On the explicit calculation of overflow moments of the high priority traffic in loss systems with selective trunk reservation

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Abstract: In this paper, systems with trunk reservation are investigated. A (primary) group with two offered traffic streams is considered. For the high priority traffic the trunk group acts as a group of full accessibility. Calls of the low priority traffic can only be switched if more than a number of \( n_{\text{res}} \) trunks (\( n_{\text{res}} \) trunks are reserved for the high priority traffic) are idle. The objective of the paper is the calculation of the overflow moments (in terms of factorial moments) of the high priority overflow traffic. For the factorial moments of the high priority overflow traffic, and in particular for the variance, explicit formulae are presented.

Keywords: trunk reservation, overflow traffic, higher order moments, variance.

1. INTRODUCTION

In recent times, systems with selective trunk reservation (STR) have found a wide spread interest [1], e.g., in mobile communication systems. In the system considered here, a high priority traffic and a low priority traffic are offered to a common trunk group. For the high priority traffic, the trunk group has full accessibility, i.e., calls can be switched if there is at least one idle trunk. Calls of the low priority traffic can, however, only be switched if there are more than \( n_{\text{res}} \) idle trunks. \( n_{\text{res}} > 0 \) can be considered as the number of reserved trunks, by which the high priority traffic is protected against the low priority traffic. The traffic streams overflowing behind the common trunk group may be offered to further (e.g., secondary) trunk groups. For the loss calculations in such secondary trunk groups it is convenient to apply, e.g., two moment methods, taking into account the variance as well as the means of overflow traffic streams. In more sophisticated investigations, higher order moments (in particular the third moment) may also be of interest. For such applications an explicit solution for the factorial moments of arbitrary order \( r \) of the high priority overflow traffic is presented in this paper. In section 2 the configuration of the considered system is indicated. The equations of state are established in section 3. Then in section 4 the higher order moments are calculated. A conclusion follows in section 5.

2. SYSTEM CONFIGURATION

Such systems with STR have been investigated by Mallin and Sanders [1]. Among other results, they present explicit formulae for the binomial moments of arbitrary order of the high priority overflow traffic for the special case \( n_{\text{res}} = 1 \), i.e., for the case that only one trunk is reserved, and an exact, recursive method for higher (arbitrary) values of \( n_{\text{res}} \).
In the present paper an explicit calculation of higher order moments of the high priority overflow traffic in the general case \( n_{\text{res}} > 0 \) is presented. The higher order moments are calculated in terms of factorial moments. From the second factorial moment, an explicit formula of the variance of the high priority overflow traffic in the general case \( (n_{\text{res}} > 0) \) follows. Further literature on STR systems can be found in [1].

In the system configuration considered here, a high priority traffic of mean \( A_H \) and a low priority traffic of mean \( A_L \) are offered to a common trunk group consisting of \( n \) trunks. Both offered traffic streams are considered to be of Poisson type, i.e. the interarrival times of arriving calls are negative exponentially distributed with the mean values \( \lambda_H \) and \( \lambda_L \) of the arrival rates, respectively, and the holding times have also a negative exponential distribution with the mean \( h \) (for both traffic streams). Calls of the high priority traffic which cannot be switched in the trunk group are overflowing to a secondary trunk group with an infinite number of trunks, \( n_2 \to \infty \). The mean of this overflow traffic is denoted as \( R_H \) or simply as \( R \) (the other overflow traffic consisting of blocked calls of the low priority traffic is not considered in this paper.) The number of busy trunks in the common trunk group is denoted by \( x_1 \) and the number of busy trunks in the infinite secondary groups by \( x_2 \). The mean values \( A_H \) and \( A_L \) of the offered traffic streams are

\[
A_H = \lambda_H \cdot h, \quad (1a) \quad A_L = \lambda_L \cdot h. \quad (1b)
\]

For the total offered traffic \( A \) holds

\[
A = A_H + A_L. \quad (1c)
\]

The number of trunks which are accessible for the low priority traffic be denoted as \( m \), where

\[
m = n - n_{\text{res}}. \quad (2)
\]

3. **EQUATIONS OF STATE**

In the general case, the states can be subdivided in four domains. Domain I for \( x_1 < m \), domain II for \( x_1 = m \), domain III for \( m < x_1 < n \) and domain IV for \( x_1 = n \).

