Connectivity Dynamics for Vehicular Ad-hoc Networks in Signalized Road Systems

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Abstract – In this paper, we analyze the connectivity dynamics of vehicular ad-hoc networks in a generic signalized urban route. Given the velocity profile of an urban route as a function of space and time, we utilize a fluid model to characterize the general vehicular traffic flow, and a stochastic model to capture the randomness of individual vehicle. From the fluid and stochastic models, we can acquire respectively the densities of the mean number of vehicles along the road and the corresponding distribution. With the knowledge of the vehicular density dynamics, we determine the probability that the communication network is fully connected, i.e., each node can communicate with every other node through a multi-hop path, and the problem is also investigated for a general case of a *k*-connected network. To closely approximate the practical road conditions, we use a density-dependent velocity profile to approximate vehicle interactions and capture the shockwave propagation at traffic signals. We confirm the accuracy of the connectivity analysis through simulations and show that the analytical results are good approximations even when vehicles interact with each other as their movement is controlled by traffic lights. We also illustrate that system engineering and planning for optimizing connectivity in the communication networks can be carried out with the results in this paper.

Index Terms – Vehicular Ad-Hoc Network, Connectivity Dynamics, Stochastic Traffic Model, Signalized Road System, Vehicle Interaction, Transmission Range Adjustment.

I. INTRODUCTION

In a Vehicular Ad-hoc Network (VANET), vehicles communicate with each other and possibly with road-side infrastructure nodes. Node connectivity and the amount of data that can be exchanged are limited by the duration and quality of the communication links established among nodes, which are determined by the space and time dynamics of moving vehicles.

To capture such dynamics in our connectivity analysis, we modify the stochastic traffic model proposed in [1,2] for modeling vehicular traffic in signalized urban road systems. The stochastic traffic model uses a deterministic fluid model to characterize the space and time dynamics of vehicle movements, which is driven by a velocity profile as a function of space and time. In real practice, empirical velocity measurements from GPS devices can serve as an input to the model. The mean density profile, again as a function of space and time, is readily computable from the conservation equations in the fluid model. The randomness of individual vehicle is captured by the Poisson Arrival Location Model (PALM). The actual number of vehicles in a given road section at a certain time instance has Poisson distribution according to previous PALM results in [1,3] given that the arrival of vehicles follow a non-homogeneous Poisson process and the velocity profile is independent of other parameters such as vehicular density.

To closely represent the practical road condition, we attempt to introduce a density-dependent velocity profile to characterize vehicle interactions and shockwave propagation at traffic signals, which has not been considered in previous PALM work [1,2]. But of course, the PALM no longer applies in principle with vehicle interactions due to the existence of dependency. So, we treat the results in the presence of vehicle interactions as approximations, and validate them through simulations in this paper.

Through the vehicular density dynamics computed from the stochastic traffic model, we determine the distribution of a node's location on the urban route, and derive the probability that the entire communication network in the road segment is connected in a multi-hop manner, and the problem is further investigated for a general case of a *k*-connected network. In essence, we show that the analysis is a valid approximation even when extra randomness is inherited from vehicular interactions and traffic signals. To illustrate the applicability of our results, we demonstrate how the connectivity knowledge can facilitate the adjustment of transmission ranges of vehicles to prevent network disconnection and boost the overall communication connectivity.

Related Work

There are a number of studies on node connectivity in Mobile Ad-hoc Networks (MANETs). For instance, [4] shows that if the radio transmission range of n nodes that are placed uniformly and independently in a disc of unit area is set to $r_0 = [(\log\ n + c(n)) / n\pi]^{1/2}$, the resulting wireless multi-hop network is asymptotically connected with probability one if and only if $c(n) \rightarrow \infty$. Reference [5] investigates the radio range assignment problem, and obtains bounds for the probability that a node is isolated and the network is connected on a one-dimensional line. On the other hand, [6] examines the node density threshold for achieving full connectivity in both 1-D and 2-D ad-hoc network. [7,8] study the relation between the minimum node degree and *k*-connectivity in a random graph, and explore the minimum radio transmission range r_0 for achieving a fully connected ad-hoc network for a given node density.

Most of the existing studies assume that nodes are uniformly random distributed in an area and they are either stationary or move according to the random waypoint model

[8], which are obviously inadequate to capture the spatial distribution of vehicles and their movements. In fact, vehicle movements, particularly in urban environments, are restricted by the road topologies, buildings, etc., and affected by traffic density, which is determined by road capacity, traffic control and driver behaviors. There are also recent works that aim to model connectivity of vehicles on a one-dimensional highway. [9] assumes the space headway between vehicles is exponentially distributed, and introduces a robustness factor to capture the effect of disturbance on VANET connectivity. [10] assumes a continuous-time mobility model where movement of each node is a function of time consisting of a sequence of random intervals that is exponential distributed, and during each interval, a node moves at a constant speed which is independently chosen from a normal distribution. Based on this synthetic mobility model, the author derived the mean cluster size and the probability that the nodes will form a single cluster under the assumption that vehicles arrive according to a Poisson process. These previous studies, however, lack of a realistic traffic model to adequately capture node density and its influence on vehicle speed, which are significant in determining connectivity, especially in urban road systems where strong interactions among neighboring vehicles exist.

The rest of the paper is organized as follows. Section II provides the background information of the stochastic traffic model (fluid and stochastic models) for computing vehicular density. Section III presents the connectivity analysis with and without the consideration of vehicle interactions. Section IV extends the investigation to the general case of *k*-connectivity in the network. Section V provides numerical results to verify our analysis. Section VI presents potential applications of the connectivity modeling on transmission range adjustment for boosting overall network connectivity. Finally, Section VII concludes the paper.

II. STOCHASTIC TRAFFIC MODEL

In this section, we define the system model and provide background information of the fluid and stochastic models, from which we can acquire the vehicular density dynamics in signalized urban routes for VANET connectivity analysis.

