On a Positive Recurrence Criterion for Multidimensional Markov Chains with Application to the Stability of Slotted-Aloha with a Finite Number of Queues

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***Abstract*— In this paper, we revisit the stability analysis of well known slotted-Aloha protocol with finite number of queues. Under standard modeling assumptions, we derive a sufficient condition for the stability by invoking stochastic dominance arguments in conjunction with recurrence criterion due to Rosberg [1]. Our sufficiency condition for stability is linear in arrival rates and does not require knowledge of the stationary joint statistics of queue lengths. We believe that the technique reported here could be useful in analyzing other stability problems in countable-space Markovian settings.**

I. INTRODUCTION

The stability analysis of slotted Aloha protocol with finite number of queues has attracted lot of attention from researchers since its formulation by Tsybakov and Mikhailov [2] in 1979. The continued interest in this protocol is chiefly due to the fact that in spite of its extreme simplicity the analytical difficulties presented by the interacting queues has yielded no general necessary and sufficient conditions for positive recurrence. The standard modeling assumptions made in the literature to analyze this protocol result in a discrete-time Markov chain model of the queue lengths over a countable set. An important performance measure of this protocol is stability, i.e., for what set of arrival rates at different queues the average delay experienced by a packet is finite. In this paper, we resurrect a positive recurrence criterion for countable space multi-dimensional Markov chain proved by Rosberg [1] that has been largely forgotten. We show usefulness of this criterion by first generalizing Rosberg's criterion [1] and then applying it to the stability analysis of buffered slotted-Aloha protocol with finite number of queues.

Recently there has been much activity in trying to understand the stability region because of the interest in multiple access schemes for wireless networks. The most common access scheme is the contention mechanism used in the IEEE 802.11 that uses a exponential window type of backoff where the window doubles if collisions occur. Due to the difficulty of analyzing the stability region much attention has ben focussed on the so called saturation throughput. This is the average throughput seen by each queue or user assuming that they always have a packet to transmit. Thus the entire question of stability is sidestepped. It is well known and by now there are many text-book accounts of it [3] that the slotted aloha scheme is unstable when the number of users (or queues) goes to infinity and there is a fixed probability of attempting to transmit. It is only the finite case that is really unknown.

The discrete-time countable space Markov chain modeling of the slotted-Aloha protocol with finite number of queues was first proposed and analyzed by Tsybakov and Mikhailov in [2]. In [2], they provided the exact stability characterization when the number of queues $J = 2$. The next effort was by Rao and Ephremides [4] in 1988 who provided *exact* stability conditions for $J = 2$, and sufficient conditions for stability for $J > 2$ using stochastic dominance arguments and assuming Bernoulli input process at each queue. In 1994, Szpankowski [5] obtained the exact stability region for $J > 2$. The stability region characterization is given in terms of the stationary probabilities of joint statistics of the queues. However, for systems with more than three queues, the necessary and sufficient condition cannot be computed explicitly since it becomes very hard to compute the joint stationary statistics of the queues. In 1999, Luo and Ephremides [6] introduced the concept of "stability rank" to obtain tight inner and outer bounds to the stability region when $J > 2$. When queue i is known to be stable, then any queue j such that $\frac{\lambda_j(1-p_j)}{p_j} \leq \frac{\lambda_i(1-p_i)}{p_i}$ is also proven to be stable. Then it immediately follows that if queue k is unstable then $\frac{\lambda_k(1-p_k)}{p_k} > \frac{\lambda_i(1-p_i)}{p_i}$. With the help of stability ranks, they computed tight inner bounds to the stability region. Unfortunately, here also it is required to determine some stationary joint probabilities but which are extremely difficult to compute.

In all these papers, the goal was to derive sufficiency conditions for a *fixed* transmission attempt probability vector p. Instead, if one considers the union of the stability regions over all possible transmission probability vectors p, one obtains the closure of the stability region. In 1991, Anantharam [7] obtained closure of the stability region for $J > 2$ albeit for a correlated arrival processes.

Recently, in [8], a simple approximate expression for the stability region was proposed by using *mean field analysis*. Indeed they show a propagation of chaos takes place when the number of interacting queues is large. This expression is proved to be exact when the number of queues grows large, and is also shown to be very accurate in the case of small-queue systems through numerical experiments.

Our approach leads to sufficient conditions that do not depend on knowing the stationary distributions, and are completely characterized by the arrival parameters and the attempt probabilities of the queues. In particular, we show that for the case of two and three interacting queues we can recover the known results. However, we have found a few instances where certain arrival rate vectors λ satisfy the sufficiency condition for stability derived by Luo and Epphremides [6] but *not* ours. However we do not need to establish the stability of any higher rank queue as they require.

The rest of the paper is organized as follows. In Section II, we present modeling details of the slotted-Aloha protocol. In Section III, we describe in detail a dominant queueing model of the protocol and present its stability analysis in Section V. In Section IV, we state and prove a generalization of Rosberg's multi-dimensional positive recurrence criterion [1]. We end the paper with conclusions in Section VI.

II. MODEL

Consider a system $\mathcal{S}_1$ of J transmitting stations. At each station, there is a queue with infinite buffer space to store incoming packets and the queue is connected to a transmitter. These J transmitters wish to send packets in their respective buffers to a common receiver over a *collision channel*. Transmitter j is assumed to be associated with a Bernoulli random process $Y_j = \left\{Y_j^n; n \geq 1\right\}$ where the random variable Y_j^n with the distribution $\Pr\left(Y_j^n = 1\right) = p_j = 1 - \Pr\left(Y_j^n = 0\right)$ models the packet transmission attempt of the transmitter j in the nth time slot. That is, the jth transmitter with non-empty queue transmits in a slot with probability p_j and does not transmit with probability $\overline{p}_j = 1 - p_j$, independent of everything else. We denote by Y^n the random vector $(Y_1^n, Y_2^n, \ldots, Y_J^n)$. The communication channel between the transmitters and the receiver is modeled by a collision channel model. Under the collision channel model, a packet transmission is successful *if and only if* at most one transmitter with a non-empty queue transmits. When more than one transmitter transmits in a slot, all packet transmissions involved in that time slot are considered to have collided and hence are lost for all practical purposes. The length of a time slot is taken to be the duration of a packet transmission. At the end of each time slot, all transmitting stations are provided with ternary feedback which tells whether the time slot was idle (no attempted transmissions), successful (exactly one transmitter transmitted), or a failure (at least two transmitters transmitted in that time slot).

To model packet arrivals into various queues, we assume that packets arrive randomly into various queues and the packet arrival process into queue j is modeled by an i.i.d. batch arrival process $\Lambda_j = \left\{\Lambda_j^n, n \geq 1\right\}$ where the random variable Λ_j^n with finite first moment $\mathsf{E}\left(\Lambda_j^n\right) = \lambda_j$ models the number of packet arrivals into queue j during the nth time slot. Define by λ the vector $(\lambda_1, \lambda_2, \ldots, \lambda_J)$ of packet arrival rates. Let $\overline{Q}_j^n$ denote the number of packets present in the queue j at the beginning of the nth time slot. Denote by $\overline{Q}^n = \left(\overline{Q}_1^n, \overline{Q}_2^n, \ldots, \overline{Q}_J^n\right)$ the queue-length vector at the beginning of the nth time slot. From the assumptions made so far, we can easily note that the queue-length process $\left\{\overline{Q}^n, n \geq 0\right\}$ is a discrete-time Markov chain over the countable space $\mathbb{Z}_+^J$, where $\mathbb{Z}_+^J$ is the set of non-negative integer vectors of dimension J.

