Transient analysis of a fluid buffer driven by a birth and death process

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Abstract. We analyze the transient behavior of a fluid queue when the instantaneous rate of the input process is governed by a general birth and death process. These fluid flows models are widely used in the performance analysis of telecommunication systems. The transient analysis of such models has already been considered in earlier works with a finite state space background process. We consider in this paper a infinite state space driving process and we study the fluid queue by using results from spectral theory.

Keywords: Fluid buffer, spectral theory, orthogonal polynomials, transient analysis

1 Introduction

In the performance evaluation of packet telecommunication networks, fluid queues with Markov modulated input rates have been widely used to represent information flows at the burst level. By ignoring the discrete nature of packet arrivals, fluid models greatly simplify the analysis of a telecommunication system (say, a multiplexer, a switch or a router), while capturing the main performance characteristics.

There is a large number of papers dealing with the analysis of stochastic fluid flow models. Most of these papers consider such models in stationary regime. In their pioneering work, Kosten [1] and Anick et al [2] analyze the fluid model for several identical exponential on-off input sources. This model has been extended and studied in details, for instance, by Mitra [3, 4], Stern and Elwalid [5]. Stationary fluid queues driven by infinite state space Markov chains has also given rise to many research papers in the recent past few years. When the driving process is the $M/M/1$ queue, Virtamo and Norros [6] solve the well-known infinite differential system by studying the continuous spectrum of a key matrix; this model has also been studied by Adan and Resing [7], Barbot and Sericola [8]. More general input processes have also been considered; see for instance Van Doorn and Scheinhart [9], Sericola and Tuffin [10], Sericola [11].

The transient analysis of fluid queues fed by finite state Markov chains has been studied by using Laplace transform by Ren and Kobayashi [12, 13] for exponential on-off sources.
These studies have been extended to the Markov modulated input rate model by Tanaka et al in [14]. In [15], Sericola has obtained a transient solution based on simple recurrence relations, which are particularly interesting for their numerical properties. More recently, Ahn and Ramaswami [16] use an approach based on an approximation of the fluid model by the amounts of work in a sequence of Markov modulated queues of the quasi birth and death type. When the driving Markov chain has an infinite state space, the transient analysis is more complicated. Sericola et al [17] consider the case of the $M/M/1$ queue by using recurrence relations and Laplace transforms.

In this paper we study the transient behavior of an infinite buffer fluid queue driven by an infinite birth and death process, which controls the input rates into the buffer. By using a double Laplace transform of the joint probability density function of the buffer level and of the state of the driving process, we obtain a matrix equation, which is solved by means of spectral properties of some linear operators.

The paper is organized as follows. In Section 2, we present the model, the notation and the system of partial differential equations satisfied by the joint probability density function of the buffer level and of the state of the birth and death process. In this same section, we establish the basic matrix equation satisfied by the Laplace transforms of the joint probability density functions. Section 3 is devoted to the resolution of this matrix equation. Some concluding remarks are presented in Section 4.

## 2 Problem formulation and preliminary results

Throughout this paper, we consider a queue fed with a fluid traffic source, whose instantaneous transmitting rate is modulated by a general birth and death process $\{A_t\}$ taking values in $\mathbb{N}$. The input rate is precisely $r(A_t)$, where $r$ is a given increasing function from $\mathbb{N}$ into $\mathbb{R}$.

The birth and death rates of the process $\{A_t\}$ are denoted by $\lambda_i > 0$ for $i > 0$ and $\mu_j > 0$ for $j \geq 1$, respectively. The infinitesimal generator is given by the infinite tridiagonal matrix

$$
\mathcal{A} = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & \cdots \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & \cdots \\ 0 & \mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}
$$

(1)

We assume that the birth and death process $\{A_t\}$ is ergodic, which amounts to assuming (see [18] for instance) that

$$
\sum_{n=0}^{\infty} \frac{1}{\lambda_n \pi_n} = \infty \quad \text{and} \quad \sum_{n=0}^{\infty} \pi_n < \infty, \quad (2)
$$

where the quantities $\pi_n$ are defined by $\pi_0 = 1$ and for $n \geq 1$, $\pi_n = \lambda_0 \ldots \lambda_{n-1}/(\mu_1 \ldots \mu_n)$. Under the above assumption, the birth and death process $\{A_t\}$ has a unique invariant probability measure, denoted by $\{p(n), n \geq 0\}$ and defined by $p(n) = \sum_{j=0}^{\infty} \pi_j$ for $n \geq 0$.