We consider traffic in a one-way, single-lane, semi-infinite signalized urban road (or route) as shown in Figure 1. Although the road is fed with traffic from adjacent streets, the one-way road under consideration is the one running from the left to the right in the figure. More complicated road topology can be represented by superposing multiple versions of urban routes. Let our location space to be the interval $[0, \infty)$, the boundary point 0 is the spatial origin, and it marks the starting point of the road. The arrival process $\{A(t) \mid -\infty < t < \infty\}$ counts the number of arrivals to the first segment of the route up to time t, which we assume is finite with probability 1, and is characterized by a non-negative and integrable external arrival rate function $\alpha(t)$.

Furthermore, the route consists of a number of road segments indexed by i = 1, 2, 3,…, and traffic lights are located at the junctions of road segments, where vehicles can leave and join the route. We let the location of the i-th junction (or traffic light) between road segments i and i+1 be x_i.

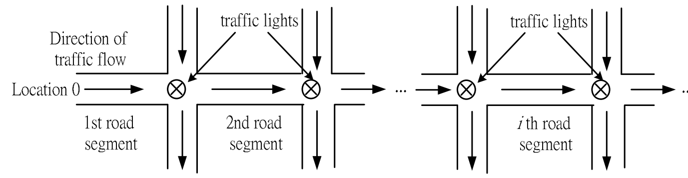


Figure 1. The road configuration considered in this paper.

A. Deterministic Fluid Model

The fluid model is a kind of continuum traffic flow models, which reduces laws of traffic to a partial differential equation (PDE) that may be studied as elegantly and simply as other physical phenomena that are also governed by PDE's.

The major difference between our fluid model and other continuum models is that we model vehicle motions with a velocity profile, vehicles at location x and time t move forward the route according to a velocity field $v(x, t)$, it can be deterministic or density dependent. Vehicles stop at road junctions for a red signal, which can be reflected and modeled by the velocity profile. However, continuum model alone is unable to capture traffic instability and the randomness of individual vehicle, therefore, we couple the fluid model with the stochastic model in the next section as a remedy.

To begin with, we will describe the fluid dynamic conservation equations and corresponding notations that hold for the general systems. Let $N(x, t)$ be the number of vehicles in location $(0, x]$ at time t, and $n(x, t)$ be the density of vehicles in location $(0, x]$ at time t. Thus,

$$n(x,t) = \frac{\partial N(x,t)}{\partial x}. \tag{1}$$

Let $Q(x, t)$ be the number of vehicles moving past position x before time t. Then the flow rate $q(x, t)$ is defined by

$$q(x,t) = \frac{\partial Q(x,t)}{\partial x}. \tag{2}$$

Let $C^+(x, t)$ and $C^-(x, t)$ be the number of vehicles arriving to and departing from the route in location $(0, x]$ during time interval $(-\infty, t]$, respectively. Then the associated rate densities are respectively

$$c^+(x,t) = \frac{\partial^2 C^+(x,t)}{\partial x \partial t} \text{ and } c^-(x,t) = \frac{\partial^2 C^-(x,t)}{\partial x \partial t}. \tag{3}$$

Assuming all traffic moves only from left to right down the positive real line, then the four variables N, Q, C^+, C^- satisfy the following conservation relation:

$$C^+(x,t) = N(x,t) + Q(x,t) + C^-(x,t) \tag{4}$$

By applying the operator $\partial^2/(\partial x \partial t)$ to (4), we have the partial differential equation

$$\frac{\partial n(x,t)}{\partial t} + \frac{\partial q(x,t)}{\partial x} = c^+(x,t) - c^-(x,t). \tag{5}$$

According to traffic flow theory [11], we have

$$q(x,t) = n(x,t)v(x,t). \tag{6}$$

By substituting (6) into (5), we have

$$\frac{\partial n(x,t)}{\partial t} + \frac{\partial \left[n(x,t)v(x,t)\right]}{\partial x} = c^+(x,t) - c^-(x,t). \tag{7}$$

The resulting partial differential equation (7) is a one-dimensional version of the generalized conservation law for fluid motion [12]. This equation governs the mean behavior of any stochastic traffic model. For the ease of solving (7), we introduce an additional assumption to convert the partial differential equation (PDE) to an ordinary differential equation (ODE). Let the location x as a time function, $x(t)$, be given by

$$\frac{dx(t)}{dt} = v(x(t), t). \tag{8}$$

By chain rule, $\frac{dn(x(t),t)}{dt} = \frac{\partial n(x,t)}{\partial x} \cdot \frac{dx(t)}{dt} + \frac{\partial n(x,t)}{\partial t}$, and by substituting (7) and (8) into this equation, we have

$$\frac{dn(x(t),t)}{dt} = c^{+}(x(t),t) - c^{-}(x(t),t) - \frac{\partial v(x,t)}{\partial x} n(x(t),t). \tag{9}$$

Due to the partial derivative of $v(x, t)$ w.r.t. x in (9), it is ODE if and only if $v(x, t)$ is not a function of $n(x, t)$. Note that though we introduce the use of a density-dependent velocity profile later on, we still solve the ODE's for the mean number of vehicles as an approximation in Section V.

We assume that vehicles arrive at the first route segment according to an external arrival rate function $\alpha(t)$. If we consider the fleet of vehicles keep moving along the route without vehicles joining or leaving at junctions (e.g., buses move along a bus route). $c^{+}(x, t) = \alpha(t)\delta(x)$ and $c^{-}(x, t) = 0$, where $\lim_{\varepsilon \to 0} \int_{x-\varepsilon}^{x+\varepsilon} \delta(y)dy = 1$ if $x = 0$ and 0 otherwise.

