mu_{i_0+1}\frac{\pi_i}{r_0}F_{i_0+1}\int_0^\infty \frac{Q_{i_0}(s;x)Q_i(s;x)}{\xi - x}d\psi_0(s;x). \quad (12)$$

Note that since $F_i(s, .)$ should have no poles in $\{\xi : \Re(\xi) \ge 0\}$, equation (12) entails

$$\sum_{j=0}^{i_0} f_j^{(0)}(\zeta_k)Q_j(s;\zeta_k) + \zeta_k\sum_{j=0}^{i_0} r_j h_j(s)Q_j(s;\zeta_k) + \mu_{i_0+1}F_{i_0+1}(s,\zeta_k)Q_{i_0}(s,r,\zeta_k(s)) = 0, \quad (13)$$

where we have used the fact that $Q_j(s; \zeta_k(s)) \ne 0$ for $k = 0, \dots, i_0$ and $0 \le j \le i_0$ [23].

Now, we show how to compute F_{i_0+i+1}, $i \ge 0$, by means of F_{i_0}. For this purpose we consider the infinite matrices $\hat{\mathbb{I}}$, $\hat{A}$ and $\hat{R}$ obtained from $\mathbb{I}$, A and R by deleting the first $(i_0 + 1)$ lines and columns, respectively. The infinite matrix $\hat{R}^{-1}(s\hat{\mathbb{I}} - \hat{A})$ defines in the Hilbert space $\hat{H}$, which is the subspace of H spanned by the vectors e_i for $i \ge i_0 + 1$, a self-adjoint operator. The Hilbert space $\hat{H}$ is composed of those series $(f_n) \in \mathbb{C}^{\mathbb{N}}$ such that $\sum_{n=0}^\infty |f_n|^2 r_{i_0+1+n}\pi_{i_0+1+n} < \infty$; the inner product is defined by

$$(f, g)_{i_0} = \sum_{n=0}^\infty f_n\bar{g}_n r_{i_0+1+n}\pi_{i_0+1+n}.$$

To apply the spectral theorem, we introduce the polynomials $Q_n(i_0+1;s;x)$ defined by: $Q_0(i_0+1;s;x)=1$, $Q_1(i_0+1;s;x)=(\lambda_{i_0+1}+\mu_{i_0+1}+s-r_{i_0+1}x)/\lambda_{i_0+1}$, and for $n\geq 0$,

$$\lambda_{i_0+1+n}Q_{i_0+n+2}(i_0+1;s;x)+\left(r_{i_0+1+n}x-(s+\lambda_{i_0+1+n}+\mu_{i_0+1+n})\right)Q_n(i_0+1;s;x) \\ +\mu_{i_0+1+n}Q_{i_0+n}(i_0+1;s;x)=0.$$

The polynomials $Q_n(i_0+1;s;x)$, $n\geq 0$, referred to as associated polynomials, form an orthogonal polynomial systems with respect to the weight measure $d\psi_1(s;x)$, which is purely discrete with atoms located at the roots of the equation $\mathcal{F}^e(s;z)=[\mathcal{F}^e(s;z)]_{i_0}$, where $[\mathcal{F}^e(s;z)]_{i_0}$ is the i_0th approximant of the continued fraction $\mathcal{F}^e(s;z)$ defined by equation (10) (see [24] for details). The polynomials $Q_n(i_0+1;s;x)$ satisfy the orthogonality relation

$$\int_0^\infty Q_n(i_0+1;s;x)Q_m(i_0+1;s;x)d\psi_1(s;x)=\frac{r_{i_0+1}\pi_{i_0+1}}{r_{i_0+n+1}\pi_{i_0+n+1}}\delta_{n,m}.$$

It can be shown that the Hilbert space $\hat{H}$ can be decomposed as $\hat{H}=\int^{\oplus}\hat{H}_x d\psi_1(s;x)$, where $\hat{H}_x$ is the vector space spanned by the vector, which ith component is $Q_i(i_0+1;s;x)$.

Denoting by $\hat{F}$ and $\hat{f}^{(0)}$ the column vectors, which ith components are $F_{i_0+1+i}/\pi_{i_0+1+i}$ and $f^{(0)}_{i_0+1+i}/\pi_{i_0+1+i}$, respectively, equation (7) can be written as

$$\left(\xi\hat{\mathbb{I}}+\hat{R}^{-1}(s\hat{\mathbb{I}}-\hat{A})\right)\hat{F}=\hat{R}^{-1}\hat{f}^{(0)}+\frac{\mu_{i_0+1}}{\pi_{i_0}r_{i_0+1}}F_{i_0}e_0,$$

since $h_i(s)\equiv 0$ for $i>i_0$. Using standard results from spectral theory, in particular the spectral identity stating that for any self-adjoint operator A with spectral measure $d\psi(x)$

$$((z\mathbb{I}-A)^{-1}f,g)=\int_{-\infty}^{\infty}\frac{(f_x,g)}{z-x}d\psi(x),$$

where f_x is the projection on the space associated with the point x in the spectrum of the operator A, we can easily obtain the following result.