The probability that \( x_1 \) trunks are busy in the primary group be denoted as \( p_1(x_1) \), and the probability that \( x_2 \) trunks are busy in the secondary group by \( p_2(x_2) \). The probability of a state with \( x_1 \) busy trunks in the common group and \( x_2 \) busy trunks in the infinite secondary group be denoted by \( p(x_1, x_2) \). Then the following relations hold true

\[
p_1(x_1) = \sum_{x_2=0}^{\infty} p(x_1, x_2), \quad (3a) \quad p_2(x_2) = \sum_{x_1=0}^{n} p(x_1, x_2). \quad (3b)
\]

It is convenient for a suitable notation to define

\[
p(x_1, x_2) = 0 \quad \text{for} \ x_1 < 0 \text{ or } x_2 < 0 \text{ or } x_1 > n. \quad (3c)
\]

With these notations the following equations of state are obtained [1].

**Domain I (\( x_1 < m \)):**

\[
(x_1 + x_2 + A)p(x_1, x_2) = Ap(x_1 - 1, x_2) + (x_1 + 1)p(x_1 + 1, x_2) + (x_2 + 1)p(x_1, x_2 + 1). \quad (4a)
\]

**Domain II (\( x_1 = m \)):**

\[
(x_1 + x_2 + A_H)p(x_1, x_2) = Ap(x_1 - 1, x_2) + (x_1 + 1)p(x_1 + 1, x_2) + (x_2 + 1)p(x_1, x_2 + 1). \quad (4b)
\]

**Domain III (\( m < x_1 < n \)):**

\[
(x_1 + x_2 + A_H)p(x_1, x_2) = A_Hp(x_1 - 1, x_2) + (x_1 + 1)p(x_1 + 1, x_2) + (x_2 + 1)p(x_1, x_2 + 1). \quad (4c)
\]

**Domain IV (\( x_1 = n \)):**
\[(x_1 + x_2 + AH) p(x_1, x_2) = AH p(x_1 - 1, x_2) + AH p(x_1, x_2 - 1) + (x_2 + 1) p(x_1, x_2 + 1). \tag{4d}\]

Furthermore, the normalizing condition holds true
\[\sum p(x_1, x_2) = 1. \tag{4e}\]

In addition to the set of equations (4a – d) the following global (or, generalized) state equation holds true
\[AH p(n, x_2) = (x_2 + 1) \cdot p_2(x_2 + 1). \tag{5}\]

This global state equation can also be obtained by summing up equations (4a – d) for all values of \(x_1\) and for the values \(x_2 = 0, \ldots, \zeta\) and then replacing \(\zeta\) by \(x_2\).

4. **CALCULATION OF THE MOMENTS**

4.1 Equations for the moments

The ordinary moments \(m_r (x_2)\) of order \(r\) and the factorial moments \(M_r (x_2 | x_1)\) of the overflow traffic \(R\) are defined as
\[m_r (x_2) = \sum_{x_2=0}^{\infty} x_2^r p_2(x_2), \tag{6}\]
\[M_r (x_2) = \sum_{x_2=0}^{\infty} \left(\frac{x_2}{r}\right) r! p_2(x_2). \tag{7}\]

For \(r = 0\) and \(r = 1\), respectively, holds
\[m_0 (x_2) = M_0 (x_2) = 1, \tag{8}\]
\[m_1 (x_2) = M_1 (x_2) = R. \tag{9}\]

The central moments \(\mu_r (x_2)\) and the variance are defined as
\[\mu_r (x_2) = \sum_{x_2=0}^{\infty} [x_2 - m_1 (x_2)]^r p_2(x_2), \tag{10}\]
\[V = \mu_r (x_2) = \sum_{x_2=0}^{\infty} [x_2 - R]^2 p_2(x_2). \tag{11}\]

For the conditional factorial moments of order \(r\) holds
\[M_r (x_2 | x_1) = \sum_{x_2=0}^{\infty} \left(\frac{x_2}{r}\right) r! p(x_1, x_2), \tag{12}\]
\[M_r (x_2) = \sum_{x_1=0}^{\infty} M_r (x_2 | x_1). \tag{13}\]