Let $\{p_0(i), i \geq 0\}$ denote the initial distribution of the birth and death process $\{A_t\}$, i.e., $\mathbb{P}(A_0 = i) = p_0(i)$. Note that if $p_0 \equiv p$, then $\mathbb{P}(A_t = i) = p(i)$ for all $t \geq 0$ and $i \in \mathbb{N}$.
We assume that the queue under consideration is drained at constant rate $c > 0$. Furthermore, we assume that $r(i) > c$ when $i$ is greater than a fixed $i_0 > 0$ (and that $r(i) \leq c$ for $0 \leq i \leq i_0$). In addition, the parameters $c$ and $r(i)$ are such that

$$
\rho = \sum_{i=0}^{\infty} \frac{r(i)}{c} p(i) < 1
$$

so that the system is stable. To avoid cumbersome special cases, we assume that $r(i) < c$ for $0 \leq i \leq i_0$, so that $r_i = r(i) - c$ is either positive or negative but always non zero. The quantity $r_i$ is the net input rate, when the modulating process $\{A_t\}$ is in state $i$.

Let $X_t$ denote the buffer content at time $t$. The process $\{X_t\}$ satisfies the following evolution equation: for $t \geq 0$,

$$
\frac{dX_t}{dt} = \begin{cases} 
  r(A_t) - c & \text{if } X_t > 0 \text{ or } r(A_t) > c, \\
  0 & \text{if } X_t = 0 \text{ and } r(A_t) \leq c.
\end{cases}
$$

(4)

Let $f_i(t, x)$ denote the joint density function defined by $f_i(t, x) = \mathbb{P}(A_t = i, X_t \in dx)$. As shown in [15], on top of its usual jump at point $x = 0$, when $X_0 = x_0 \geq 0$, the distribution function $\mathbb{P}(A_t = i, X_t \leq x)$ has a jump at points $x = x_0 + r_i t$, for $t$ such that $x_0 + r_i t > 0$, which corresponds to the case when the Markov chain $\{A_t\}$ starts and remains during the whole interval $[0, t)$ in state $i$. We thus have,

$$
\mathbb{P}(A_t = i, X_t = x_0 + r_i t) = p_0(i) e^{-(\lambda_i + \mu_i) t} \mathbb{1}_{\{x_0 + r_i t > 0\}}.
$$

The measure associated with the distribution function $\mathbb{P}(A_t = i, X_t \leq x)$ can then be written as follows: for $t \geq 0$;

$$
d\mathbb{P}(A_t = i, X_t \leq x) = f_i(t, x) dx + p_0(i) e^{-(\lambda_i + \mu_i) t} \delta_{x_0 + r_i t}(dx),
$$

(5)

where $\delta_{x_0 + r_i t}(dx)$ is the Dirac mass at point $x_0 + r_i t$. We focus in the rest of paper on the probability density function $f_i(t, x)$ for $x > 0$ along with its usual jump at point $x = 0$.

A direct consequence of the evolution equation (4) is the forward Chapman-Kolmogorov equations satisfied by $(f_i(t, x), x \geq 0, i \in \mathbb{N})$.

**Proposition 1 (Chapman-Kolmogorov equations).** The functions $(x, t) \rightarrow f_i(t, x)$ for $i \in \mathbb{N}$ satisfy the differential system (in the sense of distributions):

$$
\frac{\partial f_i}{\partial t} = -r_i \frac{\partial}{\partial x} \left( \mathbb{1}_{\{i > i_0\}} + \mathbb{1}_{\{i \leq i_0\}} \mathbb{1}_{\{x > 0\}} \right) f_i - (\lambda_i + \mu_i) f_i + \lambda_{i-1} f_{i-1} + \mu_{i+1} f_{i+1},
$$

(6)

with the convention $\lambda_{-1} = 0$, $f_{-1} \equiv 0$ and $f_i(t, x) = 0$ for $x < 0$.