For an event A, let us define the indicator function $\mathbb{I}\{A\}$ as $\mathbb{I}\{A\} = 1$ if the event A is true, and $\mathbb{I}\{A\} = 0$ otherwise. From the above discussion, it is clear that transmitter j transmits a packet *if and only if* the product $Y_j^n \mathbb{I}\left\{\overline{Q}_j^n > 0\right\} = 1$, and *no* packet transmission happens otherwise. If a packet from the jth queue is involved in collision during the nth time slot it is then retransmitted in the $(n+1)$th time slot with the same probability p_j. When there is only one transmission in a time slot we say that the transmission is successful in that it is received error free at the receiver and the corresponding queue length is decremented by 1. The queue length evolution with time is given by

$$\left.\begin{aligned} \overline{Q}_j^{n+1} &= \overline{Q}_j^n + \Lambda_j^n - \overline{D}_j^n \\ \overline{D}_j^n &= Y_j^n \mathbb{I}\left\{\overline{Q}_j^n > 0\right\} \prod_{k \neq j} \left(1 - Y_k^n \mathbb{I}\left\{\overline{Q}_k^n > 0\right\}\right) \end{aligned}\right\} \tag{1}$$

where the random variable $\overline{D}_j^n \in \{0, 1\}$ denotes the number of departures from the jth queue in the nth time slot.

III. DOMINANT SYSTEM

We now consider another system $\mathcal{S}_2$ of J queues such that when $\mathcal{S}_1$ and $\mathcal{S}_2$ have the following *identical* features **F1**, **F2**, and **F3**, then $\mathcal{S}_2$ will *dominate* the original system $\mathcal{S}_1$ in the sense that queue lengths in $\mathcal{S}_2$ will be at least as large as the respective queue lengths in $\mathcal{S}_1$ at all times. Let the random variables Q_j^n and D_j^n denote [1] the queue length of and the number

[1] Usage of an *over line* in the notation will distinguish queue length and departure random variables of $\mathcal{S}_1$ from that of $\mathcal{S}_2$. An over line in the notation is used *only* for the original system $\mathcal{S}_1$.

of departures from the jth queue for the nth time slot. The following features are assumed to be identical to both $\mathcal{S}_1$ and $\mathcal{S}_2$.

- **F1** initial state, i.e., $Q^0 = \overline{Q}^0$.
- **F2** arrival processes, i.e., arrivals into the jth queue in $\mathcal{S}_2$ occur *exactly* at the same time instants as in the original system $\mathcal{S}_1$.
- **F3** transmission attempts, i.e., the Bernoulli random vector Y^n that determines transmission attempts in the original system $\mathcal{S}_1$ for the nth time slot also determines the transmission attempts for the nth time slot in the system $\mathcal{S}_2$.

The distinguishing feature of $\mathcal{S}_2$ from $\mathcal{S}_1$ will come from the presence of *dummy packet transmissions* from $\mathcal{S}_2$, i.e., queue j of $\mathcal{S}_2$ transmits a packet, called dummy packet, with probability p_j upon becoming empty. The aspect on which the two systems will differ is the interference as seen by the individual queues in $\mathcal{S}_2$. By careful construction, we make interference for the queue j in $\mathcal{S}_2$ at least as large as the interference seen by the queue j in the original system $\mathcal{S}_1$. As a consequence, a successful transmission from the queue j in $\mathcal{S}_2$ implies a successful transmission from the queue j of $\mathcal{S}_1$ provided $\overline{Q}_j > 0$. But the converse need not be true. This fact becomes revealed when we compare the queue length evolutions (1) and (2), respectively, of $\mathcal{S}_1$ and $\mathcal{S}_2$. Henceforth, we will refer to $\mathcal{S}_2$ as a *dominant* of the original $\mathcal{S}_1$.

To be able to define the rules that will specify the interference to be seen by any individual queue in $\mathcal{S}_2$, we define two sets $\mathcal{U}_j$ and $\mathcal{V}_j$ of queues for each queue j.

$$\mathcal{U}_1 = \emptyset \text{ and } \mathcal{U}_j = \{1, 2, \ldots, j-1\} \text{ for } j \geq 2$$

$$\mathcal{V}_j = \{j+1, j+2, \ldots, J\} \text{ for } j < J \text{ and } \mathcal{V}_J = \emptyset$$

The sets $\mathcal{U}_j$ and $\mathcal{V}_j$ will be designated as, respectively, the set of *non-persistent* and the set of *persistent* queues of the jth queue for the following reason. A real packet transmission from queue j is effected by (i) *only real* packet transmissions from the queues that belong to the set of queues $\mathcal{U}_j$, and (ii) *both real and dummy* packet transmissions from the queues that belong to the set of queues $\mathcal{V}_j$. In other words, each queue j in $\mathcal{S}_2$ transmits a *dummy* packet with probability p_j upon becoming empty. But not every real packet transmission is effected by a dummy packet transmission from the jth queue. Dummy transmissions are designed only to cause interference but the successful transmission of a dummy packet from the queue j has no significance, i.e., queue length Q_j is unaffected. An interesting point and the main aspect in which our dominant system $\mathcal{S}_2$ differs from the previous constructions is that different queues have different sets of persistent and non-persistent queues in the *same time slot*.

From the discussion made above, the queue length evolution in the dominant system $\mathcal{S}_2$ can now be represented as

$$\left.\begin{aligned} Q_j^{n+1} &= Q_j^n + \Lambda_j^n - D_j^n \\ D_j^n &= \left[\prod_{k \in \mathcal{U}_j} \left(1 - Y_k^n \mathbb{I}\left\{Q_k^n > 0\right\}\right)\right] \\ &\quad \times Y_j^n \mathbb{I}\left\{Q_j^n > 0\right\} \left[\prod_{k \in \mathcal{V}_j} \left(1 - Y_k^n\right)\right] \end{aligned}\right\} \quad (2)$$

For the queue-length vector Q, we define $u_j(Q)$ as the probability that *no* real packet is transmitted from the queues of the set $\mathcal{U}_j$. That is

$$\begin{aligned} u_1(Q) &= 1 \\ u_j(Q) &= \prod_{k=1}^{j-1} \left(1 - p_k \mathbb{I}\{Q_k > 0\}\right) \qquad \text{for } j \geq 2 \end{aligned}$$