Multiplying equation (5) by the factor \(\left(\frac{x_2}{r}\right) r!\) and summing up for all values \(x_2 \geq 0\) and regarding the equations (12), (13) and (14) leads (after the rearrangement of terms) to the relations (15) and (16) [6]
\[(x_2 + 1) \left(\frac{x_2}{r+1}\right) r! = \left(\frac{x_2+1}{r+1}\right) (r+1)!, \tag{14}\]
\[M_{r+1} (x_2) = AH M_r (x_2 | n), \quad r \geq 0, \tag{15}\]
\[M_r (x_2) = AH M_{r-1} (x_2 | n), \quad r \geq 1. \tag{16}\]

In equation (16), the factorial moment \(M_r (x_2)\) is expressed as a function of the conditional factorial moment \(M_{r-1} (x_2 | n_1)\). Next, equations for the conditional factorial moments have therefore to be established.
4.2 The conditional moments

**Domain I:** Now equation (4a) is multiplied by the factor $\frac{x^2}{r^r}$ and summed up for all values of $x_2 \geq 0$ as shown above. After rearrangement of terms the following equation is obtained

\[(x_1 + A + r)M_r(x_2 \mid x_1) = (x_1 + 1)M_r(x_2 \mid x_1 + 1) + AM_r(x_2 \mid x_1 - 1), \quad 0 \leq x_1 < m, \quad (17)\]

where of course $M_r(x_2 \mid -1)$ is equal to zero. According to Brockmeyer [2], this difference equation (17) for the conditional factorial moments $M_r(x_2 \mid x_1)$ has the solution

\[M_r(x_2 \mid x_1) = M_r(x_2 \mid 0)S_r, x_1(A), \quad 0 \leq x_1 \leq m \quad (18)\]

where

\[S_r, x_1(A) = \sum_{v=0}^{\infty} \frac{A^{x_1 - v}}{(x_1 - v)!} \binom{r - 1 + v}{v}. \quad (19)\]

The S-polynomials according to equation (19) fulfil the conditions [2]

\[S_r, x = S_{r-1}, x + S_r, x-1, \quad (20) \quad \sum_{\xi=0}^{\infty} S_r, \xi = \sum_{v=0}^{\infty} \frac{A^{x_1 - v}}{(x_1 - v)!} \binom{r - 1 + v}{v}. \quad (21)\]

\[S_r, 0 = 1, \quad (22a) \quad S_0, x = \frac{A^x}{x!}. \quad (22b)\]

(The value $A$ is omitted here for the sake of simplicity.) These S-polynomials can be evaluated easily by means of successive additions according to equation (20), starting with the S-polynomials for $r = 0$ and $m = 0$, respectively [2]. Equation (18) yields for $x_1 = m - 1$ and $x_1 = m$, respectively

\[M_r(x_2 \mid m-1) = M_r(x_2 \mid 0)S_r, m-1(A), \quad (23a) \quad M_r(x_2 \mid m) = M_r(x_2 \mid 0)S_r, m(A). \quad (23b)\]

(For $x_1 = m - 1$, equation (17) yields an equation for the calculation of the moment $M_r(x_2 \mid m)$ from the values $M_r(x_2 \mid m - 1)$ and $M_r(x_2 \mid m - 2)$. Therefore equations (18) and (23b) are also valid for $x_1 = m$).

**Domain II:** Arguing along the same lines as in case of domain I, equation (4b) for $x_1 = m$ leads to the equation

\[(m + A_H + r)M_r(x_2 \mid m) = (m + 1)M_r(x_2 \mid m + 1) + AM_r(x_2 \mid m - 1) \quad (24)\]

and with equation (23a,b) to the equation

\[M_r(x_2 \mid m+1) = \frac{1}{m+1}[(m + A_H + r)S_r, m(A) - AS_r, m-1(A)]M_r(x_2 \mid 0). \quad (25)\]