Note that the differential system (6) holds for the density probability functions $f_i(t, x)$. This is the main difference with the differential system considered in [9, 19] and governing the probability distribution functions $\mathbb{P}(X_t \leq x, A_t = i), i \geq 0$. The differential system (6) is actually the equivalent of the Takács’ integro-differential formula for the $M/G/1$ queue [20]. The resolution of this differential system is addressed below.
Introduce the double Laplace transform

\[ F_i(s, \xi) = \int_0^\infty \int_0^\infty e^{-st-\xi x} f_i(t, x)dx dt = \int_0^\infty e^{-st} \mathbb{E}(-\xi X_t 1_{\{A_t = i\}}) dt \]

and define the functions \( f_i^{(0)}(\xi) \) and \( h_i(s) \) for \( i \in \mathbb{N} \) as follows

\[ f_i^{(0)}(\xi) = \int_0^\infty e^{-2\xi} \mathbb{P}\{A_0 = i, X_0 \in dx\} \quad \text{and} \quad h_i(s) = \int_0^\infty e^{-st} \mathbb{P}\{A_t = i, X_t = 0\} dt. \]

The functions \( f_i^{(0)} \) are related to the initial conditions of the system and are known functions. On the contrary, the functions \( h_i \) for \( i \leq i_0 \) are unknown and have to be determined by taking into account the dynamics of the system. In the following, we set \( h_i(s) = 0 \) for \( i > i_0 \); the buffer cannot be empty when the net input rate is positive.

By taking Laplace transforms in equation (6), we obtain the following result.

**Proposition 2.** Let \( F(s, \xi), f^{(0)}(\xi), \) and \( h(s) \) be the infinite column vectors, which \( i \)th components are \( F_i(s, \xi)/\pi_i, f_i^{(0)}(\xi)/\pi_i, \) and \( h_i(s)/\pi_i, \) respectively. Then, these vectors satisfy the matrix equation

\[ (sI + \xi R - A)F(s, \xi) = f^{(0)}(\xi) + \xi Rh(s), \]

where \( I \) is the identity matrix, \( A \) is the infinite matrix defined by equation (1), and \( R \) is the diagonal matrix with diagonal elements \( r_i, i \geq 0. \)

When we consider the stationary regime of the fluid queue, we have to set \( f^{(0)}(s) \equiv 0 \) and eliminate the term \( sI \) in equation (7), which then becomes

\[ (\xi R - A)F(\xi) = \xi Rh, \]

where \( F(\xi) \) is the vector, which \( i \)th component is \( \mathbb{E}\left[e^{-\xi X_t} 1_{\{A_t = i\}}\right]/\pi_i. \) This is the Laplace transform version of equation (12) in the paper by Van Doorn and Scheinhardt [9]. (We take in this paper the Laplace transform of the probability density function \( f_i(t, x) \) instead of the Laplace transform of the probability distribution function \( \mathbb{P}\{A_t = i, X_t \leq x\}. \) The resolution of equation (8) can be found in [9].

### 3 Resolution of the basic matrix equation

#### 3.1 Resolution of the matrix equation

In this section, we show how equation (7) can be solved. For, we analyze the structure of this equation and in a first step, we prove that the functions \( F_i(s, \xi) \) can be expressed in terms of the function \( F_{i_0}(s, \xi) \). (Recall that the index \( i_0 \) is the greatest integer such that \( r(i) - c < 0 \) and that for \( i \geq i_0 + 1, r(i) > c. \) The proof greatly relies on the spectral properties of some operators defined in adequate Hilbert spaces.