Similarly, $v_j(Q)$ will be defined as the probability that *neither* a real packet transmission *nor* a dummy packet transmission will occur from the queues of the set $\mathcal{V}_j$, i.e.,

$$\begin{aligned} v_j(Q) &= \prod_{k=j+1}^{J} \overline{p}_k \quad \text{for } 1 \leq j \leq J-1 \\ v_J(Q) &= 1 \end{aligned}$$

We note here that $v_j(Q)$ is queue-length *independent* and $u_j(Q)$ is queue-length *dependent*. Likewise, we define the *success probability* $r_j(Q)$ of the jth queue as

$$r_j(Q) \;=\; u_j(Q) p_j v_j(Q) \mathbb{I}\{Q_i > 0\} \qquad (3)$$

Define the success probability vector $r(Q) = (r_1(Q), r_2(Q), \ldots, r_J(Q))$. For the sake of notational convenience, henceforth, we denote the more expressive notation $u_j(Q)$, $v_j(Q)$, $r_j(Q)$, and $r(Q)$, respectively, as simply u_j, v_j, r_j, and r, as long as no ambiguity is caused. Some times we may also write $u_j(r)$ in place of $u_j(Q)$ or u_j. From the knowledge of r, we can immediately tell which queues are empty and which queues are non-empty and hence indices of the non-empty queues in the $\mathcal{U}_j$. Then it becomes straightforward to write down the value of $u_j(r)$. Also, in the rest of this paper, we will prefer the more convenient notation Q_j in place of Q_j^n unless explicit emphasis on the time slot is needed, and we extend this rule to other variables too. With the help of the notation introduced so far, we now state our central result on stability of the slotted-Aloha protocol.

Proposition 3.1: Let $\eta = (\eta_1, \eta_2, \ldots, \eta_J)$ denote a permutation of the set $\{1, 2, \ldots, J\}$. Define $\mathcal{C}(\eta) \subset \mathbb{R}_+^J$ to be the set of λ that satisfies

$$\frac{\lambda_{\eta_j}}{p_{\eta_j} v_{\eta_j}} + \mathbb{I}\{j \geq 2\} \sum_{k=1}^{j-1} \frac{\lambda_{\eta_k}}{v_{\eta_k}} \;<\; 1, \quad \text{for } 1 \leq j \leq J$$

Define $\mathcal{C} = \cup_\eta \mathcal{C}(\eta)$. Then the dominant system $\mathcal{S}_2$ is stable for $\lambda \in \mathcal{C}$. ∎

Since stability of $\mathcal{S}_2$ implies stability of $\mathcal{S}_1$ because of queue length dominance, we conclude that the original system $\mathcal{S}_1$ is also stable for λ that satisfies the conditions of Proposition 3.1. In Figure 1, we show a portion of the stability region $\mathcal{C}$ when $J = 3$.

We will now specialize Proposition 3.1 for $J = 2, 3$, and $J \to \infty$. For $J = 2$, we have

$$\mathcal{C}(\{1,2\}) = \left\{\lambda : \frac{\lambda_1}{p_1\overline{p}_2} < 1 \text{ and } \frac{\lambda_1}{\overline{p}_2} + \frac{\lambda_2}{p_2} < 1\right\}$$
$$\mathcal{C}(\{2,1\}) = \left\{\lambda : \frac{\lambda_2}{\overline{p}_1 p_2} < 1 \text{ and } \frac{\lambda_1}{p_1} + \frac{\lambda_2}{\overline{p}_1} < 1\right\}$$

Then $\mathcal{C} = \mathcal{C}(\{1,2\}) \cup \mathcal{C}(\{2,1\})$ reduces to the exact stability condition derived by Tsybakov and Mikhailov [2].

Let us now consider $J = 3$. We have a total of six permutations of the set of queues $\{1,2,3\}$ and the corresponding sufficient conditions for stability are as follows:

$$\mathcal{C}(\{1,2,3\}) = \left\{\lambda : \begin{array}{r} \dfrac{\lambda_1}{p_1\overline{p}_2\overline{p}_3} < 1 \\ \dfrac{\lambda_2}{p_2\overline{p}_3} + \dfrac{\lambda_1}{\overline{p}_2\overline{p}_3} < 1 \\ \dfrac{\lambda_3}{p_3} + \dfrac{\lambda_2}{\overline{p}_3} + \dfrac{\lambda_1}{\overline{p}_2\overline{p}_3} < 1 \end{array}\right\}$$

$$\mathcal{C}(\{1,3,2\}) = \left\{\lambda : \begin{array}{r} \dfrac{\lambda_1}{p_1\overline{p}_2\overline{p}_3} < 1 \\ \dfrac{\lambda_3}{\overline{p}_2 p_3} + \dfrac{\lambda_1}{\overline{p}_2\overline{p}_3} < 1 \\ \dfrac{\lambda_2}{p_2} + \dfrac{\lambda_3}{\overline{p}_2} + \dfrac{\lambda_1}{\overline{p}_2\overline{p}_3} < 1 \end{array}\right\}$$

$$\mathcal{C}(\{2,3,1\}) = \left\{\lambda : \begin{array}{r} \dfrac{\lambda_2}{p_2\overline{p}_1\overline{p}_3} < 1 \\ \dfrac{\lambda_3}{p_3\overline{p}_1} + \dfrac{\lambda_2}{\overline{p}_1\overline{p}_3} < 1 \\ \dfrac{\lambda_1}{p_1} + \dfrac{\lambda_3}{\overline{p}_1} + \dfrac{\lambda_2}{\overline{p}_1\overline{p}_3} < 1 \end{array}\right\}$$

$$\mathcal{C}(\{2,1,3\}) = \left\{\lambda : \begin{array}{r} \dfrac{\lambda_2}{p_2\overline{p}_1\overline{p}_3} < 1 \\ \dfrac{\lambda_1}{p_1\overline{p}_3} + \dfrac{\lambda_2}{\overline{p}_1\overline{p}_3} < 1 \\ \dfrac{\lambda_3}{p_3} + \dfrac{\lambda_1}{\overline{p}_3} + \dfrac{\lambda_2}{\overline{p}_1\overline{p}_3} < 1 \end{array}\right\}$$

$$\mathcal{C}(\{3,2,1\}) = \left\{\lambda : \begin{array}{r} \dfrac{\lambda_3}{p_3\overline{p}_1\overline{p}_2} < 1 \\ \dfrac{\lambda_2}{p_2\overline{p}_1} + \dfrac{\lambda_3}{\overline{p}_1\overline{p}_2} < 1 \\ \dfrac{\lambda_1}{p_1} + \dfrac{\lambda_2}{\overline{p}_1} + \dfrac{\lambda_3}{\overline{p}_1\overline{p}_2} < 1 \end{array}\right\}$$