For the case that there are vehicles join and leave the route at road junctions, let us use $\xi_i(t)$ to denote the external arrival rate of vehicles at the i-th junction at time t. Then we have

$$c^{+}(x,t) = \alpha(t)\delta(x) + \sum_i \xi_i(t)\delta(x - x_i). \tag{10}$$

As for vehicles leaving the route, we use $\rho_i(t)$ to denote the fraction of vehicles departing when they pass by the i-th junction at time t. If these departing vehicles leave at the same velocity as they move forward along the route, then

$c^{-}(x,t) = \beta(x,t)n(x,t)$, where

$$\beta(x,t) = v(x,t)\sum_i \rho_i(t)\delta(x - x_i). \tag{11}$$

B. Stochastic Model

In contrast to the deterministic fluid model, the stochastic model captures the stochastic fluctuations of the quantities of interest. When the two models are coupled with each other to form the stochastic traffic model, the solutions from the PDE's or ODE's describe the expected number of vehicles, and the actual number of vehicles is captured by the additional distribution information from the stochastic model.

From now on, the densities $n(x, t)$ and $q(x, t)$ are defined as the partial derivatives of expected values, that is,

$$n(x,t) = \frac{\partial E[N(x,t)]}{\partial x} \text{ and } q(x,t) = \frac{\partial E[Q(x,t)]}{\partial x}.$$

Similarly, the rate densities $c^{+}(x, t)$ and $c^{-}(x, t)$ are the second partial derivatives of expected values, that is,

$$c^{+}(x,t) = \frac{\partial^2 E[C^{+}(x,t)]}{\partial x \partial t} \text{ and } c^{-}(x,t) = \frac{\partial^2 E[C^{-}(x,t)]}{\partial x \partial t}.$$

The general stochastic model can be of any distributions depending on the arrival process of vehicles, and the equations in the deterministic fluid dynamic model continue to hold regardless the distribution of the stochastic model. In this paper, we specifically consider a Poisson arrival location model (PALM). Again, the fluid dynamic model is not dependent on the Poisson assumption; they hold as long as the arrival process A is an arbitrary point process with time-dependent arrival-rate function α.

With PALM, the arrival process $\{A(t) \mid -\infty < t < \infty\}$ for vehicles to arrive at the first road segment of the route is a *non-homogeneous Poisson process* with non-negative and integrable external arrival rate function $\alpha(t)$. That is, the number of arrivals in the interval $(t_1, t_2]$ is Poisson with mean $\int_{t_1}^{t_2} \alpha(s)ds$.

According to [1,3], we can construct $N(x, t)$, the random number of vehicles within the range $(0, x]$ at time t, via stochastic integration starting with the Poisson process A, where $A(t)$ counts the number of vehicles arriving to the road segment up to time t.

$$N(x,t) = \int_{\sigma(x,t)}^{t} 1_{\{L_s(t)\in(0,x]\}} dA(s) = \sum_{n=A(\sigma(x,t))}^{A(t)} 1_{\{L_{\hat{A}_n}(t)\in(0,x]\}} \tag{12}$$

Hence, for all real t, $\{N(x, t) \mid x \geq 0\}$ is a Poisson process with

$$E[N(x,t)] = \int_{\sigma(x,t)}^{t} \alpha(s)ds. \tag{13}$$

where $\hat{A}_n$ is the nth jump time of A, counting backward from time t. 1_B is an indicator function such that it returns 1 if B is true and 0 otherwise. $L_s(t)$ is the location process, which specifies the position of the vehicle on the road segment at time t that arrived at time s. Let $\sigma(x, t)$ denote the route entrance time for a vehicle to be in position x at time t. For vehicles that arrive to the route before $\sigma(x, t)$, it will be past position x by time t. On the other hand, for vehicles arrive after $\sigma(x, t)$, it will be still in position x at time t.

Therefore, as long as we model the traffic flow or even traffic signals through a deterministic velocity field as a function of space and time, and all the vehicles do not interact with each other, the Poisson distributional conclusion in [1,3] remains valid.

In Section V, we approximate vehicular interactions through a density-dependent velocity profile. In that case, the stochastic model in principle is invalid, and the Poisson property no longer holds as we cannot determine if a car arrives at time s will have passed location x at time t with the density-dependent velocity profile. However, we found from simulations that the Poisson property and stochastic independence of the model can still be primarily retained when the traffic load is not too high even in the presence of vehicle interactions. Thus, we treat the results with consideration of vehicle interactions as approximations, and confirm their accuracy through simulations. The reader is referred to Section V for details.

III. VANET CONNECTIVITY ANALYSIS

With the knowledge of the vehicular density dynamics from the stochastic traffic model, we determine in this section the probability that the network within an urban road segment is connected, and extend the investigation to k-connectivity of the network.

To model wireless transmission between vehicles, a radio link model is assumed in which each vehicle has a transmission range r, two vehicles are able to communicate (or are connected) directly with each other via a wireless link if the Euclidean distance between them is less than or equal to r. In a one-dimensional ad-hoc network, a node is said to be *disconnected from the forward network* if it is not connected to any forward neighbors in the network (e.g., node 2 in Figure 2), where '*forward neighbors'* here represents nodes that are on the right of the considered node assuming that the stream of traffic moves from left to right.

A network is said to be *connected* if for every pair of nodes there exists a path (that is composite of one or more number of communication links) between them, and otherwise it is *disconnected*. For the connectivity of a one-dimensional network, the following definition holds.

Definition 1: For a one-dimensional ad-hoc network, it is connected if and only if there does not exist any nodes in the network that are not connected to any of the forward nodes.

In another words, the network is disconnected if the separation between any two adjacent nodes is great than r as illustrated in Figure 2. Therefore, the probability that the network in a road segment is connected is equivalent to the probability that no nodes got isolated from the forward network. The reader is referred to the proof of Proposition 1, which considers the general case of a k-connected network.