In the following, we use the polynomials \( Q_n(s; x), n \geq 0, \) defined by the recursion: \( Q_0(s; x) \equiv 1, \) \( Q_1(s; x) = (s + \lambda_0 - |r_0|x)/\lambda_0 \) and for \( n \geq 1,

\[ \frac{\lambda_n}{|r_n|} Q_{n+1}(s; x) + \left(x - \frac{s + \lambda_n + \mu_n}{|r_n|}\right) Q_n(s; x) + \frac{\mu_n}{|r_n|} Q_{n-1}(s; x) = 0. \]

(9)
The polynomials $Q_n(s; x)$ for $n \geq 0$ form an orthogonal polynomial system and the polynomials \( \left\{ \frac{\lambda_1 \ldots \lambda_n}{r_1 \ldots r_n} Q_n(s; -z) \right\}_{n \geq 0} \) are the successive denominators of the continued fraction

\[
\mathcal{F}^e(s; z) = \frac{1}{z + \frac{s + \lambda_0}{|r_0|} - \frac{\mu_1 \lambda_0}{|r_0 r_1|} - \frac{\mu_2 \lambda_1}{|r_1 r_2|} - \cdots}.
\]

which is itself the even part of the continued fraction defined for $s \geq 0$ by

\[
\mathcal{F}(s; z) = \frac{\alpha_1(s)}{z + \frac{\alpha_2(s)}{1 + \frac{\alpha_3(s)}{z + \frac{\alpha_4(s)}{1 + \cdots}}}},
\]

where the coefficients $\alpha_k(s)$ are such that $\alpha_1(s) = 1$, $\alpha_2(s) = (s + \lambda_0)/|r_0|$, and for $k \geq 1$,

\[
\alpha_{2k}(s) \alpha_{2k+1}(s) = \frac{\lambda_{k-1} \mu_k}{|r_{k-1} r_k|}, \quad \alpha_{2k+1}(s) + \alpha_{2(k+1)}(s) = \frac{s + \lambda_k + \mu_k}{|r_k|}.
\]

It can be shown (see [21] for details) that the continued fraction $\mathcal{F}(z)$ is a converging Stieltjes fraction (in particular $\alpha_k(s) > 0$ for $k \geq 0$). Hence, there exists a unique bounded, increasing function $\psi(s; x)$ such that

\[
\mathcal{F}(s; z) = \int_0^\infty \frac{1}{z + x} d\psi(s; x).
\]

In addition, the polynomials $Q_n(s; x)$ are orthogonal with respect to the measure $d\psi(s; x)$ and satisfy the orthogonality relations

\[
\int_0^\infty Q_i(s; x) Q_j(s; x) d\psi(s; x) = \frac{|r_0|}{|r_i| \pi_i} \delta_{i,j},
\]

where $\delta_{i,j}$ is equal to 0 if $i \neq j$ and to 1 if $i = j$. Note that $\int_0^\infty d\psi(s; x) = 1$.

Under condition (2), it can be shown (see [22] for details) that for all $s \geq 0$, the operator defined by the infinite matrix $R^{-1}(s \mathbb{I} - A)$ is self-adjoint in the Hilbert space

\[
H = \left\{ (f_n) \in \mathbb{C}^N : \sum_{n=0}^\infty |f_n|^2 |r_n| \pi_n < \infty \right\},
\]

equipped with the scalar product $(f, g) = \sum_{n=0}^\infty f_n \overline{g_n} |r_n| \pi_n$. The operator $R^{-1}(s \mathbb{I} - A)$ is non-negative (i.e., $(R^{-1}(s \mathbb{I} - A)f, f) \geq 0$ for all $f \in H$) and its spectrum is equal to the support of the measure $d\psi(s; x)$. In the following, we denote by $e_i$ the vector with all entries equal to 0 except the $i$th one equal to 1.
In a first step, we show how we can compute $F_i$ for $i \leq i_0$ by means of $F_{i_0+1}$. Let $\tilde{I}$, $\tilde{A}$ and $\tilde{R}$ denote the finite matrices obtained from $I$, $A$ and $R$, respectively by deleting the lines and the columns with an index greater than $i_0$. Denoting by $\tilde{F}$, $\tilde{h}$, and $\tilde{f}^{(0)}$ the finite column vectors, which $i$th components are $F_i/\pi_i$, $h_i/\pi_i$, and $f_i^{(0)}/\pi_i$, respectively for $i = 0, \ldots, i_0$, equation (7) can be written as

\[
(\xi \tilde{I} + \tilde{R}^{-1}(s \tilde{I} - \tilde{A})) \tilde{F} = \tilde{R}^{-1} \tilde{f}^{(0)} + \xi \tilde{h} + \frac{\lambda_{i_0}}{r_{i_0} \pi_{i_0+1}} F_{i_0+1} e_{i_0}.
\]

(The matrix $\tilde{R}$ is invertible since $r(i) < c$ for all $i \leq i_0$.)