Lemma 2. *For $s\geq 0$, the functions $F_i(s,\xi)$ are related to the function F_{i_0} by the relation: for $i\geq 0$,*

$$F_{i_0+i+1}(s,\xi)=\frac{\pi_{i_0+i+1}}{r_{i_0+1}\pi_{i_0+1}}\sum_{j=0}^{\infty}f^{(0)}_{i_0+j+1}(\xi)\int_0^\infty\frac{Q_j(i_0+1;s;x)Q_i(i_0+1;s;x)}{x+\xi}d\psi_1(s;x) \\ +\lambda_{i_0}\frac{\pi_{i_0+i+1}}{r_{i_0+1}\pi_{i_0+1}}F_{i_0}(s;\xi)\int_0^\infty\frac{Q_i(i_0+1;s;x)}{\xi+x}d\psi_1(s;x). \quad (14)$$

In the following, we show how the function $F_{i_0}(s,\xi)$ can be derived by combining equation (14) for $i=i_0+1$ and equation (12) for $i=i_0$. For this purpose, let us define the non negative quantities $\eta_\ell(s)$, $\ell=0,\dots,i_0$, which are the (i_0+1) solutions to the equation

$$1-\frac{\lambda_{i_0}\mu_{i_0+1}\pi_{i_0}}{r_{i_0+1}r_0}\mathcal{F}_{i_0}(s;\xi)\int_0^\infty\frac{Q_{i_0}(s;x)^2}{\xi-x}d\psi_0(s;x)=0. \quad (15)$$

Then, we can state the following result, which gives a means for computing the unknown functions $h_j(s)$ for $j=0,\dots,i_0$.

Proposition 3. *The functions $h_j(s)$, $j=0,\dots,i_0$, satisfy the linear equations: for $\ell = 0,\dots,i_0$,*

$$\frac{\lambda_{i_0}\mathcal{F}_{i_0}(s;\eta_\ell(s))\eta_\ell(s)}{r_{i_0}}\left((\eta_\ell(s)\check{\mathbb{I}}+\check{R}^{-1}(s\check{\mathbb{I}}-\check{A}))^{-1}e_{i_0},h(s)\right)$$
$$=\left((\eta_\ell(s)\hat{\mathbb{I}}+\hat{R}^{-1}(s\hat{\mathbb{I}}-\hat{A}))^{-1}e_0,\hat{R}^{-1}\hat{f}^{(0)}(\eta_\ell(s))\right)_{i_0}$$
$$-\frac{\lambda_{i_0}\mathcal{F}_{i_0}(s;\eta_\ell(s))}{r_{i_0}}\left((\eta_\ell(s)\check{\mathbb{I}}+\check{R}^{-1}(s\check{\mathbb{I}}-\check{A}))^{-1}e_{i_0},\check{R}^{-1}\check{f}^{(0)}(\eta_\ell(s))\right), \quad (16)$$

where $\mathcal{F}_{i_0}(s;z)$ is the continued fraction (11).

Proof. From equation (14) for $i=i_0+1$ and equation (12) for $i=i_0$, we deduce that

$$\left(1-\frac{\lambda_{i_0}\mu_{i_0+1}\pi_{i_0}}{r_{i_0+1}r_0}\mathcal{F}_{i_0}(s;\xi)\int_0^\infty\frac{Q_{i_0}(s;x)^2}{\xi-x}d\psi_0(s;x)\right)F_{i_0+1}(s,\xi)$$
$$=\frac{1}{r_{i_0+1}}\sum_{j=0}^{\infty}f^{(0)}_{i_0+j+1}(\xi)\int_0^\infty\frac{Q_j(i_0+1;s;x)}{x+\xi}d\psi_1(s;x)$$
$$+\frac{\lambda_{i_0}\pi_{i_0}\mathcal{F}_{i_0}(s;\xi)}{r_0r_{i_0+1}}\sum_{j=0}^{i_0}(f^{(0)}_j(\xi)+\xi r_jh_j(s))\int_0^\infty\frac{Q_j(s;x)Q_{i_0}(s;x)}{\xi-x}d\psi_0(s;x). \quad (17)$$