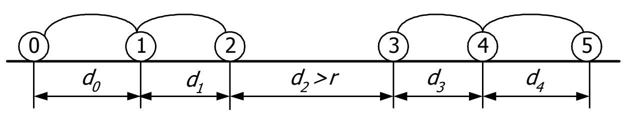


Figure 2. Disconnection in a one-dimensional network.

A. Connectivity without Vehicle Interactions

For analytical simplicity, we cease to focus on time dynamics in this subsection, and assume the system has reached a steady state with respect to time. Thus, all system variables and parameters (including the velocity profile) become independent of time. For this reason, we simply drop the variable t from our previously defined notation.

We define $N(x_a, x_b) = N(x_b) - N(x_a)$, where $x_b > x_a$, as the number of vehicles in the region $(x_a, x_b]$. According to the stochastic model, the actual number of vehicles distributed within a road region is Poisson distributed. Thus, we can acquire from it the probability that a specific number of vehicles are located within a road region, which is given by

$$P\left(N(x)=n\right)=\frac{E[N(x)]^n}{n!}e^{-E[N(x)]} \tag{14}$$

$$P\left(N(x_a,x_b)=n\right)=\frac{E[N(x_a,x_b)]^n}{n!}e^{-E[N(x_a,x_b)]} \tag{15}$$

where $E[N(x)]=\int_0^x n(u)du$.

Consider a road segment of length L, cars arrive at location 0 as a homogeneous Poisson process with mean rate α, and assume there are no cars joining and leaving the urban route at junctions. Since we assume the system has reached a steady state with respect to time, the velocity profile is only a function of space but independent of time. Thus, every car that travels through the road segment will have the same velocity-time graph with regard to its arrival time, let us denote it with $v(s)$. At time t, the physical separation between the i-th and the i+1-st cars is

$$d_{I_i}(t)=\int_{t-I_i}^{t} v(s)ds \tag{16}$$

where I_i is the inter-arrival time between the i-th and i+1-st arrivals. We are interested in finding the *critical inter-arrival time* of the road, T_c, such that if $I_i \le T_c$, the i-th and i+1-st cars will remain connected throughout the whole journey. Or on the other hand, if $I_i > T_c$, the maximum separation between the two cars in the road segment will be greater than r. We have

$$\max_{t\in\Omega}\left\{d_{T_c}(t)\right\}=\max_{t\in\Omega}\left\{\int_{t-T_c}^{t} v(s)ds\right\}=r \tag{17}$$

where Ω is the set of time instances such that the i-th and i+1-st cars co-exist in the road segment. Given the velocity profile, we are able to find T_c as a function of r. For Poisson arrival process with parameter α, $P(I_i \le T_c) = 1 - e^{-\alpha Tc}$, let it be p_c. Therefore, the probability that the entire network remains connected (conditioning on that the population size of the road is non-zero, i.e., $N(L) > 0$) is

$$P(\text{net con})=\frac{1}{1-P(N(L)=0)}\sum_{j=1}^{\infty} p_c^{\,j-1}P(N(L)=j). \tag{18}$$

Since the number of cars in the road segment has a Poisson distribution with parameter $E[N(L)]$, which can be computed from the fluid model, by substituting (14) into (18), we have

$$P(\text{net con})=\frac{\left(e^{p_c E[N(L)]}-1\right)}{p_c\left(e^{E[N(L)]}-1\right)}. \tag{19}$$

B. Connectivity with Vehicle Interactions

Subsection A above gives us an exact treatment of modeling the spatial separations between vehicles and connectivity of the network. Strictly speaking, when vehicle interactions are considered, the stochastic model is no longer valid due to the existence of dependency. So, we can only do approximation here and use simulation to evaluate its validity.

For velocity profile that is a function of both the space and time, and with vehicle interactions, we approximate the probability of connectivity based on the results of the fluid model, which captures the time and space dynamics as well as vehicular interactions. We now consider a specific time instance t_0, and thus the variable t is again simply dropped from our notation. The probability over a period of time can be obtained by taking the time-average of multiple time instances.

Again, we consider a road segment with length L in region $(0, L]$. With the knowledge of the mean density profile $n(x)$ from the fluid model, we can derive the pdf

$$f_L(x)=n(x)/E[N(L)] \tag{20}$$

such that $f_L(x)\Delta x$ represents the probability that a random node in the road segment is located in the small region $(x, x + \Delta x]$.

Given that there are j nodes located in the road segment at a time instance and assume that their locations are independent, we now regard a randomly chosen node located in the road segment. The probability that this node is disconnected from the forward network is given by the weighted sum of the probability that the other $j-1$ nodes are not located within the transmission range in front over all possible locations of the node in the road segment.

$$\begin{aligned} &P(\text{node discon}|\, j \text{ nodes}) \\ &= \int_0^{L-r} P(\text{the other } j-1 \text{ nodes are not in } (x, x+r])\, f_L(x)dx \\ &= \int_0^{L-r} \left(1-\int_x^{x+r} f_L(x)dx\right)^{j-1} f_L(x)dx. \end{aligned} \tag{21}$$

Note that we define a node is not disconnected if it is located in region $(L-r, L]$ in the road segment.

According to [8], the events that an individual node is isolated or disconnected from the forward network are almost independent from node to node with the assumptions that the number of nodes in the road segment $N >> 1$ and $r << L$. Thus, the probability that there are no disconnected nodes in the road segment or the network is connected is

$$\begin{aligned} &P(\text{net con}) = P(\text{no discon node}) \\ &= \sum_{j=0}^{\infty} \left(1-P(\text{node discon}|\, j \text{ nodes})\right)^j P(N(L)=j). \end{aligned} \tag{22}$$

By substituting (14) and (21) into (22), we are able to find this probability based on the input $n(x)$ computed from the fluid model.