Let $\zeta_k(s)$ for $k = 0, \ldots, i_0$ denote the zeros of the polynomial $Q_{i_0+1}(s; x)$, which are real, positive and simple from the general theory of orthogonal polynomials [23]. Let $Q(s; \zeta_k(s))$ denote the vector, which $i$th entry is $Q_i(s; \zeta_k(s))$ for $i = 0, \ldots, i_0$. Then, the vectors $Q(s; \zeta_k(s))$, $k = 0, \ldots, i_0$, are eigen vectors of the finite matrix $\tilde{R}^{-1}(s \tilde{I} - \tilde{A})$ and form an orthogonal basis for the vector space spanned by the vectors $e_i$ for $i = 0, \ldots, i_0$. Their weight function, denoted by $\psi_0(s, x)$, is discrete with a finite number of atoms and satisfies

\[
\int_0^\infty \frac{1}{z + x} d\psi_0(s; x) = -\frac{P_{i_0+1}(s; z)}{Q_{i_0+1}(s; x)}.
\]

where $P_n(s; z)$, $n \geq 0$, are the polynomials of the first kind associated with the polynomials $Q_n(s; x)$; those polynomials satisfy the recurrence relation (9) with the initial conditions $P_0(s; z) = 0$ and $P_1(s; z) = |r_0|/\lambda_0$. Via standard manipulations, we obtain the following result.

**Lemma 1.** The functions $F_i$ for $i \leq i_0$ are related to the function $F_{i_0+1}$ as follows: for $\xi \neq \zeta_k(s)$, $k = 0, \ldots, i_0$,

\[
F_i = \frac{\pi_i}{r_0} \sum_{j=0}^{i_0} (f_j^{(0)}(\xi) + r_j h_j(s)) \int_0^\infty \frac{Q_j(s; x)Q_i(s; x)}{\xi - x} d\psi_0(s; x)
\]

\[
+ \frac{\mu_{i_0+1}}{r_0} F_{i_0+1} \int_0^\infty \frac{Q_{i_0}(s; x)Q_i(s; x)}{\xi - x} d\psi_0(s; x).
\]

(12)

Note that since $F_i(s, \cdot)$ should have no poles in $\{\xi : \Re(\xi) \geq 0\}$, equation (12) entails

\[
\sum_{j=0}^{i_0} f_j^{(0)}(\zeta_k)Q_j(s; \zeta_k) + \zeta_k \sum_{j=0}^{i_0} r_j h_j(s)Q_j(s; \zeta_k) + \mu_{i_0+1} F_{i_0+1}(s, \zeta_k)Q_{i_0}(s, r, \zeta_k(s)) = 0,
\]

(13)

where we have used the fact that $Q_j(s; \zeta_k(s)) \neq 0$ for $k = 0, \ldots, i_0$ and $0 \leq j \leq i_0$ [23].

Now, we show how to compute $F_{i_0+i+1}$, $i \geq 0$, by means of $F_{i_0}$. For this purpose we consider the infinite matrices $I$, $A$ and $R$ obtained from $I$, $A$ and $R$ by deleting the first $(i_0 + 1)$ lines and columns, respectively. The infinite matrix $\tilde{R}^{-1}(s \tilde{I} - \tilde{A})$ defines in the Hilbert space $\tilde{H}$, which is the subspace of $H$ spanned by the vectors $e_i$ for $i \geq i_0 + 1$, a self-adjoint operator. The Hilbert space $\tilde{H}$ is composed of those series $(f_n) \in \mathbb{C}^\infty$ such that $\sum_{n=0}^\infty |f_n|^2 r_{i_0+1+n} \pi_{i_0+1+n} < \infty$; the inner product is defined by