From equation (13), we see that the points $\zeta_k(s)$ appear as removable singularities in expression (17). The quantities $h_j(s)$, $j=0,\dots,i_0$, are then determined by using the fact that the r.h.s. of equation (17) must cancel at points $\eta_\ell(s)$ for $\ell=0,\dots,i_0$. This entails that for $\ell=0,\dots,i_0$, the terms

$$\sum_{j=0}^{\infty}f^{(0)}_{i_0+j+1}(\eta_\ell(s))\int_0^\infty\frac{Q_j(i_0+1;s;x)}{x+\eta_\ell(s)}d\psi_1(s;x)$$
$$+\frac{\lambda_{i_0}\pi_{i_0}\mathcal{F}_{i_0}(s;\eta_\ell(s))}{r_0}\sum_{j=0}^{i_0}(f^{(0)}_j(\eta_\ell(s))+\eta_\ell(s)r_jh_j(s))\int_0^\infty\frac{Q_j(s;x)Q_{i_0}(s;x)}{\eta_\ell(s)-x}d\psi_0(s;x)$$

must cancel. These equations can be rewritten in matrix form as in equation (16).

By solving the system of linear equations (16), we can compute the unknown functions $h_j(s)$ for $j=0,\dots,i_0$. This gives a means of computing F_{i_0+1} by using equation (17):

$$\left(1-\frac{\lambda_{i_0}\mu_{i_0+1}\pi_{i_0}}{r_{i_0+1}r_0}\mathcal{F}_{i_0}(s;\xi)\int_0^\infty\frac{Q_{i_0}(s;x)^2}{\xi-x}d\psi_0(s;x)\right)F_{i_0+1}(s,\xi)$$
$$=\left((\xi\hat{\mathbb{I}}+\hat{R}^{-1}(s\hat{\mathbb{I}}-\hat{A}))^{-1}e_0,\hat{R}^{-1}\hat{f}^{(0)}(\xi)\right)_{i_0}$$
$$-\frac{\lambda_{i_0}\mathcal{F}_{i_0}(s;\xi)}{r_{i_0}}\left((\xi\check{\mathbb{I}}+\check{R}^{-1}(s\check{\mathbb{I}}-\check{A}))^{-1}e_{i_0},\check{R}^{-1}\check{f}^{(0)}(\xi)+h(s)\right). \quad (18)$$

The function $F_{i_0}(s,\xi)$ is computed by using equation (18) and equation (12) for $i=i_0$. The other functions F_i are computed by using Lemmas 1 and 2.

The above procedure can be applied for any value i_0 but expressions are much simpler when $i_0 = 0$, i.e., when there is only one state with negative net input rate. In that case, we have the following result, when the buffer is initially empty and the birth and death process is in state 1.

Proposition 4. *Assume that $r_0 < 0$ and $r_i > 0$ for $i > 0$. When the buffer is initially empty and the birth and death process is in the state 1 at time 0 (i.e., $p_0(i) = \delta_{1,i}$ for all $i \geq 0$), the Laplace transform $h_0(s)$ is given by*

$$h_0(s) = \frac{r_0\eta_0(s) + s + \lambda_0}{\lambda_0\eta_0(s)|r_0|} = \frac{\mu_1\mathcal{F}_0(s;\eta_0(s))}{r_1|r_0|\eta_0(s)}, \tag{19}$$

where $\eta_0(s)$ is the unique positive solution to the equation

$$1 - \lambda_0\mu_1\mathcal{F}_0(s;\xi)/(r_1(s + \lambda_0 + r_0\xi)) = 0.$$

Finally, note that by using equation (19), we obtain the expression of $F_1(s;\xi)$, namely

$$F_1(s,\xi) = \frac{\frac{1}{r_1}\left(1 + \frac{\lambda_0\xi r_0 h_0(s)}{s+\lambda_0+r_0\xi}\right)\mathcal{F}_0(s;\xi)}{1 - \frac{\lambda_0\mu_1}{r_1(s+\lambda_0+r_0\xi)}\mathcal{F}_0(s;\xi)}.$$

4 Conclusion

We have presented in this paper a general method for computing the Laplace transform of the transient probability distribution function of the content of a fluid reservoir fed with a source, whose transmission rate is modulated by a general birth and death process. This Laplace transform can be evaluated by solving a polynomial equation (see equation (15)). Once the zeros are known, the quantities $h_i(s)$ for $i = 0, \ldots, i_0$ are computed by solving the system of linear equations (16). These functions then completely determined the two critical functions F_{i_0} and F_{i_0+1}, which are then used for computing the functions F_i for $i > i_0 + 1$ and F_i for $i < i_0$ by using equations (14) and (12), respectively. Moreover, we note that the theory of orthogonal polynomials and continued fractions plays a crucial role in solving the basic equation (7).