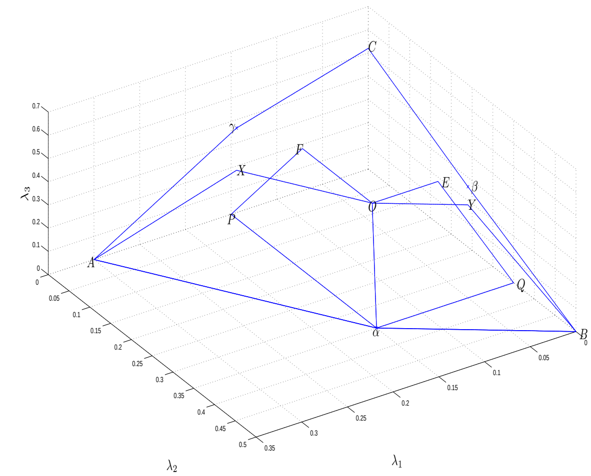

Fig. 1. Figure shows the sufficiency conditions for stability for the permutations $\{3,2,1\}$ and $\{3,1,2\}$. In the figure, $A = (p_1, 0, 0)$, $B = (0, p_2, 0)$, $C = (0, 0, p_3)$, $O = (p_1\overline{p}_2\overline{p}_3, \overline{p}_1 p_2\overline{p}_3, \overline{p}_1\overline{p}_2 p_3)$, $\alpha = (p_1\overline{p}_2, \overline{p}_1 p_2, 0)$, $\beta = (0, p_2\overline{p}_3, \overline{p}_2 p_3)$, $\gamma = (p_1\overline{p}_3, 0, \overline{p}_1 p_3)$, $P = (p_1\overline{p}_2, 0, 0)$, $Q = (0, \overline{p}_1 p_2, 0)$, $X = (p_1\overline{p}_3, 0, \overline{p}_1\overline{p}_2 p_3)$, $Y = (0, p_2\overline{p}_3, \overline{p}_1\overline{p}_2 p_3)$, $E = (0, \overline{p}_1 p_2\overline{p}_3, \overline{p}_1\overline{p}_2 p_3)$, $F = (p_1\overline{p}_2\overline{p}_3, 0, \overline{p}_1\overline{p}_2 p_3)$. When $\lambda_3 = 0$, the stability region is confined to $\lambda_1\lambda_2$-plane and the corresponding stability region boundary is described by the line segments $A\alpha$ and $B\alpha$. Similarly, the line segments $A\gamma$ and $C\gamma$ together represent the boundary of the stability region when $\lambda_2 = 0$, and the line segments $B\beta$ and $C\beta$ together represent the boundary of the stability region when $\lambda_1 = 0$. For $\lambda_3 < \overline{p}_1\overline{p}_2 p_3$, the plane segment $A\alpha OX$, bearing the plane equation $\frac{\lambda_1}{p_1} + \frac{\lambda_2}{\overline{p}_1} + \frac{\lambda_3}{\overline{p}_1\overline{p}_2} = 1$ and the plane segment $Q\alpha OE$, bearing the plane equation $\frac{\lambda_2}{\overline{p}_1 p_2} + \frac{\lambda_3}{\overline{p}_1\overline{p}_2} = 1$, together describe the sufficiency condition for stability.

$$\mathcal{C}(\{3,1,2\}) = \left\{\lambda : \begin{array}{r} \dfrac{\lambda_3}{p_3\overline{p}_1\overline{p}_2} < 1 \\ \dfrac{\lambda_1}{p_1\overline{p}_2} + \dfrac{\lambda_3}{\overline{p}_1\overline{p}_2} < 1 \\ \dfrac{\lambda_2}{p_2} + \dfrac{\lambda_1}{\overline{p}_2} + \dfrac{\lambda_3}{\overline{p}_1\overline{p}_2} < 1 \end{array}\right\}$$

Each of these six sufficient conditions for $J = 3$ strictly include the respective sufficient conditions derived in Rao and Ephremides [4]. For the asymptotic case of $J \to \infty$ and symmetric arrival rates and transmission probabilities (i.e., $\lambda_j = \lambda$ and $p_j = p$), our result recovers the well known result [9] that the system is unstable. Before we formally prove Proposition 3.1, we first present a generalization of Rosberg's positive recurrence criterion [1] for multi-dimensional queueing processes. We also provide some background material on the *usual stochastic order* of multivariate random variables.

IV. Rosberg's Positive-Recurrence Criterion

Let $\mathcal{X}$ be a countable set of states over which the irreducible, aperiodic, and discrete-time Markov chain $\{X_n, n \geq 0\}$ takes its values. For any integer $k \geq 1$, define $\left\{p_{xy}^k, x, y \in \mathcal{X}\right\}$ to be the k-step transition probability law of the Markov chain

$\{X_n, n \geq 0\}$. For any subset $B \subseteq \mathcal{X}$, we know that the $\lim_{k\to\infty} p_{xB}^k = \lim_{k\to\infty} \sum_{y\in B} p_{xy}^k = \pi(B) \geq 0$ exists and is independent of the initial state x.

For any nonnegative-valued function V on $\mathcal{X}$, define $\Delta^k V(x) \triangleq \sum_y p_{xy}^k V(y) - V(x)$ to be the k-step drift of the function V in state x. In the following Lemma 4.1, we state an expression for $\Delta^k V(x)$ in terms of the one-step drifts $\{\Delta V(x), x \in \mathcal{X}\}$.

Lemma 4.1: Let t_1 and t_2 be positive integers. Then

$$\Delta^{t_1+t_2} V(x) \quad = \quad \Delta^{t_1} V(x) + \sum_{y\in\mathcal{X}} p_{xy}^{t_1} \Delta^{t_2} V(y)$$

■

Proof:

$$\begin{aligned}
\Delta^{t_1+t_2} V(x) &= \sum_{y\in\mathcal{X}} p_{xy}^{t_1+t_2} V(y) - V(x) \\
&= \sum_{y\in\mathcal{X}} \left(\sum_{z\in\mathcal{X}} p_{xz}^{t_1} p_{zy}^{t_2} \right) V(y) - V(x) \\
&= \sum_{z\in\mathcal{X}} p_{xz}^{t_1} \sum_{y\in\mathcal{X}} p_{zy}^{t_2} V(y) - V(x) \\
&= \sum_{z\in\mathcal{X}} p_{xz}^{t_1} \left(\Delta^{t_2} V(z) + V(z) \right) - V(x) \\
&= \Delta^{t_1} V(x) + \sum_{y\in\mathcal{X}} p_{xy}^{t_1} \Delta^{t_2} V(y)
\end{aligned}$$

■

Corollary 4.1: Define $t_0 = 0$. For some integer $J \geq 1$, let $t_1, t_2, \ldots, t_J$ be positive integers. Then for $x \in \mathcal{X}$,

$$\Delta^{\sum_{j=1}^J t_j} V(x) \quad = \quad \sum_{y\in\mathcal{X}} \sum_{j=0}^{J-1} p_{xy}^{\left(\sum_{k=0}^j t_k\right)} \Delta^{t_{j+1}} V(y)$$

■

Proof: Repeated application of Lemma 4.1 gives the result. ■

Next, we establish that $\lim_{k\to\infty} \frac{1}{k} \Delta^k V(x) = c^* \geq 0$ under the assumption that the Markov chain is irreducible and aperiodic.