When the expected number of cars in the network is large, for the simplicity of calculation, we can approximate the probability above with

$$\begin{aligned} &\tilde{P}(\text{no discon node}) \\ &= \sum_{j=0}^{\infty} \left(1-P(\text{node discon})\right)^j P(N(L)=j) \end{aligned} \tag{23}$$

where

$$\begin{aligned} P(\text{node discon}) &= \int_0^{L-r} P(\text{node discon} \mid x) f_L(x)dx \\ &= \int_0^{L-r} P\left(N(x+\Delta x, x+r)=0\right) f_L(x)dx. \end{aligned}$$

Substitute (15) into it, we have

$$P(\text{node discon}) = \int_0^{L-r} e^{-E[N(x+\Delta x, x+r)]} f_L(x)dx. \tag{24}$$

By substituting (14) and (24) into (23), we have

$$\begin{aligned} &\tilde{P}(\text{net con}) = \tilde{P}(\text{no discon node}) \\ &= \sum_{j=0}^{\infty} \left(1-\int_0^{L-r} e^{-E[N(x+\Delta x, x+r)]} f_L(x)dx\right)^j \frac{E[N(L)]^j}{j!} e^{-E[N(L)]} \\ &= \exp\left(-E[N(L)]\int_0^{L-r} e^{-E[N(x+\Delta x, x+r)]} f_L(x)dx\right) \end{aligned} \tag{25}$$

To verify the tightness of approximating (22) with (25), we plot the relative error, $|\tilde{P}$ (net con) – P(net con)$|$ / P(net con) as a function of the expected number of cars in the network in Figure 3 for a homogeneous Poisson arrival with constant speed. From which we can see that the error of the approximation converge to zero as the expected number of cars in the network is large, for instance, when there are expected 100 cars in the network, the error is less than 1%. The expected number of cars in the network is a function of the arrival rate, velocity and the length of the network. In general, given that the length of the road segment L considered is long enough, we can have sufficient expected number of cars and thus P(net con) can be well approximated by (25). Further evaluation of the analytical results with consideration of vehicle interactions is shown later in Section V.

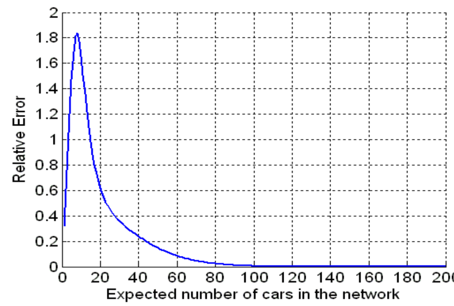


Figure 3. Relative error = $|\tilde{P}$ (net con) – P(net con)$|$ / P(net con) as a function of the expected number of cars in the network for homogenous Poisson arrival with constant velocity profile.

IV. *k*-connectivity

In the last section, we consider specifically the 1-connectivity of the network, we now proceed to investigate the general case of k-connectivity. For connected one-dimensional ad-hoc network, we can characterize the degree of connectivity by examining the *forward node degree* of nodes, which represents the number of direct single-hop forward neighbors of a node. Given that a node is located at x, we define that the forward node degree of the node as the number of nodes in the region $(x+\Delta x, x+r]$. Let $K_F(x)$ denote the forward node degree of a given node located at x, with the PALM assumptions, we have

$$\begin{aligned} &P\left(K_F(x)=k\right) = P\left(N(x+\Delta x, x+r)=k\right) \\ &= \frac{E[N(x+\Delta x, x+r)]^k}{k!} e^{-E[N(x+\Delta x, x+r)]} \end{aligned} \tag{26}$$

To associate forward node degree to connectivity in a one-dimensional communication network, we use a geometric graph $G = G(V, E)$ to represent the ad-hoc network, which consists of a set of nodes (vertices) and a set of communication links (edges). There is an edge between two vertices i and j if and only if the Euclidean distance between them $|i-j| \le r$.

With reference to [7], a graph is said to be k-connected if for each node pair there exist at least k mutually independent paths connecting them. Or, a graph is k-connected if and only if no set of $(k-1)$ nodes exists whose removal would disconnect the graph. Then, we have the following proposition.

Proposition 1: In a one-dimensional geometric graph G, let $K_{Fmin}(G)$ denotes the *minimum forward node degree* of graph G, then

$$P(G \text{ is } k\text{-connected}) = P(K_{Fmin}(G) \ge k) \tag{27}$$

Proof: We divide the proof into two parts, first, we prove that $P(G$ is k-connected$) \le P(K_{Fmin}(G) \ge k)$ followed by proving

P(G is k-connected) $\geq P(K_{Fmin}(G) \geq k)$, the combination of them proves proposition 1. □

Proposition 1.1: P(G is k-connected) $\leq P(K_{Fmin}(G) \geq k)$

Proof: This is equivalent to proving (G is k-connected) implies ($K_{Fmin}(G) \geq k$), which can be proved by contradiction. Assume on the contrary that ($K_{Fmin}(G) = k - 1 < k$), that is, every node only connects to at least $k - 1$ forward neighbors. In this case, if we remove $k - 1$ nodes whose are forward neighbors of a specific node i which has only $k - 1$ forward neighbors, node i will be disconnected from the forward network, and the network is disconnected. Hence, the network (graph) is not k-connected. □

Proposition 1.2: P(G is k-connected) $\geq P(K_{Fmin}(G) \geq k)$

Proof: This is equivalent to proving ($K_{Fmin}(G) \geq k$) implies (G is k-connected). In a one-dimensional network, assume that every node connects to at least k forward neighbors, i.e., ($K_{Fmin}(G) \geq k$), then it is trivial that no set of $k - 1$ nodes whose removal will disconnect the network. For example, in the worst case, if we remove $k - 1$ nodes whose are forward neighbors of a specific node i which has only k forward neighbors, node i is still connected to the forward network with one forward neighbor. Hence, the network (graph) is k-connected. □

Consider the case when $k = 1$ in Proposition 1, we have P(G is connected) $= P(K_{Fmin}(G) \geq 1)$, that is, the network is connected if and only if every node has at least one forward neighbor, which is equivalent to Definition 1, and the probability can be found with (25). The generalized version of (25) for k-connected network is as follows.