The above method can be used for evaluating the Laplace transform of the duration of a busy period of the fluid reservoir. As a matter of fact, following the papers by Asmussen [25], it can be shown that the law of the busy period can be related to the occupation level of a dual reservoir. This point will be addressed in further studies.

References

1. Kosten, L.: Stochastic theory of data-handling systems with groups of multiple sources. In: Proceedings of the IFIP WG 7.3/TC 6 Second International Symposium on the Performance of Computer-Communication Systems, Zurich, Switzerland. (1984) 321–331
2. Anick, D., Mitra, D., Sondhi, M.M.: Stochastic theory of a data-handling system with multiple sources. Bell System Tech. J. **61** (1982) 1871–1894

3. Mitra, D.: Stochastic fluid models. In: Proceedings of Performance'87, Brussels, Belgium. (1987) 39–51
4. Mitra, D.: Stochastic theory of a fluid model of producers and consumers coupled by a buffer. Advances in Applied Probability **20** (1988) 646–676
5. Stern, T.E., Elwalid, A.I.: Analysis of separable Markov-modulated rate models for information-handling systems. Advances in Applied Probability **23** (1991) 105–139
6. Virtamo, J., Norros, I.: Fluid queue driven by an M/M/1 queue. Queueing Systems - Theory and Application **16** (1994) 373–386
7. Adan, I., Resing, J.: Simple analysis of a fluid queue driven by an M/M/1 queue. Queueing Systems - Theory and Application **22** (1996) 171–174
8. Barbot, N., Sericola, B.: Stationary solution to the fluid queue fed by an M/M/1 queue. Journ. Appl. Probab. **39** (2002) 359–369
9. van Dorn, E., Scheinhardt, W.: A fluid queue driven by an infinite-state birth and death process. In Ramaswami, V., Wirth, P., eds.: Proc. ITC 15, Amsterdam, Elsevier (1997) 465–475
10. Sericola, B., Tuffin, B.: A fluid queue driven by a Markovian queue. Queueing Systems - Theory and Application **31** (1999) 253–264
11. Sericola, B.: A finite buffer fluid queue driven by a Markovian queue. Queueing Systems - Theory and Application **38** (2001) 213–220
12. Kobayashi, H., Ren, Q.: A mathematical theory for transient analysis of communication networks. IEICE Trans. Communications **E75-B** (1992) 1266–1276
13. Ren, Q., Kobayashi, H.: Transient solutions for the buffer behavior in statistical multiplexing. Perf. Eval. **23** (1995) 65–87
14. Tanaka, T., Hashida, O., Takahashi, Y.: Transient analysis of fluid models for ATM statistical multiplexer. Performance Evaluation **23** (1995) 145–162
15. Sericola, B.: Transient analysis of stochastic fluid models. Perf. Eval. **32** (1998) 245–263
16. Ahn, S., Ramaswami, V.: Transient analysis of fluid flow models via stochastic coupling. Comm. Statist. Stochastic Models **20** (2004) 71–101
17. Sericola, B., Parthasarathy, P.R., Vijayashree, K.V.: Exact transient solution of an M/M/1 driven fluid queue. Int. Journ. of Computer Mathematics (2005) to appear.
18. Asmussen, S.: Applied probability and queues. J. Wiley and Sons (1987)
19. Parthasarathy, P., Vijayashree, K., Lenin, R.: Fluid queues driven by a birth and death process with alternating flow rates. Mathematical Problems in Engineering **5** (2004) 469–489
20. Kleinrock, L.: Queueing Systems. Volume 1. J. Wiley (1975)
21. Henrici, P.: Applied and computational complex analysis. Vol. 2 Wiley, New York (1977)
22. Guillemin, F.: Spectral theory of birth and death processes. Submitted for publication (2005)
23. Chihara, T.: An introduction to orthogonal polynomials. Gordon and Breach, New York (1978)
24. Guillemin, F., Pinchon, D.: Excursions of birth and death processes, orthogonal polynomials, and continued fractions. J. Appl. Prob. **36** (1999) 752–770
25. Asmussen, S.: Busy period analysios, rare events and transient behavior in fluid flow models. Journal of Applied Mathematics and Stochastic Analysis **7** (1994) 269–299