\[
(f, g)_{i_0} = \sum_{n=0}^\infty f_n \bar{g}_n r_{i_0+1+n} \pi_{i_0+1+n}.
\]
To apply the spectral theorem, we introduce the polynomials \( Q_n(i_0 + 1; s; x) \) defined by: \( Q_0(i_0 + 1; s; x) = 1 \), \( Q_1(i_0 + 1; s; x) = (\lambda_{i_0+1} + \mu_{i_0+1} + s - r_{i_0+1}x)/\lambda_{i_0+1} \), and for \( n \geq 0 \),

\[
\lambda_{i_0+1+n} Q_{i_0+n+2}(i_0 + 1; s; x) + (r_{i_0+1+n}x - (s + \lambda_{i_0+1+n} + \mu_{i_0+1+n})) Q_{n}(i_0 + 1; s; x) + \mu_{i_0+1+n} Q_{i_0+n}(i_0 + 1; s; x) = 0.
\]

The polynomials \( Q_n(i_0 + 1; s; x) \), \( n \geq 0 \), referred to as associated polynomials, form an orthogonal polynomial system with respect to the weight measure \( d\psi_1(s; x) \), which is purely discrete with atoms located at the roots of the equation \( \mathcal{F}^e(s; z) = [\mathcal{F}^e(s; z)]_{i_0} \), where \( [\mathcal{F}^e(s; z)]_{i_0} \) is the \( i_0 \)th approximant of the continued fraction \( \mathcal{F}^e(s; z) \) defined by equation (10) (see [24] for details). The polynomials \( Q_n(i_0 + 1; s; x) \) satisfy the orthogonality relation

\[
\int_{0}^{\infty} Q_n(i_0 + 1; s; x) Q_m(i_0 + 1; s; x) d\psi_1(s; x) = \frac{r_{i_0+1+n} \lambda_{i_0+1} + n\pi_{i_0+n+1}}{r_{i_0+1+n} \lambda_{i_0+1} + n\pi_{i_0+n+1}} \delta_{n,m}.
\]

It can be shown that the Hilbert space \( \tilde{H} \) can be decomposed as \( \tilde{H} = \int_{0}^{\infty} \tilde{H}_x d\psi_1(s; x) \), where \( \tilde{H}_x \) is the vector space spanned by the vector, which \( i \)th component is \( Q_1(i_0 + 1; s; x) \).

Denoting by \( F \) and \( f^{(0)} \) the column vectors, which \( i \)th components are \( F_{i_0+1+i}/\pi_{i_0+1+i} \) and \( f_{i_0+1+i}/\pi_{i_0+1+i} \), respectively, equation (7) can be written as

\[
\left( \xi I + \tilde{R}^{-1}(s I - \tilde{A}) \right) F = \tilde{R}^{-1} f^{(0)} + \frac{\mu_{i_0+1}}{\pi_{i_0+1}} F_{i_0} e_0,
\]

since \( h_i(s) \equiv 0 \) for \( i > i_0 \). Using standard results from spectral theory, in particular the spectral identity stating that for any self-adjoint operator \( A \) with spectral measure \( d\psi(x) \)

\[
((z I - A)^{-1} f, g) = \int_{-\infty}^{\infty} \frac{(f_x, g)}{z - x} d\psi(x),
\]

where \( f_x \) is the projection on the space associated with the point \( x \) in the spectrum of the operator \( A \), we can easily obtain the following result.

**Lemma 2.** For \( s \geq 0 \), the functions \( F_i(s, \xi) \) are related to the function \( F_{i_0} \) by the relation: for \( i \geq 0 \),

\[
F_{i_0 + i + 1}(s, \xi) = \frac{\pi_{i_0+1+i}}{r_{i_0+1} \pi_{i_0+1}} \sum_{j=0}^{\infty} f_{i_0+j+1}^{(0)}(\xi) \int_{0}^{\infty} Q_{j}(i_0 + 1; s; x) Q_{i}(i_0 + 1; s; x) \frac{d\psi_1(s; x)}{x + \xi}
\]

\[
+ \lambda_{i_0} \frac{\pi_{i_0+1+i}}{r_{i_0+1} \pi_{i_0+1}} F_{i_0}(s, \xi) \int_{0}^{\infty} Q_{i}(i_0 + 1; s; x) \frac{d\psi_1(s; x)}{\xi + x}.
\]