To see this, assume that the drift $\Delta V(x)$ is upper bounded by a positive constant η. Then it is easy to see the existence of $\lim_{k\to\infty} \frac{1}{k} \Delta^k V(x)$ because $\lim_{k\to\infty} \frac{1}{k} \Delta^k V(x) = \lim_{k\to\infty} \frac{1}{k} \sum_{y\in\mathcal{X}} \Delta V(y) \sum_{l=0}^{k-1} p_{xy}^l = \sum_{y\in\mathcal{X}} \Delta V(y) \pi(y) \leq \eta$. Also, since $\frac{V(x)}{k} \overset{k\to\infty}{\to} 0$ and $V(x) \geq 0$, we have $\lim_{k\to\infty} \frac{1}{k} \Delta^k V(x) = \lim_{k\to\infty} \frac{1}{k} \left(\sum_{y\in\mathcal{X}} p_{xy}^k V(y) - V(x) \right) = c^* \geq 0$.

We need the following three assumptions to be able to establish positive recurrence of the Markov chain.

Assumption 4.1: Let $J \geq 2$ be an integer. Then there exists

(i) a collection $\mathcal{P} = \{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_J\}$ of partitions of the set $\mathcal{X}$ where $\mathcal{P}_j = \left\{\mathcal{X}_j, \mathcal{X}_j^c\right\}$ [2].

(ii) There exist nonnegative-valued functions $\{V_j(x), x \in \mathcal{X}\}$ for $1 \leq j \leq J$ such that the drift $\Delta V_j(x)$ of the function V_j in state x has the following form:

$$\Delta V_j(x) \quad \leq \quad \begin{cases} \eta_j & \text{for } x \in \mathcal{X} \\ -\epsilon_j & \text{for } x \in \mathcal{X}_j^c, \end{cases} \tag{4}$$

where $\epsilon_j > 0$ and $\eta_j \geq 0$.

Assumption 4.2: For $1 \leq j \leq J$, let there exist partitions $\left\{\mathcal{A}_{j,k}, \mathcal{A}_{j,k}^c\right\}$, $k \geq 1$, of the set $\mathcal{X}$ with the following two properties: for $k \geq 1$

(i) $p_{xy}^l = 0$, $0 \leq l \leq k-1$, for $x \in \mathcal{A}_{j,k}^c$ and $y \in \mathcal{X}_j$

(ii) $\cap_j \mathcal{A}_{j,k}$ is a finite set

In other words, we need *at least* k transitions to reach the set $\mathcal{X}_j$ when started from a state in $\mathcal{A}_{j,k}^c$.

Assumption 4.3: The sequence $\left\{\frac{\Delta^k V_j(x)}{k}, k \geq 1\right\}$ is said to be uniformly upper bounded (UUB), if for any $\delta > 0$, there exists a positive integer K such that $\frac{\Delta^k V_j(x)}{k} < c_j^* + \delta$ for $k \geq \mathsf{K}$ and $x \in \mathcal{A}_{j,k}$.

Theorem 4.1: Let Assumptions 4.1, 4.2, and 4.3 hold. Then the Markov chain $\{X_n, n \geq 1\}$ is positive recurrent. ■

Remark 4.1: Rosberg [1] assumed a *fixed* (say J) dimensional Markov chain and then considered an equal number J of partitions of the countable space and also an equal number J of Lyapunov functions. However, in our generalization of his criterion, we do not require the countable space to have a fixed predetermined dimension. We believe this generalization will be useful for the reason that in many situations of interest one does not obtain a countable space of some fixed predetermined dimension. Also, even in the context of a Markov chain with some fixed dimension it may not be appropriate to consider an equal number J of Lyapunov functions and hence an equal number J of partitions of the state space $\mathcal{X}$.

Before we prove Theorem 4.1, we prove few supporting lemmas.

Lemma 4.2: For any non-negative random variables $Y_1, Y_2, \ldots, Y_m$ and a constant $a > 0$,

$$p\left(\max_{1\leq i\leq m} Y_i \geq a\right) \quad \leq \quad \frac{1}{a} \sum_{i=1}^m \mathbb{E}(Y_i)$$

Proof:

$$\begin{aligned}
p\left(\max_{1\leq i\leq m} Y_i \geq a\right) &= p\left(\bigcup_{i=1}^m \{Y_i \geq a\}\right) \\
&\leq \sum_{i=1}^m p(Y_i \geq a) \overset{(b)}{\leq} \frac{1}{a} \sum_{i=1}^m \mathbb{E}(Y_i),
\end{aligned}$$

where (b) follows from Markov inequality. ■

Lemma 4.3: Let $k \geq 1$. Then $\Delta^k V_j(x) \leq -k\epsilon_j$, $1 \leq j \leq J$, for $x \in \mathcal{A}_{j,k}^c$.

Proof: Let $x \in \mathcal{A}_{j,k}^c$. Then $\Delta^k V_j(x)$

$$\begin{aligned}
&= \sum_{y\in\mathcal{X}} \Delta V_j(y) \sum_{l=0}^{k-1} p_{xy}^l \\
&= \sum_{y\in\mathcal{X}_j} \Delta V_j(y) \sum_{l=0}^{k-1} p_{xy}^l + \sum_{y\in\mathcal{X}_j^c} \Delta V_j(y) \sum_{l=0}^{k-1} p_{xy}^l \\
&\overset{(a)}{\leq} -k\epsilon_j
\end{aligned}$$

[2] We will be using the same notation $\mathcal{X}$ to denote both the state space and its subsets. The distinction is made through the usage of subscripts, i.e., $\mathcal{X}_j$ denotes a subset

where (a) follows from Assumption 4.2. ∎

Let us fix an arbitrary $\delta > 0$. Next for $1 \leq j \leq J$ and $x \in \mathcal{X}$, let us assume that there exits a positive integer K such that $\frac{\Delta^k V_j(x)}{k} \leq c_j^* + \delta$ for $k \geq \mathsf{K}$. Let us pick one such K, and then introduce the set of functions $\left\{g_j^{\mathsf{K}}(x), x \in \mathcal{X}\right\}$ such that the following holds good:

$$\Delta^{\mathsf{K}} V_j(x) \quad = \quad -g_j^{\mathsf{K}}(x) + \mathsf{K}\left(c_j^* + \delta\right) \tag{5}$$

Two observations on the functions g_j^{K} are in order: the first and the obvious observation is that $g_j^{\mathsf{K}}(x) \geq 0$ for $x \in \mathcal{X}$. Also, since $\Delta^{\mathsf{K}} V_j(x) \leq -\mathsf{K}\epsilon_j$ for $x \in \mathcal{A}_{j,\mathsf{K}}^c$, we have that $g_j^{\mathsf{K}}(x) \geq \left(c_j^* + \delta + \epsilon_j\right)$ for $x \in \mathcal{A}_{j,\mathsf{K}}^c$. Set $\epsilon = \min_j \epsilon_j$ and $\delta = \min_j \delta_j$.