For a randomly chosen node in the road segment, the probability that it has less than k forward neighbors is

$$P(K_F < k) = \int_0^{L-r} P(K_F(y) < k \mid x) f_L(x)dx$$

$$= \int_0^{L-r} \sum_{i=0}^{k-1} \frac{E[N(x+\Delta x, x+r)]^i}{i!} e^{-E[N(x+\Delta x, x+r)]} f_L(x)dx. \quad (28)$$

By Proposition 1 and following similar steps in the derivation of (25), we have

$$P(k\text{-connected}) = P(K_{F\min} \geq k) = \exp\left(-E[N(L)]P(K_F < k)\right)$$

$$= \exp\left(-E[N(L)]\int_0^{L-r} \sum_{i=0}^{k-1} \frac{E[N(x+\Delta x, x+r)]^i}{i!} e^{-E[N(x+\Delta x, x+r)]} f_L(x)dx\right). \quad (29)$$

Therefore, given the density profile $n(x)$ from the fluid model, the communication range r, the probability that the network in a one-dimensional road segment is k-connected can be found. We are going to evaluate the analytical results by comparing them with simulated results in the next section, especially for the case with vehicle interactions.

V. NUMERICAL RESULTS

Scenario without Vehicle Interactions

Let us first consider an illustrative example for the case without vehicle interactions. Consider a road segment of length 10 km, and assume that cars only enter the route at location 0 with a rate of 30 cars/min and no cars join or depart at junctions.

Given the velocity profile $v(x)$ of the road as shown in Figure 4a, we can find that the *critical inter-arrival time*, $T_c = r$ according to the definition in (17), and we can compute from the fluid model (by solving the ODE's) that $E[N(L)] = 481$ for α = 30 cars/min. Thus, according to (19), we plot the probability that the network in the road segment is connected as a function of the transmission range r in Figure 4b. The reader is referred to Section VI for additional examples and applications regarding the tradeoff between transmission range and connectivity. In real practice, velocity profile can be obtained through empirical measurements, and more complicated velocity profiles can be handled by our models.

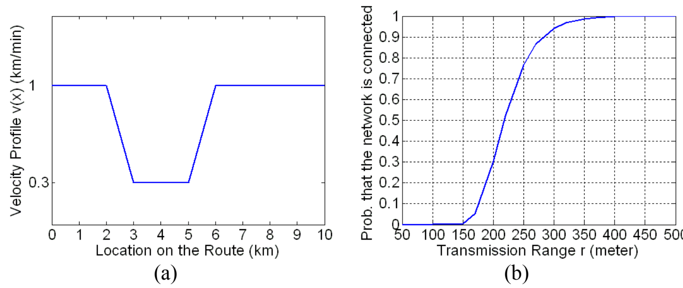


Figure 4. a) The velocity profile; and b) the corresponding probability that the network is connected as a function of transmission range r.

Scenario with Traffic Signals and Vehicle Interactions

We now evaluate the robustness of our analytical results with more complicated traffic signal systems, specifically, in a generic urban route with the ramifications of consecutive traffic signals, and even with vehicle interactions.

In the simulation, we capture vehicle interactions or the propagation of shockwave through the fundamental relationship between vehicular density and velocity. In the followings, we propose the front-density-dependent velocity field, which is modified based on the Greenshield's model [13] for such propose:

$$v(x,t) = v_f\left(1 - \frac{n(x+\Delta x, t)}{k_j}\right) \quad (30)$$

where v_f represents the mean free speed and k_j denotes the jamming density. If we define l_c as the average space occupied by a car at stationary, then $k_j = 1/\, l_c$.

Eq. (30) illustrates our front-density-dependent velocity profile, we have the velocity as a function of density Δx in front, the velocity decreases as the density in front increases. It is analogous to the car-following mechanism [14] in transport studies, when the vehicular density in front becomes high (i.e., cars in front decelerate), we should decelerate as well. Such general traffic flow model is applicable to most of the urban route scenarios, no matter there are slowing down, stopping, start moving of vehicles or not.

With the front-density-dependent velocity profile, we can approximate the interactions between vehicles in the fluid model iteratively. The general idea is, initially, we assume there are no vehicles on the road, i.e., $n(x, 0) = 0$ for all x belongs to X, where X is the location space. Therefore, the initial velocity will be the mean free speed v_f according to (30). Based on these initial conditions, we solve the differential equations in the fluid model for the vehicular density along the route. We can then compute the new velocity profile for the next time slot from the vehicular density profile according to (30), and so on.

In the simulation, we assume there are no cars joining and leaving the urban route at junctions, cars only arrive at location 0 at a constant rate (denoted by α) of 30 cars/min, with the mean free speed, v_f = 1 km/min, using the front-density-dependent velocity profile with Δx = 0.02 km as described in (30) unless otherwise specified. Assume the space occupied by a car at stationary as 4 m, so the jamming density k_j = 250 cars/km.

Figure 5 depicts the new street configuration for simulations. We consider that there are two traffic signals that are 500 m apart, located at the 2 km and 2.5 km locations respectively. Both of them have a 1 min cycle time, with 30 sec green, 3 sec amber, and 27 sec red signal periods.

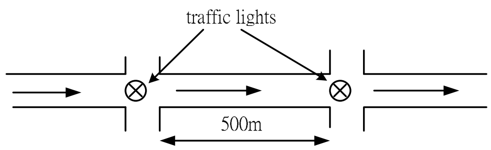


Figure 5. Street configuration test network.