(14)

In the following, we show how the function \( F_{i_0}(s, \xi) \) can be derived by combining equation (14) for \( i = i_0 + 1 \) and equation (12) for \( i = i_0 \). For this purpose, let us define the non negative quantities \( \eta_\ell(s) \), \( \ell = 0, \ldots, i_0 \), which are the \( (i_0 + 1) \) solutions to the equation

\[
1 - \frac{\lambda_{i_0} \mu_{i_0+1} \pi_{i_0} F_{i_0}(s; \xi)}{r_{i_0+1} r_{i_0}} \int_{0}^{\infty} Q_{i_0}(s; x)^2 \frac{d\psi_0(s; x)}{\xi - x} = 0.
\]

(15)

Then, we can state the following result, which gives a means for computing the unknown functions \( h_j(s) \) for \( j = 0, \ldots, i_0 \).
**Proposition 3.** The functions $h_j(s)$, $j = 0, \ldots, i_0$, satisfy the linear equations: for $\ell = 0, \ldots, i_0$,

$$
\frac{\lambda_{i_0} F_{i_0}(s; \eta_\ell(s))}{r_{i_0}} \frac{\eta_\ell(s)}{r_{i_0}^2} \left( (\eta_\ell(s) \mathbf{I} + \tilde{R}^{-1}(\mathbf{I} - \tilde{A}))^{-1} e_{i_0}, h(s) \right)
$$

$$
= \left( (\eta_\ell(s) \mathbf{I} + \tilde{R}^{-1}(\mathbf{I} - \tilde{A}))^{-1} e_0, \tilde{R}^{-1} f^{(0)}(\eta_\ell(s)) \right)_{i_0}
$$

$$
- \frac{\lambda_{i_0} F_{i_0}(s; \eta_\ell(s))}{r_{i_0}} \left( (\eta_\ell(s) \mathbf{I} + \tilde{R}^{-1}(\mathbf{I} - \tilde{A}))^{-1} e_{i_0}, \tilde{R}^{-1} f^{(0)}(\eta_\ell(s)) \right), \quad (16)
$$

where $F_{i_0}(s; z)$ is the continued fraction (11).

**Proof.** From equation (14) for $i = i_0 + 1$ and equation (12) for $i = i_0$, we deduce that

$$
\left( 1 - \frac{\lambda_{i_0} \mu_{i_0+1} \pi_{i_0}}{r_{i_0+1} r_0} F_{i_0}(s; \xi) \int_0^\infty Q_{i_0}(s; x)^2 \frac{d\psi_0(s; x)}{\xi - x} \right) F_{i_0+1}(s, \xi)
$$

$$
= \frac{1}{r_{i_0+1}} \sum_{j=0}^\infty f_{i_0+j+1}(\xi) \int_0^\infty Q_j(i_0 + 1; s; x) \frac{d\psi_1(s; x)}{x + \xi} + \frac{\lambda_{i_0} \pi_{i_0} F_{i_0}(s; \xi)}{r_0} \sum_{j=0}^{i_0} (f_j^{(0)}(\eta_\ell(s)) + \eta_\ell(s) r_j h_j(s)) \int_0^\infty Q_j(s; x) Q_{i_0}(s; x) \frac{d\psi_0(s; x)}{\eta_\ell(s) - x} \quad (17)
$$

From equation (13), we see that the points $\zeta_k(s)$ appear as removable singularities in expression (17). The quantities $h_j(s)$, $j = 0, \ldots, i_0$, are then determined by using the fact that the r.h.s. of equation (17) must cancel at points $\eta_\ell(s)$ for $\ell = 0, \ldots, i_0$. This entails that for $\ell = 0, \ldots, i_0$, the terms

$$
\sum_{j=0}^\infty f_{i_0+j+1}(\eta_\ell(s)) \int_0^\infty Q_j(i_0 + 1; s; x) \frac{d\psi_1(s; x)}{x + \eta_\ell(s)}
$$

$$
+ \frac{\lambda_{i_0} \pi_{i_0} F_{i_0}(s; \eta_\ell(s))}{r_0} \sum_{j=0}^{i_0} (f_j^{(0)}(\eta_\ell(s)) + \eta_\ell(s) r_j h_j(s)) \int_0^\infty Q_j(s; x) Q_{i_0}(s; x) \frac{d\psi_0(s; x)}{\eta_\ell(s) - x}
$$

must cancel. These equations can be rewritten in matrix form as in equation (16).