As a result, we have the obvious deduction that $\max_j g_j^{\mathsf{K}}(x) \geq \min_j \mathsf{K}\left(c_j^* + \delta + \epsilon_j\right) = (c^* + \delta + \epsilon)$ for $x \in \cup_j \mathcal{A}_{j,\mathsf{K}}^c$. Hence $\max_j g_j^{\mathsf{K}}(x) < \min_j \mathsf{K}\left(c^* + \delta + \epsilon\right)$ implies [3] that $x \in \cap_j \mathcal{A}_{j,\mathsf{K}}$.

A. Proof of Theorem 4.1

Proof: Denote by $\mathsf{E}_x\left(g_j^{\mathsf{K}}(X_n)\right)$ the expectation of $g_j^{\mathsf{K}}(X_n)$ given that $X_0 = x$ and by $p_x(X_n \in A)$ the probability that $X_n \in A$ given that $X_0 = x$. Now $\frac{\Delta^{n\mathsf{K}} V_j(x)}{n}$

$$\begin{aligned}
&= \sum_{y \in \mathcal{X}} \frac{1}{n} \sum_{l=0}^{n-1} p_{xy}^{l\mathsf{K}} \Delta^{\mathsf{K}} V_j(y) \\
&\overset{(a)}{=} \sum_{y \in \mathcal{X}} \frac{1}{n} \sum_{l=0}^{n-1} p_{xy}^{l\mathsf{K}} \left[-g_j^{\mathsf{K}}(y) + \mathsf{K}(c_j^* + \delta)\right] \\
&= -\sum_{y \in \mathcal{X}} \frac{1}{n} \sum_{l=0}^{n-1} p_{xy}^{l\mathsf{K}} g_j^{\mathsf{K}}(y) + \mathsf{K}(c_j^* + \delta) \\
&= -\frac{1}{n} \sum_{l=0}^{n-1} \mathsf{E}_x\left(g_j^{\mathsf{K}}\left(X_{l\mathsf{K}}\right)\right) + \mathsf{K}(c_j^* + \delta)
\end{aligned}$$

where (a) follows from (5).

Since $\lim_{n\to\infty} \frac{\Delta^{n\mathsf{K}} V_j(x)}{n\mathsf{K}} = c_j^*$, we have that $\frac{1}{n}\sum_{l=0}^{n-1} \mathsf{E}_x\left(g_j^{\mathsf{K}}\left(X_{l\mathsf{K}}\right)\right) = \mathsf{K}\delta$. Now $\liminf_{n\to\infty} \frac{1}{n} p_x\left(\max_j g_j^{\mathsf{K}}(x) < \mathsf{K}\left(c^* + \delta + \epsilon\right)\right)$

$$\begin{aligned}
&\overset{(b)}{\geq} 1 - \limsup_{n\to\infty} \frac{1}{n} \sum_{l=0}^{n-1} \sum_{j=1}^{J} \frac{\mathsf{E}_x\left(g_j^{\mathsf{K}}\left(X_{l\mathsf{K}}\right)\right)}{\mathsf{K}\left(c^* + \delta + \epsilon\right)} \\
&\geq 1 - \frac{1}{\mathsf{K}\left(c^* + \delta + \epsilon\right)} \times \\
&\quad \sum_{j=1}^{J} \limsup_{n\to\infty} \frac{1}{n} \sum_{l=0}^{n-1} \mathsf{E}_x\left(g_j^{\mathsf{K}}\left(X_{l\mathsf{K}}\right)\right) \\
&= 1 - \frac{J\mathsf{K}\delta}{\mathsf{K}\left(c^* + \delta + \epsilon\right)} \\
&= 1 - \frac{J\delta}{\left(c^* + \delta + \epsilon\right)},
\end{aligned}$$

[3]We should observe that $x \in \cap_j \mathcal{A}_{j,k}$ *need not imply that* $\max_j g_j^{\mathsf{K}}(x) < \mathsf{K}\left(c^* + \delta + \epsilon\right)$.

where (b) follows from Lemma 4.2.

We note that there exists a $\delta_0 > 0$ such that $1 - \frac{J\delta}{(c^* + \delta_0 + \epsilon)} > 0$. Define the set

$$A_0 \quad = \quad \left\{x \in \mathcal{X} : \max_j g_j^{\mathsf{K}}(x) < \mathsf{K}\left(c^* + \delta_0 + \epsilon\right)\right\}$$

We can observe that $A_0 \subseteq \cap_j \mathcal{X}_j$ is a finite set. Hence it follows that for the finite set A_0,

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{l=0}^{n-1} p_x\left(X_{l\mathsf{K}} \in A_0\right) \quad > \quad 0$$

Since the chain is assumed to be irreducible and aperiodic, it follows that the Markov chain is positive recurrent. ∎

B. Multivariate Stochastic Order for Random Variables

In this subsection, we briefly discuss a multivariate stochastic order known as *usual multivariate stochastic order* [10]. Let $x = (x_1, x_2, \ldots, x_J)$ and $y = (y_1, y_2, \ldots, y_J)$ be any two vectors in $\mathbb{R}^J$; then we denote $x \leq y$ if $x_j \leq y_j$ for $1 \leq j \leq J$. A set $\mathcal{B} \subseteq \mathbb{R}^J$ is called (i) *Upper* if $y \in \mathcal{B}$ whenever $y \geq x$ and $x \in \mathcal{B}$ and (ii) *Lower* if $y \in \mathcal{B}$ whenever $y \leq x$ and $x \in \mathcal{B}$. We say that an $\mathbb{R}^J$-valued random variable X is *stochastically larger* than another $\mathbb{R}^J$-valued random variable $Y \in \mathbb{R}^J$ if

$$\Pr(X \in \mathcal{B}) \quad \geq \quad \Pr(Y \in \mathcal{B}), \forall \text{ Upper sets } \mathcal{B} \subseteq \mathbb{R}^J$$

When X is stochastically larger than Y, we write $X \geq_{\text{st}} Y$. We say that X is stochastically smaller than Y, i.e., $X \leq_{\text{st}} Y$, if

$$\Pr(X \in \mathcal{B}) \quad \geq \quad \Pr(Y \in \mathcal{B}), \forall \text{ Lower sets } \mathcal{B} \subseteq \mathbb{R}^J$$

An important characterization of the usual stochastic order is given in the following theorem.