**Domain III:** Analogously, equation (4c) leads to the following equation if the conditional factorial moments of domain III are denoted by $M^*_r(x_2 \mid x_1)$

\[(x_1 + A_H + r)M^*_r(x_2 \mid x_1) = (x_1 + 1)M^*_r(x_2 \mid x_1 + 1) + A_H M^*_r(x_2 \mid x_1 - 1), \quad m < x_1 < n, \quad (26a)\]

\[M^*_r(x_2 \mid x_1) = M_r(x_2 \mid x_1), \quad m \leq x \leq n. \quad (26b)\]
For \( x_1 \geq m \), the values \( M_r^*(x_2 \mid x_1) \) are identical to the values \( M_r(x_2 \mid x_1) \). For \( x_1 < m \), the values \( M_r^*(x_2 \mid x_1) \) represent hypothetical values, which, however, can also be calculated according to equation (26a). Then it can be seen that, despite the fact that the conditional moments \( M_r(x_2 \mid -1) \) of domain I and \( M_0^*(x_2 \mid -1) \) are equal to zero, the hypothetical values \( M_r^*(x_2 \mid -1) \) for \( r > 0 \) are not equal to zero. Therefore the S-polynomials according to equation (19) are not a solution of equation (26a) for \( r > 0 \). A solution can, however, be found by means of generating functions, as is shown in the appendix. According to the appendix, the solution for equation (26a) is equation (41a) where the values \( U_r, x_1 \) are given by equation (41b) in the appendix. The values \( k_r \) and \( M_r^*(x_2 \mid -1) \) must be determined such that the conditional moments for \( x_1 = m \) and \( x_1 = m + 1 \) in equation (41a) are in accordance with the equations (23b and 25). This leads to the equations

\[
k_r S_r, m(A_H) + M_r^*(x_2 \mid -1) U_{r, m} = M_r(x_2 \mid m),
\]

(42)

\[
k_r S_r, m+1(A_H) + M_r^*(x_2 \mid -1) U_{r, m+1} = M_r(x_2 \mid m+1).
\]

(43)

For a more convenient notation it is suitable to normalize all moments as well as the constants \( k_r \) with respect to the value \( M_r(x_2 \mid 0) \). The normalized values be denoted by \( L, K_{1, r} \) and \( K_{2, r} \) as follows:

\[
L_r, x_1 = M_r^*(x_2 \mid x_1) / M_r(x_2 \mid 0),
\]

(44a)

\[
L_r, x_1 = M_r(x_2 \mid x_1) / M_r(x_2 \mid 0),
\]

(44b)

\[
L_r, x_1 = M_r^*(x_2 \mid x_1) / M_r(x_2 \mid 0),
\]

(44c)

\[
L_r, x_1 = L_r, x_1, m \leq x_1 \leq n ,
\]

(44d)

\[
K_{1, r} = k_r / M_r(x_2 \mid 0),
\]

(44e)

\[
K_{2, r} = L_r^*(-1)
\]

(44f)

with (according to equations (13),(19) and (25))

\[
L_r, x_1 = S_r, x_1(A), x_1 \leq m,
\]

(44g)

\[
L_r = \sum_{x_1=0}^{m} L_{r, x_1} + \sum_{x_1=m+1}^{n} L^*_{r, x_1},
\]

(44h)

\[
L_{r, m+1} = \frac{1}{m+1} \left[ (m + A_H + r) S_{r, m}(A) - A S_{r, m-1}(A) \right].
\]

(44i)

With these notations equations (42) and (43) have the form

\[
K_{1, r} S_{r, m}(A_H) + K_{2, r} U_{r, m} = L_{r, m},
\]

(45)

\[
K_{1, r} S_{r, m+1}(A_H) + K_{2, r} U_{r, m+1} = L_{r, m+1}.
\]

(46)

The solution of this set of two linear equations yields

\[
K_{1, r} = \frac{U_{r, m+1} L_{r, m} - U_{r, m} L_{r, m+1}}{U_{r, m+1} S_{r, m}(A_H) - U_{r, m} S_{r, m+1}(A_H)}, \quad r > 0
\]

(47a)

\[
K_{2, r} = \frac{S_{r, m}(A_H) L_{r, m+1} - S_{r, m+1}(A_H) L_{r, m}}{U_{r, m+1} S_{r, m}(A_H) - U_{r, m} S_{r, m+1}(A_H)}, \quad r > 0
\]