To explain the relation between the traffic signals and velocity, consider the traffic signal at location 2 km on the route, we allow a further 0.012 km distance behind the traffic signal as the length of the junction. Thus, during the red signal period, $v = 0$ if $2 \le x < 2.012$ as shown in (31). In addition, we also include extra 0.02 km in front and behind the zero-velocity region so as to ensure that the partial derivative of v w.r.t. x in (9) is finite. For other regions during the red signal period, the front-density-dependent velocity field v applies.

$$v(x,t) = \begin{cases} v & \text{if } x < 1.98 \\ (v/0.02)*(2-x) & \text{if } 1.98 \le x < 2 \\ 0 & \text{if } 2 \le x < 2.012 \\ (v/0.02)*(x-2.02) & \text{if } 2.012 \le x < 2.032 \\ v & \text{if } x \ge 2.032 \end{cases} \tag{31}$$

When the signal turns green, we instantaneously set the velocity in the junction (2, 2.012] to be the mean free speed v_f (for a very short period of time) to represent that the first car in the queue move down the road with the highest speed when the signal turns green. The following cars then move according to the front-density-dependent velocity field.

We introduce an amber signal period to model the stopping motion of vehicles more rigorously, such that the velocity within the junction region (2, 2.012] gradually decreases to zero when the signal turns red.

To evaluate the stochastic independence of the model with vehicle interactions, we examine the dependency of the number of cars in two non-overlapping regions in the presence of vehicle interactions, we raise the arrival rate α of vehicles to the urban route in steps from 10 to 30 cars/min, and examine the correlation between the number of vehicles in two consecutive regions (2, 2.2] and (2.2, 2.4] in time interval (4, 6]. We can see from Figure 6 that as the arrival rate decreases, the correlation decreases. For α = 30 cars/min, the average correlation is 0.2486 (the closer the coefficient to either −1 or 1, the stronger the correlation between the variables), while for 20 and 10 cars/min, it drops to 0.057 and 0.003 respectively. The reason behind is that there are less interactions between vehicles with lower traffic load, and thus the stochastic independency of the PALM is better maintained. Therefore, given that the arrival rate is not too high, the PALM distributional results can be treated as an approximation to the actual number of cars in a road section even in the presence of vehicle interactions.

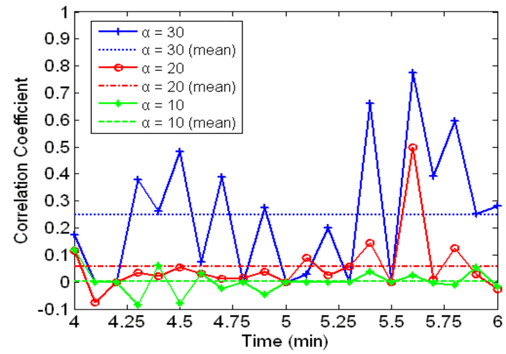


Figure 6. Correlation coefficient in time interval (4, 6] in the two-traffic-light scenario with different arrival rate (phase shift = 15 sec).

We verify the analytical results in (25) and (29) by comparing them with simulated results. For the cascaded traffic light scenario with the phase shift between the two traffic lights set to be 22.5 sec, we plot the simulated and analytical results in Figure 7 for the probability that the network is k-connected (for k = 1 and 2) as a function of the transmission range r. We consider the network in the road region (1.5, 2.5] at time instance 4.5 min. From the figure, we can see that the analytical and simulated results are close to each other even when vehicle interactions are considered by (30), and the k-connected probability grows to one when the transmission range is large enough.

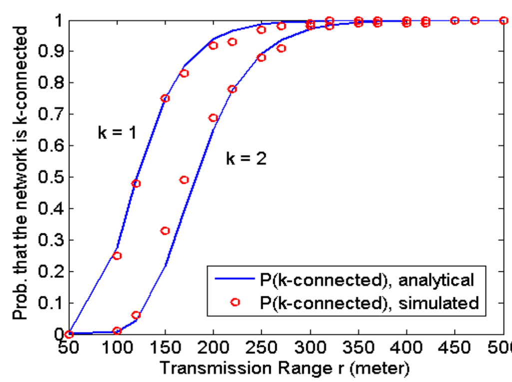
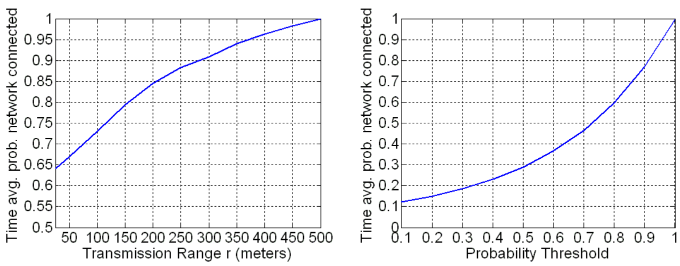


Figure 7. The analytical and simulated probability that the network in road region (1.5, 2.5] at time instance 4.5 min in the two-traffic-light scenario is k-connected as a function of transmission range r.

VI. TRANSMISSION RANGE ADJUSTMENT

Many system engineering and network management issues can be investigated with the knowledge of connectivity dynamics. As an illustrative example, we demonstrate with numerical results in this section of how the connectivity information facilitates transmission range adjustment of mobile nodes for boosting the overall network connectivity.

For road segments that are likely to be disconnected, we can adjust the transmission ranges of vehicles to achieve an almost surely connected network. As illustrated in Figure 7, given the transmission range r, we can calculate from (29) the probability that the network in the road segment is k-connected at a time instance. We can then take the time average of the probabilities to determine the *critical transmission range* for achieving certain degree of connectivity with high probability over a period of time. For instance, for the two-traffic-light scenario described in Section V with a traffic load of 3 cars/min (off-peak hour case), we plot in Figure 8a the time-averaged probability that the network in road region (2, 2.5] is connected over time interval (3, 5] as a function of transmission range, from which we can see that as the transmission range of vehicles increases beyond 370 m, the probability the network is connected is greater than 95%.