Theorem 4.2 (Theorem 6.B.1. in [10]): The random vectors X and Y satisfy $X \leq_{\text{st}} Y$ if, and only if, there exists two random vectors $\hat{X}$ and $\hat{Y}$, defined on the same probability space, such that $\hat{X} =_{\text{st}} X$, $\hat{Y} =_{\text{st}} Y$, and $P\left\{\hat{X} \leq \hat{Y}\right\} = 1$. ∎

V. Stability of the Queue Length Process $\{Q^n, n \geq 1\}$

We now prove Proposition 3.1 for the particular permutation $\eta = (1, 2, \ldots, J)$. The proof consists of verifying validity of the Assumptions 4.1, 4.2, and 4.3 of Theorem 4.1 for the Markov chain $\{Q^n, n \geq 1\}$. Then Theorem 4.1 implies the sufficiency condition of Proposition 3.1. But to facilitate the presentation, we need to introduce some more notation. We note that the distribution of the departure random variable D_j associated with the jth queue is $p(D_j = 1) = r_j = 1 - p(D_j = 0)$. Hence follows that

$$\mathbb{E}(D_j) \;=\; r_j \tag{6}$$

Let us define the set $\mathcal{R}$ of success probability vectors $r(Q)$ as $\mathcal{R} = \left\{r(Q) : Q \in \mathbb{Z}_+^J\right\}$ where the components of the vector $r(Q)$ are as defined in (3). defined in. For $1 \leq j \leq J$, we define a partition $\{\mathcal{R}_j, \mathcal{R}_j^c\}$ of the set $\mathcal{R}$ as

$$\mathcal{R}_j = \{r \in \mathcal{R} : r_j = 0\} \quad \text{and} \quad \mathcal{R}_j^c = \{r \in \mathcal{R} : r_j > 0\}$$

Define the mapping $g : \mathbb{Z}_+^J \to \mathcal{R}$ by $g(Q) = (r_1(Q), r_2(Q), \ldots, r_J(Q))$,

where $g(Q)$ is the vector of success probabilities when the queue length vector is Q. Since the knowledge of which queues are empty and which queues are non-empty alone is sufficient to determine the success probability vector r, we can group all queue length vectors Q that result in the same r. This is done by defining the *set-valued* map $g^{-1} : \mathcal{R} \to \mathbb{Z}_+^J$ as

$$g^{-1}(r) \;\triangleq\; \left\{Q \in \mathbb{Z}_+^J : g(Q) = r\right\}$$

We note that the collection $\left\{g^{-1}(r), r \in \mathcal{R}\right\}$ of sets defines a partition of the space queue length vectors, $\mathbb{Z}_+^J$.

Next we prove that, conditioned on the event $\{Q_j \geq 1\}$, there exist positive weights such that *the sum of weighted expected number of departures from the queues 1 to* j is equal to 1. We establish this fact for every j.

Lemma 5.1: For $1 \leq j \leq J$,

$$\mathbb{E}\left(\frac{D_j}{p_j v_j} + \mathbb{I}\{j \geq 2\}\sum_{k=1}^{j-1}\frac{D_k}{v_k}\,\middle|\, Q_j \geq 1\right) \;=\; 1$$

$$\mathbb{E}\left(\frac{D_j}{p_j v_j} + \mathbb{I}\{j \geq 2\}\sum_{k=1}^{j-1}\frac{D_k}{v_k}\,\middle|\, Q_j = 0\right) \;=\; 1 - u_j$$

■

Proof: Suppose that for some k_1 and k_2 such that $1 \leq k_1 < k_2 \leq j \leq J$, let it be true that (i) $Q_{k_1} > 0$ and $Q_{k_2} > 0$, and (ii) every other queue l, $k_1 < l < k_2$, is such that $Q_l = 0$. Then it is easy to note that $u_{k_2} = \overline{p}_{k_1} u_{k_1}$. As a consequence, we can immediately note that $u_{k_2} + p_{k_1} u_{k_1} = u_{k_1}$. Then $\mathbb{E}\left(\frac{D_j}{p_j v_j} + \mathbb{I}\{j \geq 2\}\sum_{k=1}^{j-1}\frac{D_k}{v_k}\,\middle|\, Q_j \geq 1\right)$

$$\stackrel{(a)}{=} \quad u_j + \mathbb{I}\{j \geq 2\}\sum_{k=1}^{j-1} p_k u_k \mathbb{I}\left\{Q_k > 0\right\} \stackrel{(b)}{=} 1$$

where (a) follows from (6) and (b) follows by repeatedly applying the observation made above. Almost on the similar lines, we can also note that $\mathbb{E}\left(\frac{D_j}{p_j v_j} + \mathbb{I}\{j \geq 2\}\sum_{k=1}^{j-1}\frac{D_k}{v_k}\,\middle|\, Q_j = 0\right)$

$$= \quad \sum_{k=1}^{j-1} p_k u_k \mathbb{I}\left\{Q_k > 0\right\} = 1 - u_j$$

■

Now, we are at a stage to verify the Assumption 4.1. To verify the Assumption 4.1, consider the Lyapunov functions V_j, $1 \leq j \leq J$, defined as

$$V_j(Q) \;=\; \frac{Q_j}{v_j p_j} + \mathbb{I}\{j \geq 2\}\sum_{k=1}^{j-1}\frac{Q_k}{v_k}$$

Over the state space $\{Q_j \geq 1\}$, the drift $\Delta V_j(Q)$ can be written as

$$\begin{aligned}\Delta V_j(Q) \;&=\; \frac{\lambda_j}{v_j p_j} + \mathbb{I}\{j \geq 2\}\sum_{k=1}^{j-1}\frac{\lambda_k}{v_k} \\ &\quad -\mathbb{E}\left(\frac{D_j}{p_j v_j} + \mathbb{I}\{j \geq 2\}\sum_{k=1}^{j-1}\frac{D_k}{v_k}\,\middle|\, Q_j \geq 1\right) \\ &=\; \frac{\lambda_j}{v_j p_j} + \mathbb{I}\{j \geq 2\}\sum_{k=1}^{j-1}\frac{\lambda_k}{v_k} - 1 \end{aligned} \tag{7}$$

Likewise, the drift $\Delta V_j(Q)$ over the state space $\{Q_j = 0\}$ can be written as

$$\begin{aligned}\Delta V_j(Q) \;&=\; \frac{\lambda_j}{v_j p_j} + \mathbb{I}\{j \geq 2\}\sum_{k=1}^{j-1}\frac{\lambda_k}{v_k} \\ &\quad -(1 - u_j)\end{aligned} \tag{8}$$

where (7) and (8) follow from Lemma 5.1.

Remark 5.1: For a given $Q \in \mathbb{Z}_+^J$, we can note that the drift $\Delta V_i(Q)$ depends on Q *only* through the mapping g. Hence for $Q, Q' \in \mathbb{Z}_+^J$ such that $Q \neq Q'$ but $g(Q) = g(Q')$, we have that $\Delta V_i(Q) = \Delta V_i(Q')$. For this reason, we sometimes use the notation $\Delta V_i(r)$ instead of $\Delta V_i(Q)$ as long as no confusion arises.

We now verify Assumption 4.2. Define $\mathcal{A}_{j,k}^c = \{Q \in \mathbb{Z}_+^J : Q_j \geq k\}$ and $\mathcal{A}_{j,k} = \{Q \in \mathbb{Z}_+^J : Q_j < k\}$. We can easily see that $\cap_{j=1}^J \mathcal{A}_{j,k}$ is a finite set. Before we proceed to verify Assumption 4.3, we first state Lemma 5.2 in which we establish the usual stochastic order of the queue length process $\{Q^n, n \geq 1\}$ in the initial state Q^0. For any random variable Z and an event A, let $(Z|A)$ denote any random variable that has its distribution the conditional distribution of Z given A.