(47b)
It can easily be shown that equation (49) also holds true for \( r = 0 \) with the constants
\[
K_{1,0} = \frac{S_{0,m}(A)}{S_{0,m}(AH)} = \frac{A^m}{A_H^m}, \quad \text{(48a)} \quad K_{2,0} = 0. \quad \text{(48b)}
\]

Now all normalized conditional moments \( L_{r,x_1} \) and \( L_{r,x_1}^* \) can be calculated according to equations (44b) and (49). As equation (49) is used only for \( m \leq x_1 \leq n \), the asterisk can be omitted
\[
L_{r,x_1} = L_{r,x_1}^* = K_{1,r} \cdot S_{r,x_1}(AH) + K_{2,r} U_{r,x_1}, \quad m \leq x_1 \leq n. \quad \text{(49)}
\]

**Domain IV:** The equations referring to domain IV are not used for the calculation of the conditional moments and are therefore not considered here.

### 4.3 The moments \( M_r(x_2) \)

If the primary group with \( n \) trunks is considered separately, it is very easy to establish a formula for the probability \( p_1(n) \) that all \( n \) trunks are busy, in analogy to the well-known formula by Erlang (Erlang-B formula)[5]. One obtains
\[
p_1(n) = M_0^*(x_2 \mid n) = M_0(x_2 \mid n) = \frac{A^m A_H^{n-m}}{n! \sum_{i=0}^{m} A^i/i! + \sum_{i=m+1}^{n} A^m A_H^{i-m}/i!}.
\]

This value can be used as a basis for the following calculations. By dividing equations (44c) and (44a), the following equation is obtained with equation (44d)
\[
M_r(x_2 \mid n) = M_r(x_2) \frac{L_{r,n}}{L_r}.
\]

Taking into account equation (15) yields
\[
M_{r+1}(x_2) = A_H \cdot \frac{L_{r-1,n}}{L_{r-1}} M_r(x_2) \quad \text{(52a)}
\]

or
\[
M_r(x_2) = A_H \cdot \frac{L_{r-1,n}}{L_{r-1}} M_{r-1}(x_2), \quad r \geq 1. \quad \text{(52b)}
\]

From equations (8) and (50) follows
\[
\frac{L_{0,n}}{L_0} = p_1(n). \quad \text{(53)}
\]

For all other values \( r > 1 \) the terms \( L_r \) and \( L_{r,n} \) are known according to the equations (44fg,i), and (49). From equations (8) and (52b), the following formulae for the moments are obtained directly
These equations (54) and (55) can easily be generalized. Then the following equation for the factorial moments is obtained, which can easily be proved by induction

\[ M_r(x_2) = A_H^r \cdot \prod_{i=0}^{r-1} \frac{L_{i,n}}{L_i}, \quad r \geq 1. \]  

Equation (56) represents an explicit formula for the factorial moments of the overflow traffic of arbitrary order \( r \).

4.4 Special cases and example

**Special case \( r = 1 \)**

According to equations (50) and (52b) holds

\[ M_1(x_2) = R = A_H p_1(n) \]  

with \( p_1(n) \) according to equation (50).

**Special case \( r = 2 \)**

According to equation (16) holds

\[ M_2(x_2) = A_H M_1(x_2 | n). \]

Therefore the conditional moments for the order \( r = 1 \) have to be considered. For \( r = 1 \), equation (41c) yields for all values of \( x_1 \)

\[ U_{r,x_1} = 1. \]

With this equation and equation (20), equations (47a,b) can be simplified as follows:

\[ K_{1,1} = \frac{A_H S_{1,m}(A) - A S_{1,m-1}(A)}{(m+1) S_{0,m+1}(A_H)}, \]  

\[ K_{2,1} = S_{1,m}(A) - K_{1,1} S_{1,m}(A_H). \]

The values \( L_{1,x_1} \) and \( L_1 \) are obtained according to equations (44d,g,h) and (49) as

\[ L_{1,x_1} = S_{1,x_1}(A), \quad x_1 \leq m, \]  

\[ L_1 = \sum_{x_1=0}^{n} L_{1,x_1}. \]

Now the second factorial moment can be calculated according to equation (55).