(a) (b)

Figure 8. Probability that the network in road region (2, 2.5] is connected time-averaged over interval (3, 5] as a function of a) transmission range r; and b) the probability threshold ξ.

Instead of having all vehicles to use large transmit power and thus transmission range, we can adjust the transmission range of individual vehicle dynamically according to its location and the corresponding vehicular density $n(x, t)$ so as to prevent disconnection of the network.

At a time instance, given $f_L(x)$, which is derived from $n(x)$, we can adjust r to $r^*(x)$ such that $P(N(x+\Delta x, x+r^*(x)) \geq 1) \geq \xi$ for all x, where ξ is certain probability threshold that is close to one, to ensure connectivity. Hence, we have

$$e^{-E[N(x+\Delta x, x+r^*(x))]} \leq 1-\xi \qquad (32)$$

Substitute (32) into (25), the probability that the network in the road segment $(0, L]$ is connected after such manipulation is

$$P(\text{net con}) \geq \exp\left(-E[N(L)](1-\xi)\int_0^{L-r} f_L(x)dx\right) \qquad (33)$$

Figure 8b depicts the probability that the network is connected for the off-peak hour case averaged over time interval (3, 5] as a function of ξ. For instance, we can see from the figure that with $\xi = 0.95$, the time-averaged probability of connectivity is great than 0.9.

VII. CONCLUSION

Nowadays, most of the vehicles are installed with GPS devices that can collect velocity information. Such information can serve as an initial input to the stochastic traffic model for computing the vehicular density. The future evolution of the traffic can then be approximated by the front-density-dependent velocity profile, and the propagation of shock wave, compression and rarefaction of the traffic stream due to traffic signals can also be automatically captured given the corresponding signaling information.

In this paper, we have modeled connectivity dynamics in vehicular ad-hoc networks in a generic signalized urban route based on density information provided by the stochastic traffic model. Specifically, we have determined the probability that the network in a road segment is k-connected, and demonstrated the applicability of our models and analysis on network planning such as the adjustment of transmission range of vehicles for maintaining connectivity. In essence, we have shown that the connectivity analysis is a good approximation even when vehicles interact with each other as their movement is controlled by traffic lights.

Our work on connectivity modeling is the first in the literature to take account of traffic signals and vehicle interactions in VANETs. In general, the connectivity analysis in this paper is applicable to more elaborated urban traffic models. For example, routes with more number of segments and traffic signals, and with arrival and departure of vehicles at road junctions. As other extensions, connectivity of roads with multiple lanes, bi-directional traffic and more complicated urban road network (e.g., two-dimensional road topology) can be represented by superposing multiple versions of urban routes.

ACKNOWLEDGEMENT

The work reported in this paper forms part of the MESSAGE project. MESSAGE is a three-year research project which started in October 2006 and is funded jointly by the UK Engineering and Physical Sciences Research Council and the UK Department for Transport. The project also has the support of nineteen non-academic organizations from public sector transport operations, commercial equipment providers, systems integrators and technology suppliers. More information is available from the web site www.message-project.org. The views expressed in this paper are those of the authors and do not represent the view of the Department for Transport or any of the non-academic partners of the MESSAGE project.

REFERENCES

[1] W. A. Massey and W. Whitt, "A Stochastic Model to Capture Space and Time Dynamics in Wireless Communication Systems," *Prob. in the Eng. & Inf. Sci*, vol. 8, pp. 541-569, 1994.

[2] K. K. Leung, W. A. Massey, and W. Whitt, "Traffic Models for Wireless Communication Networks," *IEEE J-SAC*, vol. 12, no. 8 1994.

[3] W. A. Massey and W. Whitt, "Network of Infinite Server Queues with Nonstationary Poisson Input," *Queuing Syst.*, vol. 13, pp. 183-250, 1993.

[4] P. Gupta and P. R. Kumar, "Critical Power for Asymptotic Connectivity in Wireless Networks," *IEEE Conference on Decision and Control*, 1998.

[5] P. Santi, D. M. Blough, and F. Vainstein, "A Probabilistic Analysis for the Radio Range Assignment Problem in Ad Hoc Networks," *ACM MobiHoc*, 2001.

[6] L. Wang, B. Liu, D. Goeckel, D. Towsley, and C. Westphal, "Connectivity in Cooperative Wireless Ad Hoc Networks," *ACM MobiHoc*, 2008.

[7] C. Bettstetter, "On the Minimum Node Degree and Connectivity of a Wireless Multihop Network," *ACM MobiHoc*, 2002.

[8] C. Bettstetter, "On the Connectivity of Ad Hoc Networks," *The Computer Journal*, vol. 47, no. 4 2004.

[9] S. Ukkusuri and L. Du, "Geometric Connectivity of Vehicular Ad Hoc Networks: Analytical Characterization," *Transportation Research Part C*, vol. 16, pp. 615-634, 2008.

[10] M. Khabazian and M. K. Mehmet Ali, "A Performance Modeling of Connectivity in Vehicular Ad Hoc Networks," *IEEE Transactions on Vehicular Technology*, vol. 57, no. 4 2008.

[11] C. F. Daganzo, *Fundamentals of Transportation and Traffic Operations* Pergamon Press-Elsevier Science, 1997.

[12] K. R. Symon, *Mechanics*, 3rd ed Addison-Wesley, 1971.

[13] C. J. Khisty and B. K. Lall, *Transportation Engineering*, 2nd ed Prentice Hall, 1998.

[14] M. Brackstone and M. McDonald, "Car-following: a historical review," *Transportation Research Part F: Traffic Psychology and Behaviour*, vol. 2, no. 4, pp. 181-196, 1999.