Lemma 5.2: Let $Q^0 \leq \hat{Q}^0$. Then $\left(Q^n\middle| Q^0\right) \leq_{\text{st}} \left(\hat{Q}^n\middle|\hat{Q}^0\right)$ for $n \geq 1$. ■

Proof: The proof consists of verifying Theorem 4.2 in the present context. Consider the *common probability space* formed by the vector processes $\{Y^n, n \geq 1\}$ and $\{\Lambda^n, n \geq 1\}$. We start with the Assumption that $Q^n \leq \hat{Q}^n$ and then show that $Q^{n+1} \leq \hat{Q}^{n+1}$ for any *fixed* sample path of the above common probability space. Equivalently, we show that $\hat{D}_j^n \leq D_j^n$. Using inductive arguments, given $Q^0 \leq \hat{Q}^0$, we deduce that $\left(Q^{n+1}\middle| Q^0\right) \leq \left(\hat{Q}^{n+1}\middle|\hat{Q}^0\right)$ for $n \geq 1$. Finally, application Lemma 5.2 guarantees that $\left(Q^n\middle| Q^0\right) \leq_{\text{st}} \left(\hat{Q}^n\middle|\hat{Q}^0\right)$ for $n \geq 1$. For the queue j, we can make the following observations.

1) If $Q_j^n + 1 \leq \hat{Q}_j^n$, then nothing needs to be proved because $\hat{Q}_j^{n+1} \geq Q_j^n \geq Q_j^{n+1}$.

2) If $Q_j^n = \hat{Q}_j^n = 0$, then again nothing needs to be proved.
3) Let $Q_j^n = \hat{Q}_j^n \geq 1$. The assumption that $Q^n \leq \hat{Q}^n$ implies $\mathbb{I}\left\{\hat{Q}_j^n > 0\right\} \geq \mathbb{I}\left\{Q_j^n > 0\right\}$ for $1 \leq j \leq J$. Then, we can deduce that $D_j^n \geq \hat{D}_j^n$. ■

To verify the Assumption 4.3, we show that $\frac{1}{k}\Delta^k V_j(Q) \leq \frac{1}{k}\Delta^k V_j(0)$ for $1 \leq j \leq J$, $k \geq 1$, and $\forall Q$. The key idea to accomplish this is to write $\frac{\Delta V_j^k(Q)}{k}$ as a weighted sum of probabilities of Lower sets (introduced in Section IV-B) in $\mathbb{Z}_+^J$ which are such that $Q_j = 0$. Before we do that, we first need to establish the following simple fact.

Lemma 5.3: For $r \in \mathcal{R}_j$, the set $\mathcal{B}_j(r) = \bigcup_{r' \in \mathcal{R}_j : u_j(r') \geq u_j(r)} g^{-1}(r')$ is a Lower set. ■

Proof: To prove this, we need to show that $Q' \in \mathcal{B}_j(r)$ whenever $Q' \leq Q$ and $Q \in \mathcal{B}_j(r)$. But $u_j(Q') \geq u_j(Q)$. Hence $Q' \in \mathcal{B}_j(r)$. ■

For $Q \in \mathbb{Z}_+^J$, $\frac{1}{k}\Delta^k V_j(Q)$

$$\begin{aligned}
&= \sum_{Q' \in \mathbb{Z}_+^J} \Delta V_j(Q') \frac{1}{k} \sum_{l=0}^{k-1} p_{Q,Q'}^l \\
&= \sum_{r \in \mathcal{R}} \sum_{Q' \in g^{-1}(r)} \Delta V_j(Q') \frac{1}{k} \sum_{l=0}^{k-1} p_{Q,Q'}^l \\
&= \sum_{r \in \mathcal{R}} \Delta V_j(r) \frac{1}{k} \sum_{l=0}^{k-1} p_{Q,g^{-1}(r)}^l \\
&= \sum_{r \in \mathcal{R}_j} \Delta V_j(r) \frac{1}{k} \sum_{l=0}^{k-1} p_{Q,g^{-1}(r)}^l + \\
&\quad \sum_{r \in \mathcal{R}_j^c} \Delta V_j(r) \frac{1}{k} \sum_{l=0}^{k-1} p_{Q,g^{-1}(r)}^l \\
&= \sum_{r \in \mathcal{R}_j} u_j(r) \frac{1}{k} \sum_{l=0}^{k-1} p_{Q,g^{-1}(r)}^l \\
&\quad + \Delta V_j(r \in \mathcal{R}_j^c) \qquad (9)
\end{aligned}$$

Let there be N elements in the set $\mathcal{R}_j$. Let $\left\{r^1, r^2, \ldots, r^N\right\}$ denote a permutation of the set $\mathcal{R}_j$ such that $u_j(r^1) < u_j(r^2) < \cdots < u_j(r^N)$. Then define $\mathsf{C}_j^1 = u_j(r^1)$ and $\mathsf{C}_j^k = u_j(r^k) - u_j(r^{k-1})$ for $2 \leq k \leq N$. Define the sets

$$\mathcal{B}_j^k = \bigcup_{r \in \mathcal{R}_j : u_j(r) \geq u_j(r^k)} g^{-1}(r), \quad 1 \leq k \leq N$$

Now, equation (9) can be expressed as $\frac{1}{k}\Delta^k V_j(Q)$

$$\begin{aligned}
&= \sum_{r \in \mathcal{R}_j} u_j(r) \frac{1}{k} \sum_{l=0}^{k-1} p_{Q,g^{-1}(r)}^l + \Delta V_j(r \in \mathcal{R}_j^c) \\
&= \sum_{i=1}^{N} \mathsf{C}_j^i \frac{1}{k} \sum_{l=0}^{k-1} p_{Q,\mathcal{B}_j^i}^l + \Delta V_j(r \in \mathcal{R}_j^c) \\
&\overset{(a)}{\leq} \sum_{i=1}^{N} \mathsf{C}_j^i \frac{1}{k} \sum_{l=0}^{k-1} p_{0,\mathcal{B}_j^i}^l + \Delta V_j(r \in \mathcal{R}_j^c) \\
&= \frac{1}{k}\Delta^k V_j(0)
\end{aligned}$$

where (a) follows from Lemma 5.2 when we set $Q^0 = 0$, the zero state.

VI. Conclusion

In this paper, we have revisited the stability analysis of slotted-Aloha protocol with finite number of queues by applying Rosebrg's positive recurrence criterion [1]. An aim in this paper has been to illustrate how stochastic monotonicity arguments in conjunction with Lyapunov-drift properties can be used in establishing positive recurrence of a Markov chain in a countable space setting. We have seen that two steps are involved in verifying Theorem 4.1. The first step involves verifying Assumptions 4.1 and 4.2. The second step is about verifying Assumption 4.3 and is also the stage where we invoke stochastic monotonicity arguments of the underlying Markov chain. Our experience so far has been that one of these two steps is hard to verify, if not both, depending on the problem. A simplifying feature of this positive recurrence criterion we believe is that it allows one to think of Lyapunov-drift properties confined to certain *proper* subsets of the state space, which is relatively simpler, rather than the entire state space, which is harder.

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