**Variance of the high priority overflow traffic \( R \).**

If the equations (7) for \( r = 2 \) and equation (11) are written out in full and combined, then the following explicit formula for the variance of high priority overflow traffic is found

\[ V = M_2(x_2) + R - R^2. \]
Special case $n_{res} = 1$

In the case $n_{res} = 1$, i.e. $m = n - 1$, corresponding to [1], the domain III does not exist. The values $L_1, x_1$ can be determined according to equations (44 g,i), so that the calculation of the moments is much easier than in the general case with $n_{res} > 1$.

Numerical example

In the sequel, a very simple example is considered with $A = 12$ Erlangs, $A_H = 6$ Erlangs, $n = 4$ trunks and $n_{res} = 2$, i.e. $m = 2$. These parameters have been chosen such that the results can easily be evaluated and checked without the aid of a computer. For this example, the following values are obtained

$$p_1(n) = \frac{216}{445} = 0.485393; \quad L_{1,0} = 1, \quad L_{1,1} = 13, \quad L_{1,2} = 85, \quad L_{1,3} = 203,$$

$$K_{1,1} = \frac{59}{18} = 3.2777... \quad K_{2,1} = \frac{55}{18} = 3.0555..., \quad L_{1,4} = 380, \quad L_4 = 682,$$

$$M_1(x_2) = R = \frac{296}{445} = 2.912360 \text{ Erlargs,} \quad M_2(x_2) = \frac{295488}{30349} = 9.736344, \quad V = 4.1689.$$ 

Higher order moments

In the calculation of higher order moments, it may be easier to determine the relative conditional moments $M^*_r(x_2 \mid x_1)$ or, respectively, $L_{r,x_1}$ recursively according to equations (26a) and (44c) instead of using the explicit formulae presented here, particularly if the number $n_{res}$ of reserved trunks is small.

5. CONCLUSION

This paper deals with overflow moments of the high priority traffic in loss systems with selective trunk reservation. After establishing the equations of state, these equations are transformed into equations of the factorial moments. With the aid of S-polynomials according to [2] and generating functions these equations can be solved. Finally an explicit formulae for the factorial moments of order $r$ of the high priority of overflow traffic are obtained, and in particular an explicit formulae for the variance of this overflow traffic, which can, e.g., be used in the applications of a two moments method [3]. Explicit formulae for the overflow moments of the low priority traffic and of the total traffic remain as subjects for possible future investigations.

REFERENCES

APPENDIX

In this appendix a solution for the difference equation (26a) is derived for \( r \geq 1 \). For the sake of simplicity, the notation of some valuables is abbreviated as follows:

\[
x_1 \Rightarrow x, \quad A_H \Rightarrow A, \quad M_r^* (x_2 \mid x_1) \Rightarrow M(x).
\] (27)

Then equation (26a) reads (after rearrangement of terms)

\[
(x + 1)M(x + 1) = (x + A + r)M(x) - AM(x - 1).
\] (28)

Now the generating function

\[
G[M(x)] = \sum_{x=0}^{\infty} M(x)t^x
\] (29)

is introduced, for which the following well-known formulae are valid

\[
G[M(x - 1)] = t \cdot G[M(x)] - M(-1), \quad G[xM(x)] = t \cdot G'[M(x)],
\] (30a)

\[
G[(x + 1)M(x + 1)] = G'[M(x)], \quad \frac{d}{dt} G[M(x)] = G'[M(x)].
\] (30b)

Multiplying equation (28) by \( tx \) and summing up for all values of \( x \), regarding equations (29) and (30) transforms equation (28) into an equation for the generating function:

\[
(1 - t)G' + (A + r)G - A[G - M(-1)]
\] (31)

where

\[
G[M(x)] \Rightarrow G
\] (32)

or

\[
(1 - t)G' = A(1 - t)G + rG - AM(-1).
\] (33)

The corresponding homogenous differential equation (i.e., without the term \( AM(-1) \) has according to [3] the solution

\[
G = ce^{At} \frac{1}{(1-t)^r}.
\]