By solving the system of linear equations (16), we can compute the unknown functions $h_j(s)$ for $j = 0, \ldots, i_0$. This gives a means of computing $F_{i_0+1}$ by using equation (17):

$$
\left( 1 - \frac{\lambda_{i_0} \mu_{i_0+1} \pi_{i_0}}{r_{i_0+1} r_0} F_{i_0}(s; \xi) \int_0^\infty Q_{i_0}(s; x)^2 \frac{d\psi_0(s; x)}{\xi - x} \right) F_{i_0+1}(s, \xi)
$$

$$
= \left( (\xi \mathbf{I} + \tilde{R}^{-1}(\mathbf{I} - \tilde{A}))^{-1} e_0, \tilde{R}^{-1} f^{(0)}(\xi) \right)_{i_0}
$$

$$
- \frac{\lambda_{i_0} F_{i_0}(s; \xi)}{r_{i_0}} \left( (\xi \mathbf{I} + \tilde{R}^{-1}(\mathbf{I} - \tilde{A}))^{-1} e_{i_0}, \tilde{R}^{-1} f^{(0)}(\xi) + h(s) \right). \quad (18)
$$

The function $F_{i_0}(s; \xi)$ is computed by using equation (18) and equation (12) for $i = i_0$. The other functions $F_i$ are computed by using Lemmas 1 and 2.
The above procedure can be applied for any value $i_0$ but expressions are much simpler when $i_0 = 0$, i.e., when there is only one state with negative net input rate. In that case, we have the following result, when the buffer is initially empty and the birth and death process is in state 1.

**Proposition 4.** Assume that $r_0 < 0$ and $r_i > 0$ for $i > 0$. When the buffer is initially empty and the birth and death process is in the state 1 at time 0 (i.e., $p_0(i) = \delta_{1,i}$ for all $i \geq 0$), the Laplace transform $h_0(s)$ is given by

$$h_0(s) = \frac{r_0 \eta_0(s) + s + \lambda_0}{\lambda_0 \eta_0(s) |r_0|} = \frac{\mu_1 F_0(s; \eta_0(s))}{r_1 |r_0| \eta_0(s)},$$

(19)

where $\eta_0(s)$ is the unique positive solution to the equation

$$1 - \lambda_0 \mu_1 F_0(s; \xi)/(r_1(s + \lambda_0 + r_0 \xi)) = 0.$$

Finally, note that by using equation (19), we obtain the expression of $F_1(s; \xi)$, namely

$$F_1(s, \xi) = \frac{1}{r_1} \left( 1 + \frac{\lambda_0 \xi r_0 \eta_0(s)}{s + \lambda_0 + r_0 \xi} \right) F_0(s; \xi)$$

$$- \frac{\lambda \mu_1}{r_1(s + \lambda_0 + r_0 \xi)} F_0(s; \xi).$$

4 Conclusion

We have presented in this paper a general method for computing the Laplace transform of the transient probability distribution function of the content of a fluid reservoir fed with a source, whose transmission rate is modulated by a general birth and death process. This Laplace transform can be evaluated by solving a polynomial equation (see equation (15)). Once the zeros are known, the quantities $h_i(s)$ for $i = 0, \ldots, i_0$ are computed by solving the system of linear equations (16). These functions then completely determined the two critical functions $F_{i_0}$ and $F_{i_0+1}$, which are then used for computing the functions $F_i$ for $i > i_0 + 1$ and $F_i$ for $i < i_0$ by using equations (14) and (12), respectively. Moreover, we note that the theory of orthogonal polynomials and continued fractions plays a crucial role in solving the basic equation (7).

The above method can be used for evaluating the Laplace transform of the duration of a busy period of the fluid reservoir. As a matter of fact, following the papers by Asmussen [25], it can be shown that the law of the busy period can be related to the occupation level of a dual reservoir. This point will be addressed in further studies.

References