A solution for the inhomogeneous equation (33) can be obtained by means of a variation of the constant \( c \):

\[
G = c(t)e^{At} \frac{1}{(1-t)^r}, \quad G' = c(t) \left[ \frac{Ae^{At}}{(1-t)^r} + e^{At} \frac{r}{(1-t)^{r+1}} \right] + c'(t)e^{At} \frac{1}{(1-t)^r}, \quad r > 0.
\] (34)

If this expression \( G \) and its derivative \( G' \) are inserted in equation (33), the following equation for \( c'(t) \) is obtained

\[
c'(t) = -Ae^{-At} (1-t)^{r-1} M(-1), \quad r > 0.
\] (35)

The integration of equation (35) yields
\begin{align*}
c(t) &= k_r + M(-1)e^{-At} \sum_{\eta=0}^{r-1} \frac{(-1)^\eta \cdot (r-1)!}{\eta! \cdot (r-1-\eta)!} (1-t)^{r-1-\eta} \\
which can be proved by induction. Inserting equations (30) and equation (36) in equation (34) leads to the following formula for the generating function, if \( \eta \) is replaced by \( \xi - 1 \):
\begin{align*}
G \left[ M_r^*(x_2 \mid x_1) \right] &= k_r e^{At} \frac{1}{(1-t)^r} + \sum_{\xi=1}^{r} M(-1) \frac{(-1)^{\xi-1} \cdot (r-1)!}{A^{\xi-1} \cdot (r-\xi)!} \cdot \frac{1}{(1-t)^{\xi}}.
\end{align*}
(37)

By a re-transformation the values of the moments can be determined from the generating function (37). For the first term (with \( k_r \)) in equation (37) the result of the re-transformation is well-known [2,3]:
\begin{align*}
k_r \cdot S_{r,x_1}.
\end{align*}
(38)
The re-transformation of the terms in the summation can be obtained by development into a power series. It holds [4]
\begin{align*}
\frac{1}{(1-t)^{\xi}} &= \sum_{x=0}^{\infty} \left( \frac{x + \xi - 1}{\xi - 1} \right) t^x.
\end{align*}
(39)
The application of this formula to equation (37) leads to the following generating function for a term in this summation with the value \( \xi \):
\begin{align*}
G_\xi &= \sum_{x=0}^{\infty} M(-1) \frac{(-1)^{\xi-1}}{A^{\xi-1}} \cdot \frac{(r-1)!}{(r-\xi)!} \cdot \left( \frac{x + \xi - 1}{\xi - 1} \right) t^x.
\end{align*}
(40a)
According to equation (29) holds
\begin{align*}
G_\xi &= \sum_{x=0}^{\infty} M(x) t^x.
\end{align*}
(40b)
A comparison of equations (40a) and (40b) and regarding the term (38) yields the following formula for the moments \( M(x) \)
\begin{align*}
M(x) &= k_r S_{r,x} + M(-1) \sum_{\xi=1}^{r} \frac{(-1)^{\xi-1}}{A^{\xi-1} H} \cdot \frac{(r-1)!}{(r-\xi)!} \left( \frac{x_1 + \xi - 1}{\xi - 1} \right).
\end{align*}
(Returning to the original notations, one obtains
\begin{align*}
M_r^*(x_2 \mid x_1) &= k_r S_{r,x_1} (A_H) + M_r^*(x_2 \mid -1) \cdot U_{r,x_1}, \ r > 0, \tag{41a}
\end{align*}
where
\begin{align*}
U_{r,x_1} &= \sum_{\xi=1}^{r} \frac{(-1)^{\xi-1}}{A^{\xi-1} H} \cdot \frac{(r-1)!}{(r-\xi)!} \left( \frac{x_1 + \xi - 1}{\xi - 1} \right), \ r > 0. \tag{41b}
\end{align*}
Examples:
\begin{align*}
U_{1,x_1} &= 1, \quad U_{2,x_1} = 1 - \frac{x_1 + 1}{A_H}, \quad U_{3,x_1} = 1 - \frac{2(x + 1)}{A_H} + \frac{(x + 2)(x + 1)}{A_H^2}. \tag{41c, d, e}
\end{align*}